Math Circle Nexus: Connecting points on a circumference and counting regions

From “Out of the Labyrinth” by Robert and Ellen Kaplan, chapter 4.3 (lightly edited)

with additional questions and thoughts by our community, and you, in the right column.

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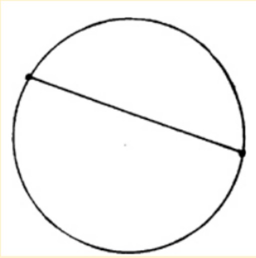
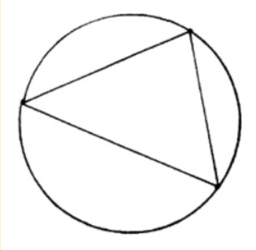
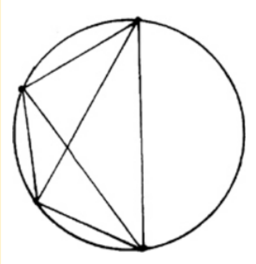


Put two points anywhere on a circle’s circumference and draw the chord connecting them.

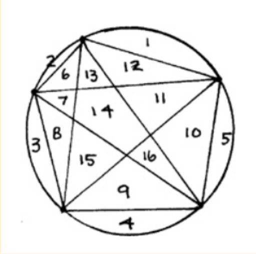
Into how many regions has the circle’s interior been divided? Clearly two.



And with three points? Obviously four.



Four points give eight regions, when we draw all possible chords,



and five yield sixteen.

More questions, notes, thoughts:

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Anyone who hasn't conjectured by now that n points yield $2^n - 1$ regions has been reading too quickly. Just to make assurance doubly sure, go back to the n = 1 case: does one point on the circumference produce $2^1 - 1 = 1$ region? Yes. Having seen five examples, the time is past to test—but ripe to prove. Oddly enough, this isn't as easy as it should be: why the number of regions should double with each successive point on the boundary isn't clear.

More questions, notes, thoughts:

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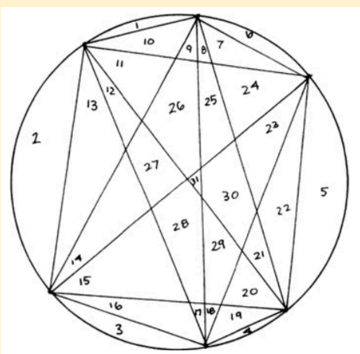

Eventually you doodle your way to another example, for want of anything better to try. Perhaps when n = 6 the cause behind the doubling will become clear.

For n = 6 we get thirty-one regions—which must simply be an error in our drawing—too many lines in too small a space

More questions, notes, thoughts:

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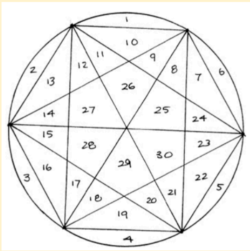

When we posed this problem to a Math Circle class of college students, one even came up with just thirty regions, because she had drawn a regular hexagon to start with, and its regularity caused a region to shrink to an internal crossing point.

This must be the answer then ("Save the hypothesis!"). The question should really have been put this way: what is the maximum number of regions that all possible chords among n boundary points create in a circle's interior? Our thirty-one regions must have come from the six points not having been distributed maximally enough. As a friend of ours once remarked, nothing in mathematics is as difficult as counting. For some reason or other, however, none of us were able to come up with the desired thirty-two, no matter how we arranged our six points.

Bob adds, remember how long the Ptolemaic astronomers added epicycles onto epicycles, to keep some sort of fit between data and theory. Facts dent our hypothesis but fail to break it.

More questions, notes, thoughts:

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Two students got hold of a large piece of drawing paper and decided to see what would happen with seven points on their circle's boundary. Meanwhile, the students with their seven points could come up with no more than fifty-seven regions instead of the predicted sixty-four.

