Marking Scheme Prelim 2020 AM P2

1 The roots of the quadratic equation $5x - 2x^2 = 2 - k$ are α and β. The roots differ by $4\frac{1}{2}$.

(i) Show that $k = 9$. [4]

$2x^2 - 5x + (2-k) = 0$

Let α be the larger root.

Sum of roots: $\alpha + \beta = \frac{5}{2}$ **[M1]**

Product of roots: $\alpha\beta = \frac{2-k}{2}$ **[M1]**

Since roots differ by $4\frac{1}{2}$, $\alpha - \beta = \frac{9}{2}$

$(\alpha+\beta)^2 = (\alpha-\beta)^2 + 4\alpha\beta$

$\left(\frac{5}{2}\right)^2 = \left(\frac{9}{2}\right)^2 + 4\left(\frac{2-k}{2}\right)$ **[M1]**

$\frac{25}{4} = \frac{81}{4} + 4 - 2k$

$2k = 18$

$k = 9$ (shown)

[A1]

Alternative:

Let α be the larger root.

Sum of roots: -- (1) **[M1]**

Since roots differ by , -- (2)

+ (2):
[M1]

Product of roots:

[M1]

[A1]

(ii) Hence find a quadratic equation with integer coefficients whose roots are α^3 and β^3. [4]

Sum of roots:

$\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$ **[M1]**

$= (\alpha+\beta)[(\alpha-\beta)^2 + \alpha\beta]$

$= \frac{5}{2}\left[\left(\frac{9}{2}\right)^2 + \frac{2-9}{2}\right]$

$= \frac{335}{8}$ **[M1]**

Product of roots:

Alternative:

Sum of roots:

Product of roots:

The equation is

$$\alpha^3\beta^3 = \left(-\frac{7}{2}\right)^3$$

$$= -\frac{343}{8} \qquad \textbf{[M1]}$$

The equation is $8x^2 - 335x - 343 = 0$ **[A1]**

(accept any equivalent equation with integers coefficients)

2 **(a)** It is given that $\tan(A+B)=8$ and $\tan B=2$. **Without using a calculator**, find the exact value of $\cot A$. [3]

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Since $\tan(A+B)=8$ and $\tan B=2$,

$$8=\frac{\tan A+2}{1-2\tan A}$$ **[M1]**

$$8-16\tan A=\tan A+2$$
$$17\tan A=6$$

$$\tan A=\frac{6}{17}$$ **[M1]**

$$\cot A=\frac{17}{6}$$ **[A1]**

(b) **(i)** Prove that $\sin 2x(\cot x-\tan x)=2\cos 2x$. [3]

$$\text{LHS}=\sin 2x(\cot x-\tan x)$$
$$=2\sin x\cos x\left(\frac{\cos x}{\sin x}-\frac{\sin x}{\cos x}\right)$$ **[M1]**
$$=2(\cos^2 x-\sin^2 x)$$ **[M1]**
$$=2\cos 2x$$ **[A1]**
$$=\text{RHS (proven)}$$

(ii) Hence solve the equation $\sin 2x(\cot x-\tan x)=\sec 2x$ for $0\le x\le$.
[4]

$$\sin 2x(\cot x-\tan x)=\sec 2x$$

$$2\cos 2x=\frac{1}{\cos 2x}$$

$$\cos^2 2x=\frac{1}{2}$$ **[M1]**

$$\cos 2x = \pm\sqrt{\frac{1}{2}}$$ **[M1]**

Basic angle = $\frac{\pi}{4}$ rad

$2x = \frac{\pi}{4}, \pi, -\frac{\pi}{4}, 2\pi + \frac{\pi}{4}, -\frac{\pi}{4}$

$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **[M1]**

$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ **[A1]**

3 The equation of circle C_1, with centre A, is $x^2 + y^2 - 8x - 4y + 11 = 0$.

(i) Find the coordinates of A and the radius of C_1. [3]

