Exam 3 Review

Chapters 6, 7, 8, 9 and 10

Determine whether each proposition is true or false, and then prove or disprove it. Clearly indicate the method of proof.

1. Proposition. Given an integer a, then a is even if and only if $a^3 + 3a^2 + 5a$ is even.
2. Proposition. For every integer n, either $4|n^2$ or $4|(n^2 - 1)$.
3. Proposition. The number $\sqrt{15}$ is irrational.
4. Proposition. The set $A = \{a \in \mathbb{N}: a\ is\ prime\ \wedge a \geq 100 \wedge a \leq 110\}$ has cardinality greater than 2.
5. Proposition. If $A, B,$ and C are sets, then $(A \cup B) - C = (A - C) \cup (B - C)$.
6. Proposition. If A and B are sets, then $(A - B) \times B = (A \times B) - (B \times B)$.
7. Proposition. If a irrational and ab is rational then b is irrational.
8. Proposition. If $A, B,$ and C are sets, then $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
9. Proposition. If A and B are sets, then $P(A) - P(B) \subseteq P(A - B)$.
10. Proposition. If $x, y \in \mathbb{R}$ and $x^2 < y^2$ then $x < y$.
11. Proposition. $\forall n \in \mathbb{N}, 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + ... + n \cdot (n + 2) = \frac{n(n+1)(2n+7)}{6}$.
12. Proposition. For any integer $n \geq 0$, it follows that $9|(4^{3n} + 8)$.
13. Proposition. Concerning the Fibonacci sequence, prove that for every n in the natural numbers, $F_1 + F_3 + F_5 + ... + F_{2n-1} = F_{2n}$.

Exam 3 Review ANSWER KEY

If you discover an error please let me know, either in class, on the OpenLab, or by email to jreitz@citytech.cuny.edu. Corrections will be posted on the "Exam Reviews" page.

NOTE 1: For problems requiring you to prove something, there is usually more than one correct answer, and it is often possible to use more than one different type of proof (direct, contrapositive, or contradiction) correctly. The following are examples of correct solutions, yours may be different.
NOTE 2: As we have been working with proofs for several weeks, there are a few facts that we have used many times - for example the definitions of even and odd number - and which have become familiar and second-nature. You will notice that I will start moving away from stating explicitly when I employ these definitions, leaving it up to you to (mentally) fill in the justification when I say something like "n is even, so $n = 2a$ for some integer a".

1. Proposition. Given an integer a, then a is even if and only if $a^3 + 3a^2 + 5a$ is even.
   TRUE.
   Proof. (Forward direction ⇒, direct proof). Suppose a is even. Then $a = 2b$ for some integer b. So $a^3 + 3a^2 + 5a = (2b)^2 + 3(2b)^2 + 5(2b) = 2(4b^3 + 6b^2 + 5b)$, which is even.
   (Backward direction ⇐, contrapositive proof). Suppose a is not even. Then a is odd, so $a = 2b + 1$ for some integer b. So
   $a^3 + 3a^2 + 5a = (2b + 1)^3 + 3(2b + 1)^2 + 5(2b + 1) = 2(4b^3 + 12b^2 + 14b + 4) + 1$, which is odd. ☐
2. Proposition. For every integer n, either $4|n^2$ or $4|(n^2 - 1)$.
   TRUE.
   *Proof. (*Direct proof). Suppose n is an integer. Then n is either even or odd.
   Case 1. Suppose n is even. Then $n = 2a$ for some integer a, and so $n^2 = (2a)^2 = 4a^2$. Thus $4|n^2$.
   Case 2. Suppose n is odd. Then $n = 2a + 1$ for an integer a, and $n^2 = (2a + 1)^2 = 4a^2 + 4a + 1$. Subtracting one from both sides, we see that $n^2 - 1 = 4(a^2 + a)$, and so $4|(n^2 - 1)$. This completes the proof. ☐
3. Proposition. The number $\sqrt{15}$ is irrational.
   TRUE.
   Proof. (Proof by contradiction). Suppose that $\sqrt{15}$ is rational. Then $\sqrt{15} = \frac{a}{b}$ for some integers a and b with no common factors. It follows that $15 = \frac{b^2}{a^2}$ and so $15a^2 = b^2$ and $3(5a^2) = b^2$. Thus 3 divides b^2, and by Euclid's Lemma it follows that 3 divides b, giving $b = 3k$ for some integer k. Substituting, we have $15a^2 = (3k)^2 = 9k^2$, and dividing by 3 we get $5a^2 = 3k^2$. We have shown that 3 divides $5a^2$ and, applying Euclid's Lemma twice, we see that 3 divides a. This

