Calculus
Date:
Period:

Name:

1.4 Continuity and One-Sided Limits

Definition of continuity:

f(x) is continuous at x = a if and only if (iff) $\lim_{x \to a} f(x) = f(a)$

In other words, f(x) is continuous at x = a iff limit = output

Process to determine if f(x) is continuous at x = a. GOAL: Does $\lim_{x\to a} f(x) = f(a)$? Yes – continuous No – not continuous

1. What is f(a)?

2. What is $\lim_{x\to a} f(x)$?

Is the same "rule" used for f(x) on both sides of x = a?

- YES
  - Use 1.3 limit techniques to find the limit.
  - Does $\lim_{x\to a} f(x)$ exist?
    - YES
      - Does $\lim_{x\to a} f(x) = f(a)$?
        - YES
          - f(x) is continuous at x = a.
        - NO
          - f(x) has **removeable discontinuity** at x = a.
    - NO
      - f(x) has **nonremoveable discontinuity** at x = a.
- NO
  - Find $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.
  - Does $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$?
    - YES
      - $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x)$
      - Does $\lim_{x\to a} f(x) = f(a)$?
        - YES
          - f(x) is continuous at x = a.
        - NO
          - f(x) has **removeable discontinuity** at x = a.
    - NO
      - $\lim_{x\to a} f(x)$ does not exist
      - f(x) has **nonremoveable discontinuity** at x = a.

Nonremovable discontinuity: $\lim_{x\to a} f(x)$ does not exist

Removeable discontinuity: $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} f(x) \neq f(a)$ i.e. "hole at $x=a$"

1. Is $f(x) = \begin{cases} x+3 & if\ x < 1 \\ 2 & if\ x = 1 \\ 2x+2 & if\ x > 1 \end{cases}$ continuous at x = 1? $\rightarrow$ $\boxed{f(1)=2}$

- Different rules at $x=1$ $\Rightarrow$ left & right limits

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+3) = 1+3 = 4$$

means $x<1$ so use "top line"

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x+2) = 2(1)+2 = 4$$

means $x>1$ so use "bottom line"

since $\lim\limits_{x\to 1^-} f(x) = 4 = \lim\limits_{x\to 1^+} f(x)$,

we know $\lim\limits_{x\to 1} f(x) = 4$.

However $f(1)=2$ so $\lim\limits_{x\to 1} f(x) \neq f(1)$.

$x=1$ is removable discontinuity on $f(x)$

($\lim\limits_{x\to 1} f(x)$ exists but $\lim\limits_{x\to 1} f(x) \neq f(1)$)

2. For what x-values is f(x) not continuous when $f(x) = \frac{x-1}{x^2+x-2}$? Classify each discontinuity type.

$$f(x) = \frac{x-1}{x^2+x-2} = \frac{x-1}{(x+2)(x-1)}$$

$f(x)$ is discontinuous at $x=1$ and $x=-2$

(because both cause division by 0)

Classify

$$\lim_{x\to 1} \frac{x-1}{x^2+x-2} = \lim_{x\to 1} \frac{x-1}{(x+2)(x-1)} = \lim_{x\to 1} \frac{1}{x+2} = \frac{1}{1+2} = \frac{1}{3}$$

$\therefore$ $x=1$ is removeable discontinuity

because $\lim\limits_{x\to 1} f(x)$ exists but $\lim\limits_{x\to 1} f(x) \neq f(1)$

($f(1)$ doesnot exist because $x=1$ causes $\div$ by 0)

$$\lim_{x\to -2} \frac{x-1}{x^2+x-2} = \lim_{x\to -2} \frac{x-1}{(x+2)(x-1)} = \lim_{x\to -2} \frac{1}{x-2}$$ = limit does not exist

(unbounded behavior)

$\therefore$ $x=-2$ is nonremoveable discontinuity

because $\lim\limits_{x\to -2} f(x)$ does not exist

3. Why must $f(x) = x^2 - 3x - 2$ have at least one real zero on the interval [0,7].
(A real zero is an input where the output is zero. In other words, a real zero is an x-intercept on the graph of f(x).)

* we know $f(x) = x^2 - 3x - 2$ is continuous

and $f(0) = 0^2 - 3(0) - 2 = -2$ and $f(7) = 7^2 - 3(7) - 2 = 26$

f(x) is continuous with ~~and~~ $f(0) = -2$ and $f(7) = 26$ so f(x) goes from a negative to a positive so there must be at least one input (0, 7) where $f(c) = 0$.

Theorem 1.13 **Intermediate Value Theorem**

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

#3 is an example of Intermediate Value Theorem.

* f(x) must be continuous on [a, b]

Then there must be at least one input, c on (a, b)

such that $f(c) = k$ for all values of k between f(a) & f(b)