$x^2 + y^2 - 8x - 4y + 11 = 0$

$(x-4)^2 - 16 + (y-2)^2 - 4 + 11 = 0$

$(x-4)^2 + (y-2)^2 = 9$ **[M1]**

coordinates of A = (4, 2) **[A1]**

radius of C_1 = 3 units **[A1]**

(ii) Explain why the line $x = 1$ is a tangent to C_1. [1]

Since the leftmost coordinates of the circle = (1, 2), $x = 1$ is a tangent to C_1. **[B1]**

The equation of a second circle C_2, with centre B, is $(x+3)^2 + (y-2)^2 = R^2$.

(iii) Find the length of AB. [2]

coordinates of B = $(-3, 2)$ **[M1]**

length of AB = 7 units **[A1]**

(iv) State the possible value(s) of R such that C_1 and C_2 touch each other at exactly one point. [2]

$R = 7 - 3 = 4$ **[B1]** or $R = 7 + 3 = 10$ **[B1]**

4 **(a)** Given that $\log_{\sqrt{2}} x = m$ and $\log_4 y = n$, express $\dfrac{\sqrt{y}}{x^6}$ in terms of m and n. [4]

$\log_{\sqrt{2}} x = m$

$x = \left(\sqrt{2}\right)^m$ --- (1)

$\log_4 y = n$

[M1 for both (1) and (2)**]**

$y = 4^n$ --- (2)

$\dfrac{\sqrt{y}}{x^6} = \dfrac{\sqrt{4^n}}{\left(\sqrt{2}\right)^{6m}}$ **[M1]**

$= \dfrac{2^n}{2^{3m}}$ **[M1]**

$= 2^{n-3m}$ **[A1]**

(b) Solve the equation $\log_2(3x-2) = \log_4\left(x^2+1\right) + \dfrac{2}{\log_{\sqrt{2}} 2}$. [6]

$\log_2(3x-2) = \log_4\left(x^2+1\right) + \dfrac{2}{\log_{\sqrt{2}} 2}$

$\log_2(3x-2) = \dfrac{\log_2\left(x^2+1\right)}{\log_2 4} + \dfrac{2\log_2\sqrt{2}}{\log_2 2}$ **[M1** change of base**]**

$\log_2(3x-2) = \dfrac{1}{2}\log_2(x^2+1) + 1$ **[M1]**

$\log_2(3x-2) = \log_2(x^2+1)^{\frac{1}{2}} + \log_2 2$

$\log_2(3x-2) = \log_2 2(x^2+1)^{\frac{1}{2}}$

[M1]

$3x-2 = 2(x^2+1)^{\frac{1}{2}}$

$(3x-2)^2 = 4(x^2+1)$

$9x^2 - 12x + 4 = 4x^2 + 4$

Alternative:

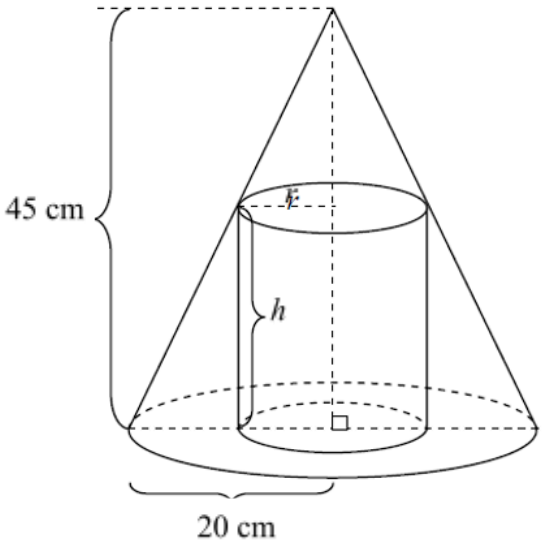
$5x^2 - 12x = 0$ **[M1]**

$x(5x - 12) = 0$

$x = 0$ (reject because $3x - 2 > 0$) **[A1** for rejecting with reason**]**

or $x = \frac{12}{5}$ **[A1]**

5 The diagram shows a solid cylinder of radius r cm and height h cm inscribed in a hollow cone of height 45 cm and base radius 20 cm. The cylinder rests on the base of the cone and the circumference of the top surface of the cylinder touches the curved surface of the cone.



