Topology: introduction (2011)

short address for this document: https://goo.gl/Yr71Mm

-- There is a mailing list for the course; please e-mail me at leyenson@gmail.com to get onto the list

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== Short Program ==

1. Fundamental group and examples. The Kunneth Formula. Fundamental group of 2-manifolds.

2. Algebraic equations (in one complex variable) of degree 3, $z^3 + pz + q = 0$: topological Galois theory

3. Category of covers of a given space; universal cover.

4. First homology group: an idea.

== Bibliography, partial ==

1. D.B. Fuchs, V.L. Gutenmacher and A.T. Fomenko, "Homotopic topology", 1986, QA612.7 .F8413 (openlibrary.org) (Google Books) (Amazon)

2. Hatcher, "Algebraic topology", (online here)

== Detailed program ==

Lecture #1

* fundamental group.
  Examples: * fundamental group of a circle S^1 (with proof);

relation to the complex analytic map z |-> z^p;

* pi_1 of the wedge of several circles (~ plane minus several points);

* pi_1 of the torus:

  ** via uniformization R^2 -> R^2/(Z+Z);

  ** via identification of edges of a square;

  ** via T = S^1 x S^1 -- see the next topic.

* Guessing the Kunneth formula:

defining the maps

pi_1(X x Y) -> pi_1(X) x pi_1(Y),

pi_1(X) -> pi_1(X x Y) ,
pi_1(Y) -> pi_1(X x Y).

Homework 1: Do the images commute? (They should, as the case of a torus suggests.)

Homework 2: genus 2 surface has "canonical" cycles a_1, b_1, a_2, b_2, as discussed in class;

Study of the commutator [a_1 b_1 a_1^{-1} b_1^{-1}],
and “the middle section” of X.

Finish the reasoning: is this commutator trivial?

* (At the request of Yair: defining RP^2 as a space of lines in R^3.

Homework - 3: Gluing RP^2 out of a square.

Lecture # 2

we considered the polynomial

$$f(z,t) = z^3 - 3 z + 2 t = 0$$

which gives a curve C in the complex plane with coordinates (z,t). C can be considered as a "universal polynomial of degree 3 in z".

C has a map to a line, $(z,t) \mapsto t$, which is a ramified cover of degree 3.

We computed the critical points and critical values of this map, and the corresponding monodromy representation

$$\pi_1(C - \{2 \text{ points}\}) \to Sym_3$$

which we proved to be epimorphic.

At a later point we will discuss the relation of this maps with the algebraic problem of solving equations of degree 3.

Lecture #3

We discussed the notion of smooth manifold and the classification of smooth oriented (compact) manifolds of dimension two; they are classified by one integer, the genus g.

Let X_g be such a surface of genus g. We computed previously the fundamental group of a torus X_1, and this time the fundamental group of the X_2, in 2 ways:

(a) by studying the connected sum of 2 toruses, $X_2 = X_1 \# X_1$

(Meanwhile, we discussed the operation of a connected sum of manifolds. Remark: the Wikipedia article on "Connected sum" is rather good".)

(b) by gluing X_2 out of a polygon with 8 sides (in “the standard way”).

Homework 4:

(1) compute pi_1(RP^2)

(2) for two spaces X,Y, we constructed the maps

i_1: pi_1(X) -> pi_1(X x Y),
i_2: pi_1(Y) -> pi_1(X x Y).

Prove that their images commute.

(We also constructed maps

p_1: pi_1(X x Y) -> pi_1(X),
p_2: pi_1(X x Y) -> pi_1(X))

(3)^* compute the commutator of the group G = pi_1(X_2) for a genus 2 (compact orientable) surface X.

Lecture # 4

1) We continued study of the fundamental group of a sphere with two handles (genus 2 surface, let us call it X_2):

we computed its abelianization (pi_1 / [pi_1, pi_1]), introduced the first homology group H_1 as the abelianization pi_1 (note that this is not the standard/usual definition), and discussed its geometric meaning (rather briefly)

(2) We discussed the projective plane RP^2 in details (as a moduli of lines in R^3; as a factor of S^2 by an involution; as a result of gluing of opposite sides of a square; and by obtaining it from upper hemi-sphere by identifying opposite points on the equator circle).

We computed the fundamental group π_1 (isomorphic to $Z/2$) in two different models.

We also gave an interpretation of the non-trivial element of fundamental group of RP^2 in terms of rotations of lines (we start rotating a sphere until 2 opposite points get exchanged, and lines joining them switches its orientation); thus we constructed a non-trivial element in the group $SO(3)$ which we also discussed.

Lecture # 5

1. Let $p: X \to Y$ be a cover with a fiber F, (where Y is connected). We discussed the map

$p_*: \pi_1(X) \to \pi_1(Y)$, and proved the "exact sequence"

$$1 \to \pi_1(X) \to \pi_1(Y) \to F$$

(the first two terms are groups, and the last one, the fiber F, is a set with a fixed point.)

We also proved that X is connected if and only if the last map is an epimorphism (of sets).

All this makes sense after we fix a point x in X, y in Y (x is over y), the fiber $F = F_y$ and the fundamental groups are computed with respect to the base points x and y.

We then gave a definition of regular (or Galois, or normal) covers.

We then discussed two problems: given X, describe all covers X -> Y; and, given Y, describe all covers X -> Y.

For the second problem, we constructed the map

φ: (covers p: X -> Y for a fixed Y) ->

(subgroups in pi_1(Y)),

where phi(p) = pi_1(X) as a subgroup of pi_1(Y).

Lecture # 6

* topological characterization of normal covers X -> Y: a loop in Y, given a point x in the fiber F_y, lifts either to a loop for each x , or to a path (for each x), but not both ways for various points in F_y;

* correspondence (covers) <-> subgroups requires subgroup {1}, i.e.,existence of the universal cover;

* we constructed the universal cover;

* We constructed a regular cover H -> H/Gamma for a free action of a group Gamma (with a particular case of universal cover).