contradicts the assumption that a and b have no common factors.☐

4. The set $A = \{a \in \mathbb{N}: a\ is\ prime\ \wedge a \geq 100 \wedge a \leq 110\}$ has cardinality greater than 2.
   TRUE.
   Proof. (Proof by example!) The numbers 101, 103, 107 and 109 are all primes between 100 and 110. Thus they are all members of A, and so A has cardinality greater than 2. ☐
5. Proposition. If A, B, and C are sets, then $(A \cup B) - C = (A - C) \cup (B - C)$.
   TRUE.
   Proof. (Forward direction, ⊆, direct proof). Suppose $a \in (A \cup B) - C$. Then a is a member of A or B, but $a \notin C$. If $a \in A$ then $a \in (A - C)$, and if $a \in B$ then $a \in (B - C)$. In either case, we have $a \in (A - C) \cup (B - C)$. Therefore $(A \cup B) - C \subseteq (A - C) \cup (B - C)$
   (Backward direction, ⊇, direct proof). Conversely, suppose $a \in (A - C) \cup (B - C)$. Then a is either in $(A - C)$ or in $(B - C)$. If $a \in (A - C)$ then $a \in A$ and $a \notin C$, and if $a \in (B - C)$ then $a \in B$ and $a \notin C$. In either case, $a \notin C$, and so we have shown that a is a member of either A or B, but $a \notin C$. Thus $a \in (A \cup B) - C$, and so
   $(A \cup B) - C \supseteq (A - C) \cup (B - C)$.
   Therefore $(A \cup B) - C = (A - C) \cup (B - C)$. ☐
6. Proposition. If A and B are sets, then $(A - B) \times B = (A \times B) - (B \times B)$.
   TRUE.
   Proof. (Forward direction, ⊆, direct proof). Suppose $(a, b) \in (A - B) \times B$. Then $a \in A$, $a \notin B$ and $b \in B$. Since $a \in A$ and $b \in B$, we have $(a, b) \in (A \times B)$, and since $a \notin B$ we have $(a, b) \notin (B \times B)$. Thus $(a, b) \in (A \times B) - (B \times B)$, and so
   $(A - B) \times B \subseteq (A \times B) - (B \times B)$
   (Backward direction, ⊇, direct proof). Conversely, suppose $(a, b) \in (A \times B) - (B \times B)$. From $(a, b) \in (A \times B)$ we conclude that $a \in A$ and $b \in B$. Since $(a, b) \notin (B \times B)$we must have either $a \notin B$ or $b \notin B$, and since we have shown that $b \in B$ it follows that $a \notin B$. Thus $a \in (A - B)$, and so $(a, b) \in (A - B) \times B$. This shows that
   $(A - B) \times B \supseteq (A \times B) - (B \times B)$
   Therefore $(A - B) \times B = (A \times B) - (B \times B)$.☐
7. Proposition. If a irrational and ab is rational then b is irrational.
   TRUE.
   Proof. (Proof by contradiction). Suppose that a is irrational and ab is rational, and b is rational. Then $ab = \frac{p}{q}$ and $b = \frac{r}{s}$ where $a, b, c, d \in \mathbb{Z}$ (by the definition of rational), and so $\frac{p}{q} = ab = a \cdot \frac{r}{s}$. Thus $\frac{p}{q} = a \cdot \frac{r}{s}$, so solving for a give $a = \frac{ps}{qr}$. Since ps and qr are integers, it follows that a is rational, which contradicts our assumption that a is irrational. ☐
8. Proposition. If A, B, and C are sets, then $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
   TRUE.
   Proof. (Forward direction, ⊆, direct proof). Suppose $a \in (A \cap B) \cup C$, so either $a \in A \cap B$ or $a \in C$.
   Case 1. If $a \in A \cap B$ then $a \in A$ and $a \in B$, and it follows that $a \in (A \cup C)$ and $a \in (B \cup C)$, so we have $a \in (A \cup C) \cap (B \cup C)$.
   Case 2. If $a \in C$, then $a \in (A \cup C)$ and $a \in (B \cup C)$, so we have $a \in (A \cup C) \cap (B \cup C)$.
   Therefore $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$.