(i) Show that the volume, V cm^3, of the cylinder is given by $V = 45\pi r^2 - \frac{9}{4}\pi r^3$. [3]

By similar triangles,

$\frac{45 - h}{45} = \frac{r}{20}$ **[M1]**

$20(45 - h) = 45r$

$900 - 20h = 45r$

$h = \frac{900 - 45r}{20}$

$= 45 - \frac{9}{4}r$ **[M1]**

$$V = \pi r^2 h$$
$$= \pi r^2 \left(45 - \frac{9}{4} r\right)$$
$$= 45\pi r^2 - \frac{9\pi}{4} r^3 \text{ (shown)}$$
[A1]

(ii) Given that r can vary, find the maximum volume of the cylinder, leaving your answer in terms of π. [4]

$$\frac{dV}{dr} = 90\pi r - \frac{27\pi}{4} r^2$$
[M1]

When V is maximum, $\frac{dV}{dr} = 0$

$$90\pi r - \frac{27\pi}{4} r^2 = 0$$

$$\pi r \left(90 - \frac{27}{4} r\right) = 0$$

$$r = \frac{40}{3} \text{ (since } r > 0\text{)}$$
[M1]

$$\frac{d^2V}{dr^2} = 90\pi - \frac{27\pi}{2} r$$

When $r = \frac{40}{3}$,

$$\frac{d^2V}{dr^2} = 90\pi - \frac{27\pi}{2} \left(\frac{40}{3}\right) < 0$$
[M1]

Since $\frac{d^2V}{dr^2} < 0$, volume of cylinder is maximum when $r = \frac{40}{3}$.

$$\text{maximum volume} = 45\pi \left(\frac{40}{3}\right)^2 - \frac{9\pi}{4} \left(\frac{40}{3}\right)^3$$

$$= 2666\frac{2}{3} \text{ cm}^3$$
[A1]

(iii) Hence show that the cylinder occupies at most $\frac{4}{9}$ of the volume of the cone. [2]

volume of cone = $\frac{1}{3}\pi(20^2)(45)$

$= 6000\pi \text{ cm}^3$ **[M1]**

When the volume of cylinder is the maximum,

$$\frac{\text{volume of cylinder}}{\text{volume of cone}} = \frac{2666\frac{2}{3}}{6000}$$

$$= \frac{4}{9} \quad \textbf{[A1]}$$

Therefore, the cylinder occupies at most $\frac{4}{9}$ of the volume of the cone.

6 **(i)** Express $\frac{2x^3 - 20x^2 - 17x - 10}{(x^2 - 4)(2x^2 + 1)}$ in partial fractions. [5]

$$\frac{2x^3 - 20x^2 - 17x - 10}{(x^2 - 4)(2x^2 + 1)} = \frac{2x^3 - 20x^2 - 17x - 10}{(x - 2)(x + 2)(2x^2 + 1)}$$

$$\frac{2x^3 - 20x^2 - 17x - 10}{(x - 2)(x + 2)(2x^2 + 1)} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{2x^2 + 1} \quad \textbf{[M1]}$$

$$2x^3 - 20x^2 - 17x - 10 = A(x + 2)(2x^2 + 1) + B(x - 2)(2x^2 + 1) + (Cx + D)(x - 2)(x + 2)$$