More questions, notes, thoughts:

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The students regretfully gave up their beloved powers of two and began to look at how boundary points, and the chords among them, create regions. Forced back and back, they had eventually to ask what a chord was and how it arose—and this clearing of vision proved to be decisive. Any two points on the boundary produce a chord; or differently put, a chord is, in effect, a pair of points. How many chords will n points therefore produce? One for every two points, so n points taken two at a time—oh!—${}_nC_2$.

Avital clarifies, ${}_nC_2$ (pronounced "n choose 2") is the number of handshakes in a room with n people. [or stated more generally, the number of pairs from a set of n elements]. ${}_nC_2$ turns out to equal ½n(n-1). [much more to say about ${}_nC_2$, to be explored in a future Nexus case study]

Tifin points out, while writing a table with the number of chords for various numbers of points, that all of the numbers of chords are all of the triangular numbers.

More questions, notes, thoughts:

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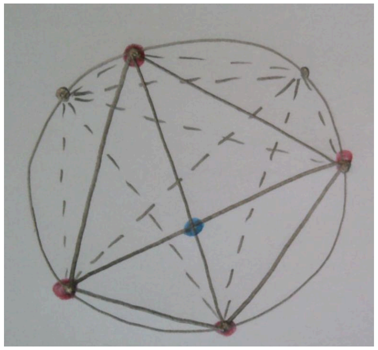
That doesn't give all the regions, however, since many new ones are created where two chords intersect. And what does that mean? A new region - one for every four points: ${}_nC_4$. The total number of regions made by n boundary points should therefore be ${}_nC_2 + {}_nC_4$.

Avital clarifies the segment: Every chord that we draw splits a region into two. **That** [counting all chords] **doesn't give all the regions, however, since many new ones are created where two chords intersect**: as you draw a new chord, immediately as it intersects any existing chord, a region is also split into two**. And what does that mean?** The total number of regions made by n boundary points shall be equal to (the total number of chords) + (the total number of intersections). We already figured out how to count chords (${}_nC_2$), but how do we count intersections?



Each intersection can be identified as the intersection of two chords. But not all two chords intersect. Thinking of the case of n=4 for insight, one notices that every set of 4 points on the circumference define a single intersection point. Vice versa is also true: each intersection is unique identified by the four circumference points that are the edges of the chords that go through the point. That is, there is a 1-to-1 correspondence between any **new region that is formed by a new intersection** and any set of **four points**, of which there are ${}_n\mathbf{C}_4$ [the number of ways to choose sets of 4 elements out of n items, which turns out to be equal to n(n-1)(n-2)(n-3)/24 [TODO: Add Nexus link here]]. Combining this insight with the previous formula for the total number of regions, we find that **the total number of regions made by n boundary points should therefore be** ${}_n\mathbf{C}_2 + {}_n\mathbf{C}_4$.

More questions, notes, thoughts:

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Try it. For n = 3, for example, ${}_3C_2 + {}_3C_4 = 3$ (since ${}_3C_2 = 3!/2!$, and ${}_3C_4$ doesn't mean anything, so adds no regions), but this is one less than the needed 4. We need a last bit of fine tuning: there was, after all, the one region of the circle we began with, prior to drawing chords—so the number of regions made by all chords drawn among n boundary points should be $1 + {}_nC_2 + {}_nC_4$.

More questions, notes, thoughts:

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When Jim Tanton presented this problem to a group of eight- to ten-year-olds in The Math Circle, in 2001, "They decided," he writes, "to count everything: the number of circles C (always 1!), the number of dots along the boundary, the number of paths drawn L (lines), the number of intersections I, and the number of regions P (pieces). It soon appeared that C + L + I = P and 'CLIP Theory', as they called it, was born.

They noted that the theory was true for very simple diagrams and seemed to remain true if an extra path were added to a diagram. They reasoned as follows—as soon as this line intersects a pre-existing path, two things occur: the count of intersections increases by one, and a region is split in two, increasing the count of pieces by one. The formula C + L + I = P remains balanced. This equation also remains valid when the path eventually returns to the circumference of the circle: the path is completed, thereby increasing the number of lines by one, which is balanced by the fact that a final region is split in two."

Avital asks, how would a group come up with the 'CLIP Theory'?

More questions, notes, thoughts:

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