(Backward direction, $\supseteq$, direct proof). Conversely, suppose $a \in (A \cup C) \cap (B \cup C)$. Then a is in either A or C, and in either B or C. I will consider separately the cases $a \in C$ and $a \notin C$.
Case 1. If $a \in C$, then $a \in (A \cap B) \cup C$.
Case 2. If $a \notin C$, then it follow that a is in A (since it is in either A or C), and similarly it follows that a is in B. Thus $a \in A \cap B$, and so $a \in (A \cap B) \cup C$. This shows $(A \cap B) \cup C \supseteq (A \cup C) \cap (B \cup C)$.
Therefore $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.□

9. Proposition. If A and B are sets, then $P(A) - P(B) \subseteq P(A - B)$.
FALSE.
Disproof. (Counterexample) Consider the sets $A = \{1, 2\}$ and $B = \{1\}$. The set $\{1, 2\}$ is a subset of A but not a subset of B, and so it it in $P(A) - P(B)$. However, it is not a subset of $A - B = \{2\}$, and so it is not in $P(A - B)$.□

10. Proposition. If $x, y \in \mathbb{R}$ and $x^2 < y^2$ then $x < y$.
FALSE.
Disproof. (Counterexample) Suppose $x = 1$ and $y =- 2$. Then $x^2 < y^2$ since $1 < 4$, but $x > y$. □

11. Proposition. $\forall n \in \mathbb{N}, 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 +...+ n \cdot (n + 2) = \frac{n(n+1)(2n+7)}{6}$.
TRUE
Proof. (Proof by induction)
Base step. If $n = 1$ then we have $1 \cdot 3 = \frac{1(1+2)(2\cdot1+7)}{6}$, or $3 = \frac{18}{6}$, which is true.
Inductive step. For a natural number k, we assume
$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 +...+ k \cdot (k + 2) = \frac{k(k+1)(2k+7)}{6}$. Adding $(k + 1)(k + 3)$ to both sides, we obtain:
$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 +...+ k \cdot (k + 2) + (k + 1)(k + 3) = \frac{k(k+1)(2k+7)}{6} + (k + 1)(k + 3)$. Note that the left hand matches the left hand side of the $n = k + 1$ case, so we will focus on the right hand side.

$$\frac{k(k+1)(2k+7)}{6} + (k + 1)(k + 3) = \frac{2k^3+9k^2+7k}{6} + \frac{6(k^2+4k+3)}{6}$$
$$= \frac{2k^3+15k^2+31k+18}{6}$$
$$= \frac{(k+1)(k+2)(2(k+1)+7)}{6}$$

Thus $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 +...+ k \cdot (k + 2) + (k + 1)(k + 3) = \frac{(k+1)(k+2)(2(k+1)+7)}{6}$.
Therefore by induction we have shown $\forall n \in \mathbb{N}$,
$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 +...+ n \cdot (n + 2) = \frac{n(n+1)(2n+7)}{6}$.

12. Proposition. For any integer $n \geq 0$, it follows that $9|(4^{3n} + 8)$.
TRUE
Proof. (Proof by induction)
Base step. If $n = 0$, then $4^{3\cdot0} + 8 = 9$, and we have $9|9$.
Inductive step. Assume $9|4^{3k} + 8$. Then $4^{3k} + 8 = 9a$ for some integer a. Multiplying both

sides by $4^3 = 64$, we have:

$4^3 \cdot 4^{3k} + 64 \cdot 8 = 64 \cdot 9a$

$4^{3k+3} + 512 = 576a$

Subtracting 504 from both sides, we obtain

$4^{3(k+1)} + 8 = 576a - 504$

$4^{3(k+1)} + 8 = 9(64a - 56)$

and so $9|4^{3(k+1)} + 8$.

Thus by induction we have $\forall n \in \mathbb{N}, 9|(4^{3n} + 8)$.☐

13. Proposition. Concerning the Fibonacci sequence, prove that for every *n* in the natural numbers, $F_1 + F_3 + F_5 + ... + F_{2n-1} = F_{2n}$.

TRUE

Proof. (Proof by induction).

Base step. $F_1 = F_2$, or 1=1.

Inductive step. Suppose $F_1 + F_3 + F_5 + ... + F_{2k-1} = F_{2k}$. Adding F_{2k+1} to both sides, we have

$F_1 + F_3 + F_5 + ... + F_{2k-1} + F_{2k+1} = F_{2k} + F_{2k+1}$

From the definition of the Fibonacci sequence, we have $F_{2k} + F_{2k+1} = F_{2k+2}$, so

$F_1 + F_3 + F_5 + ... + F_{2k-1} + F_{2k+1} = F_{2k+2}$

Carefully rewriting subscripts on both sides we obtain

$F_1 + F_3 + F_5 + ... + F_{2k-1} + F_{2(k+1)-1} = F_{2(k+1)}$.

Thus by induction we have shown that for every *n* in the natural numbers,

$F_1 + F_3 + F_5 + ... + F_{2n-1} = F_{2n}$. ☐