Let $x = 2$, $-108 = 36A$

$A = -3$ **[M1]**

Let $x = -2$, $-72 = -36B$

$B = 2$ **[M1]**

Let $x = 0,\ A = -3,\ B = 2$, $-10 = -6 - 4 - 4D$

$D = 0$

Let $x = 1,\ A = -3,\ B = 2,\ D = 0$, $-45 = -27 - 6 - 3C$

$$C = 4 \quad \textbf{[M1]}$$

Hence $\dfrac{2x^3 - 20x^2 - 17x - 10}{\left(x^2 - 4\right)\left(2x^2 + 1\right)} = -\dfrac{3}{x-2} + \dfrac{2}{x+2} + \dfrac{4x}{2x^2+1}$ **[A1]**

(ii) Differentiate $\ln\left(2x^2 + 1\right)$ with respect to x. [2]

$$\frac{d}{dx}\left(\ln\left(2x^2+1\right)\right) = \frac{1}{2x^2+1}\left(4x\right) \quad \textbf{[M1]}$$

$$= \frac{4x}{2x^2+1} \quad \textbf{[A1]}$$

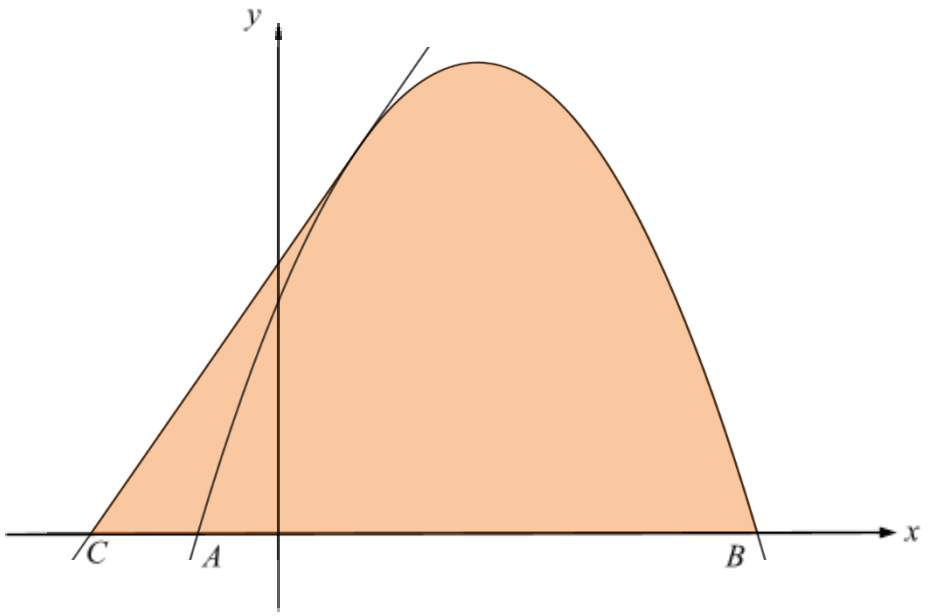
(iii) Using the results from part **(i)** and **(ii)**, find $\displaystyle\int \frac{2x^3 - 20x^2 - 17x - 10}{\left(x^2 - 4\right)\left(2x^2 + 1\right)} dx$. [3]

$$\int \frac{2x^3 - 20x^2 - 17x - 10}{\left(x^2 - 4\right)\left(2x^2 + 1\right)} dx = \int -\frac{3}{x-2} + \frac{2}{x+2} + \frac{4x}{2x^2+1} dx \quad \textbf{[M1]}$$

$$= -3\ln(x-2) + 2\ln(x+2) + \ln(2x^2+1) + c$$

[**A2** for all correct, A1 for 2 correct terms, award mark if ecf from part (i) or (ii)]

7



The diagram shows the curve $y = \mathrm{f}(x)$ which intersects the x-axis at points A and B. The tangent to the curve at the point $\left(\frac{1}{2},\ 10\right)$ intersects the x-axis at point C. It is given that $\frac{dy}{dx} = -8x + 10$.

(i) Show that B is the point $(3,\ 0)$ and find the coordinates of C. [5]

$$\frac{dy}{dx} = -8x + 10$$

$$y = \int (-8x + 10)dx \quad \textbf{[M1]}$$

$$= -4x^2 + 10x + c$$

At the point $(\frac{1}{2},\ 10)$,

$$10 = -4(\frac{1}{2})^2 + 10(\frac{1}{2}) + c$$

$$c = 6$$

$$y == -4x^2 + 10x + 6 \quad \textbf{[M1]}$$

$$y = -2(2x^2 - 5x - 3)$$

$$= -2(2x + 1)(x - 3)$$

When $y = 0,\ x = -\frac{1}{2}$ or $x = 3$

Coordinates of $B = (3, 0)$ (shown) **[A1]**

At the point $(\frac{1}{2},\ 10)$, $\frac{dy}{dx} = 6$

Equation of tangent line:

$y - 10 = 6(x - \frac{1}{2})$ **[M1]**

$y = 6x + 7$

When $y = 0$, $x = -\frac{7}{6}$

Coordinates of $C = (-\frac{7}{6},\ 0)$ **[A1]**

(ii) Find the area of the shaded region. [5]

Area of shaded region $= \frac{1}{2}(10)(\frac{1}{2} + \frac{7}{6}) + \int_{\frac{1}{2}}^{3}\left(-4x^2 + 10x + 6\right)dx$ **[M1 + M1]**

$= 8\frac{1}{3} + \left[-\frac{4x^3}{3} + 5x^2 + 6x\right]_{\frac{1}{2}}^{3}$ **[M1]**

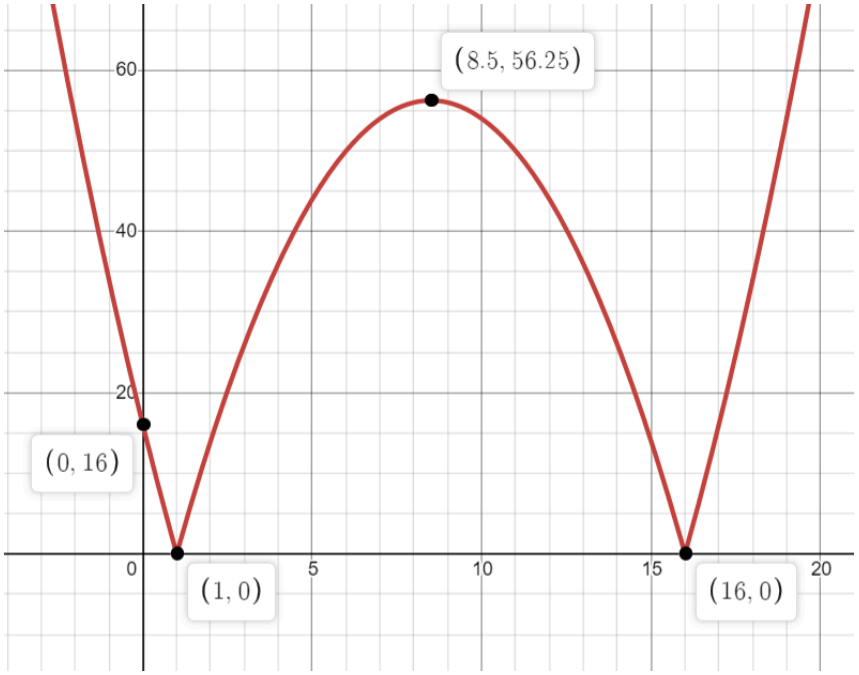
$= 8\frac{1}{3} + \left[-\frac{4(3)^3}{3} + 5(3)^2 + 6(3)\right] - \left[-\frac{4(\frac{1}{2})^3}{3} + 5(\frac{1}{2})^2 + 6(\frac{1}{2})\right]$

[M1]

$= 8\frac{1}{3} + 27 - 4\frac{1}{12}$

$= 31.25$ units^2 **[A1]**

8 **(i)** Sketch the graph of $y = |x^2 - 17x + 16|$. [3]



Shape of graph **[C1]**

Turning point **[C1]**

x and y-intercept(s) **[C1]**

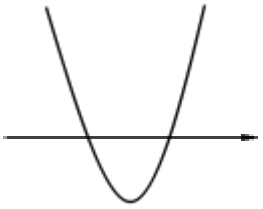
(ii) Find the range of values of x for which $\left|x^2-17x+16\right|>54$. [5]

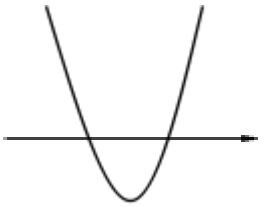
$\left|x^2-17x+16\right|>54$

$x^2-17x+16>54$ or $-x^2+17x-16>54$ **[M1]**

$x^2-17x-38>0$ or $x^2-17x+70<0$

$(x-19)(x+2)>0$ **[M1]** or $(x-7)(x-10)<0$ **[M1]**





$x<-2$ or $x>19$ **[A1]** $7<x<10$ **[A1]**

The range of values of x is $x<-2$, $7<x<10$ and $x>19$.

Alternative:

For $\left|x^2-17x+16\right|=54$,

$x^2-17x+16=54$ or $x^2-17x+16=-54$ **[M1]**

$x^2-17x-38=0$ or $x^2-17x+70=0$

$(x-19)(x+2)=0$ or $(x-7)(x-10)=0$

$x=19$ or $x=-2$ **[M1]** or $x=7$ or $x=10$ **[M1]**

For $\left|x^2-17x+16\right|>54$,

the range of values of x is $x<-2$, $7<x<10$ and $x>19$.

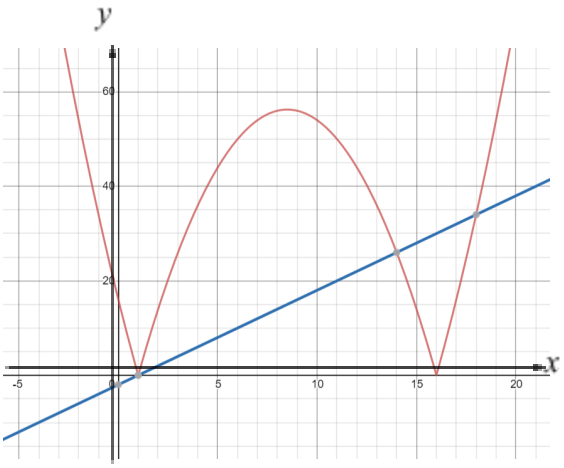
[**A1** for $x<-2$ or $x>19$ and **A1** for $7<x<10$]

(iii) Given that $|x^2 - 17x + 16| = k$ has more than 2 distinct solutions, state the range of values of k. [2]

The range of values of k is $0 < k \leq 56\frac{1}{4}$. **[B1 + B1]**

(iv) Determine the number of solutions of the equation $|x^2 - 17x + 16| = 2x - 2$. Justify your answer. [2]

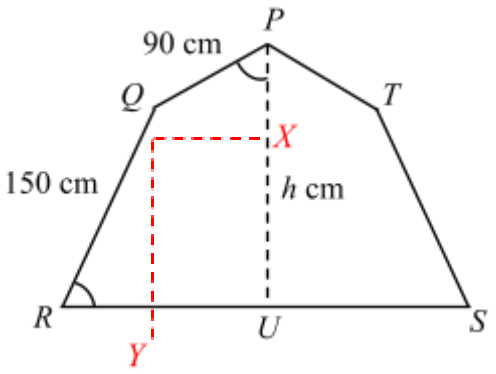
For $y = 2x - 2$, when $y = 0$, $x = 1$.



[M1 for sketching $y = 2x - 2$, clearly showing the x-intercept**]**

Since the line $y = 2x - 2$ intersects the curve at (1, 0) and the gradient is 2, the number of solutions is 3. **[A1]**

9 The diagram shows the side view $PQRST$ of a tent. The tent rests with RS on horizontal ground. $PQRST$ is symmetrical about the vertical PU, where U is the midpoint of RS. Angle QPU = angle QRU = θ radians and the lengths of PQ and QR are 90 cm and 150 cm respectively. The vertical height of P from the ground is h cm.



(i) Explain clearly why $h = 90\cos\theta + 150\sin\theta$. [2]

Let the foot of the perpendicular from Q to PU and RU be X and Y respectively.

$PX = 90\cos\theta$ } **[M1** for both**]**

$QY = 150\sin\theta$

$h = PX + QY$

$= 90\cos\theta + 150\sin\theta$ **[A1]**

(ii) Express h in the form $R\cos(\theta - \alpha)$, where $R > 0$ and α is an acute angle. [4]

$R\cos\alpha = 90$

$R\sin\alpha = 150$ **[M1]**

$\tan\alpha = \frac{150}{90}$ **[M1]**

$\alpha = 1.0304$

$R = \sqrt{90^2 + 150^2}$

$= 30\sqrt{34}$ **[M1]**

$h = 30\sqrt{34}\cos(\theta - 1.03)$ **[A1 -** accept if $R = 175$ (3s.f.)**]]**

(iii) Find the greatest possible value of h and the value of θ at which this occurs. [3]

Greatest value of h occurs when $\cos(\theta - 1.03) = 1$ **[M1]**

Greatest value of $h = 30\sqrt{34}$ **[A1** – Accept 175**]**

when $\theta = 1.03$. **[A1]**

(iv) Find the values of θ when $h = 160$. [3]

$30\sqrt{34}\cos(\theta - 1.0304) = 160$

$$\cos(\theta - 1.0304) = \frac{160}{30\sqrt{34}}$$ **[M1]**

Basic angle = 0.41613

$\theta - 1.0304 = -0.41613$ or 0.41613

$\theta = 0.614$ or 1.45 (to 3s.f.) **[A1, A1]**

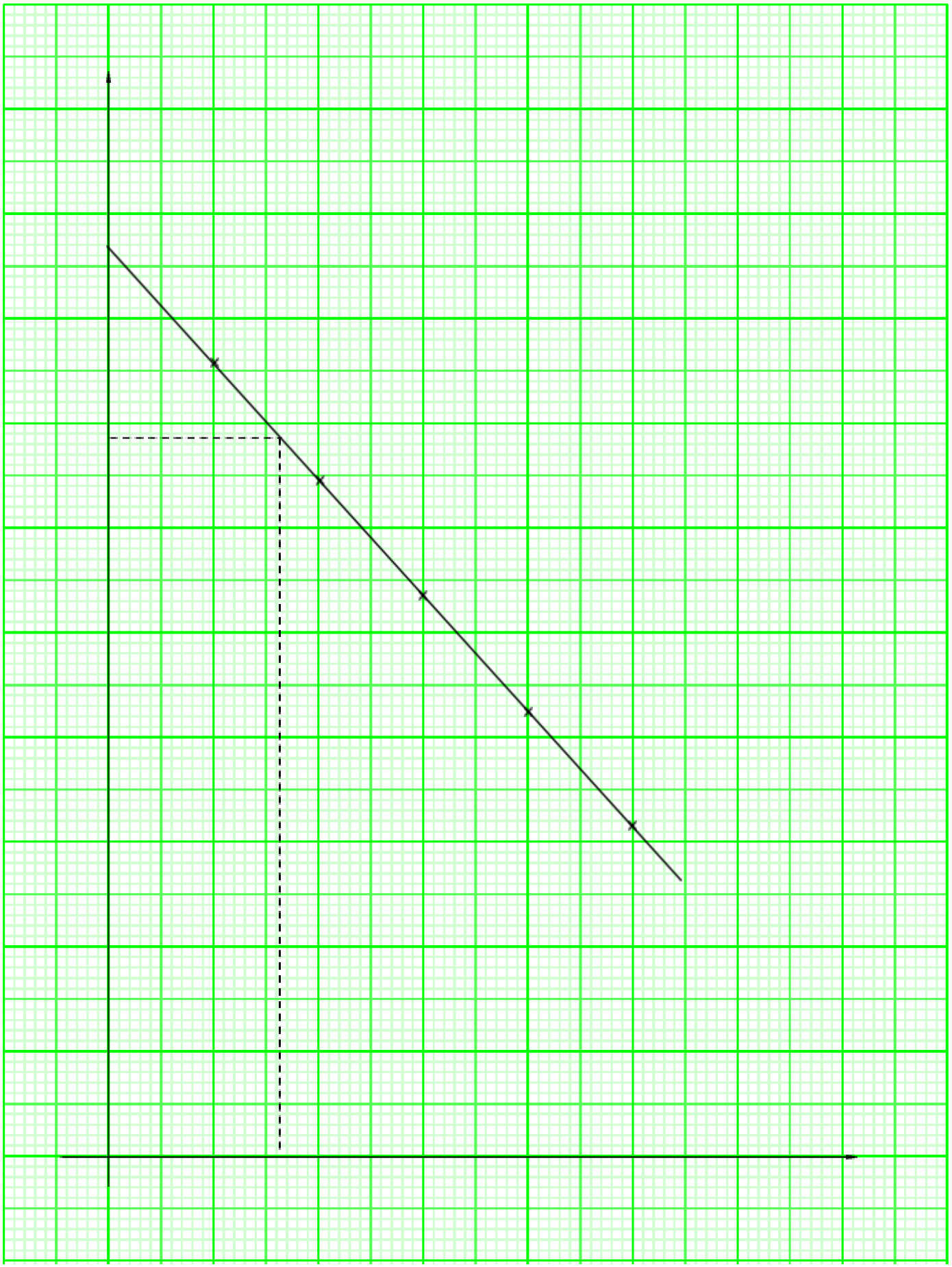
10 A bowl of liquid is heated to a high temperature. It subsequently cools in such a way that its temperature, $T°\text{C}$, is given by $T = 15 + Ae^{-kt}$, where t minutes is the time of cooling and A and k are constants. The table below shows corresponding values of t and T.

t	5	10	15	20	25
T	58.8	40.3	29.6	23.4	19.9

(i) Draw the graph of $\ln(T - 15)$ against t. [3]

t	5	10	15	20	25
T	58.8	40.3	29.6	23.4	19.9
$\ln(T - 15)$	3.78	3.23	2.68	2.13	1.59

[B1 for table**]**



(ii) Use the graph to estimate the value of each of the constants A and k. [5]

$T = 15 + Ae^{-kt}$

$\ln(T-15) = -kt + \ln A$ **[M1]**

gradient = $-k$, vertical-intercept = $\ln A$ **[M1]**

$\ln A = 4.3$

$A = e^{4.325}$

=75.6 (accept 73.7 to 77.5) **[A1]**

$-k = -0.11$ **[M1** for finding gradient**]**

$k = 0.11$ (accept 0.105 to 0.115) **[A1]**

(iii) State the initial temperature of the liquid. [1]

$T = 15 + 75.6e^{-0.11(0)}$

$= 90.6$

The initial temperature is $90.6°\text{C}$. (accept 88.7 to 92.5) **[B1]**

(iv) Use the graph to estimate the time taken for the temperature of the liquid to drop to half of its original temperature. [2]

When the temperature of the liquid is half of its original temperature,

$$\ln(T-15) = \ln\left(\frac{90.6}{2} - 15\right)$$

$= 3.41$ **[M1]**

Time taken = 8.25 minutes (accept 8 to 8.5 minutes) **[A1]**