Unit 3 : Equations of Motion and The Solution

Tutorial No : 06	**Aircraft Equations of Motion and State Space Model**
Aim (What I want students to learn)	● Understand the basic equations of motion for an aircraft. ● Derive the state space representation of an aircraft's dynamics. ● Analyze the stability and control characteristics of an aircraft using state space models.
Rationale (Why this is useful to them)	State space models provide a powerful framework for analyzing and controlling complex systems like aircraft. Understanding how to derive and use state space models is essential for aerospace engineers and researchers.
Learning outcomes (What I want them to be able to do at the end – and how I will know this is successfully achieved)	● Write down the equations of motion for an aircraft in longitudinal and lateral-directional flight. ● Convert the equations of motion into state space form. ● Determine the stability of an aircraft using the eigenvalues of the state space model. ● Use state space models to design control systems for aircraft.
Learning activities (The activities I will use to best enable my students to meet their learning outcomes)	Classroom Discussion, Tutorial and Numerical Simulation
Assessment (I will assess their learning by)	Tutorial Assignment and Quiz
Resources (What I use to support student learning)	MATLAB and Tutorial Sheet

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Aircraft Equations of Motion and State Space Model

Consider a simplified aircraft model with longitudinal dynamics. The aircraft is assumed to be flying in a straight and level path at a constant airspeed.

The state variables are:

- θ: Pitch angle (rad)
- q: Pitch rate (rad/s)
- α: Angle of attack (rad)
- V: Airspeed (m/s)

The inputs to the system are:

- δe: Elevator deflection (rad)

The equations of motion are given by:

```
θ˙= q
q̇ = (C_mα * α + C_mδe * δe) / I_y
α˙= q - (C_Lα * α + C_Lδe * δe) / (m * V)
V̇ = (T - D - L * sin(θ)) / m
```

where:

- Pitching moment coefficient due to angle of attack, **C_mα** = **-0.01**
- Pitching moment coefficient due to elevator deflection, **C_mδe = 0.02**
- Lift coefficient due to angle of attack, **C_Lα = 4.5**
- Lift coefficient due to elevator deflection, **C_Lδe = 0.5**
- Pitch moment of inertia, **I_y = 5000 kgm²**
- Aircraft Mass**, m = 1000 kg**.
- Equilibrium airspeed, **Veq = 50 m/s**
- **T** is the thrust.
- **D** is the drag.
- **L** is the lift.

Tasks:

1. **Linearize the equations of motion** around the equilibrium flight condition.
2. **Derive the state space model** for the linearized system.
3. **Determine the system's eigenvalues** to assess stability.
4. **Analyze the system's response** to a step input in elevator deflection.

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1. **Linearize the equations of motion** around the equilibrium flight condition.

   Nonlinear equations can be complex and difficult to analyze directly. Linearization reduces them to a simpler form, making it easier to study the system's behavior.

   a. Define Equilibrium Condition
   b. Perturb the Variables
   c. Substitute into Equations of Motion -
   d. Linearize - Neglect higher-order terms involving the perturbations. This is justified assuming small perturbations.

   Equilibrium Flight Condition:

   - $\theta_{eq} = 0$ (straight and level flight)
   - $q_{eq} = 0$ (constant pitch rate)
   - $\alpha_{eq} = 0$ (constant angle of attack)
   - V_{eq} (constant airspeed)

   Linearization - To linearize the equations of motion, we can assume small perturbations from the equilibrium condition:

   - $\theta = \theta_{eq} + \delta\theta$
   - $q = q_{eq} + \delta q$
   - $\alpha = \alpha_{eq} + \delta\alpha$
   - $V = V_{eq} + \delta V$

   Substituting these into the original equations and neglecting higher-order terms,

```
θ˙= q
q̇ = (C_mα * α + C_mδe * δe) / I_y
α˙= q - (C_Lα * α + C_Lδe * δe) / (m * V)
V̇ = (T - D - L * sin(θ)) / m
```

   we get:

```
δθ˙= δq
δq̇ = (C_mα * δα + C_mδe * δe) / I_y
δα˙= δq - (C_Lα * δα + C_Lδe * δe) / (m * V_eq)
δV̇ = (-C_Lα * δα - C_Lδe * δe) / m
```

2. **Deriving the State Space Model**

The state-space representation of a system is a mathematical model that describes the system's behavior using a set of first-order differential equations. It is a powerful tool for analyzing and controlling dynamic systems.

a. Define the State Variables
b. Express the Equations of Motion
c. Arrange in State-Space Form

The state variables are:

- **x1 = δθ**
- **x2 = δq**
- **x3 = δα**
- **x4 = δV**

The input is:

- **u = δe**

The output can be chosen based on the desired system response. We choose the pitch angle (δθ) as the output. The state space model is:

$$\frac{dx}{dt} = [A]x + [b]u$$
$$y = [C]x + [D]u$$

Where,

```
A = [0 1 0 0;
     C_mα/I_y 0 0 0;
     0 1 -C_Lα/(m*V_eq) 0;
     0 0 -C_Lα/m 0]

B = [0; C_mδe/I_y; -C_Lδe/(m*V_eq); 0]

C = [1 0 0 0]
D = 0
```


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3. **Determine the system's eigenvalues** to assess stability.

Eigenvalues are characteristic values associated with a linear system. They provide valuable information about the system's stability. In the context of a state-space model:

- **Real part of the eigenvalue:** Determines the rate of growth or decay of the system's response.
  - Negative real part: Stable (response decays over time)
  - Positive real part: Unstable (response grows exponentially)
  - Zero real part: Marginally stable (response neither grows nor decays)
- **Imaginary part of the eigenvalue:** Determines the oscillatory behavior of the system.
  - Non-zero imaginary part: Oscillatory behavior

Calculating Eigenvalues

The eigenvalues of a system are calculated by finding the roots of the characteristic equation:

$$\det(sI - A) = 0$$

Where:

- `s` is the Laplace variable
- `I` is the identity matrix
- `A` is the system matrix from the state-space model

Once the eigenvalues are calculated, we can assess the system's stability:

- **All eigenvalues have negative real parts:** The system is asymptotically stable. It will return to equilibrium after a disturbance.
- **At least one eigenvalue has a positive real part:** The system is unstable. It will diverge from equilibrium.
- **At least one eigenvalue has a zero real part:** The system is marginally stable. It will neither converge nor diverge, but may exhibit oscillatory behavior.

The Eigen Values are,

- `0.0000 + 0.0000i`
- `-0.0001 + 0.0000i`
- `0.0000 + 0.0014i`
- `0.0000 - 0.0014i`

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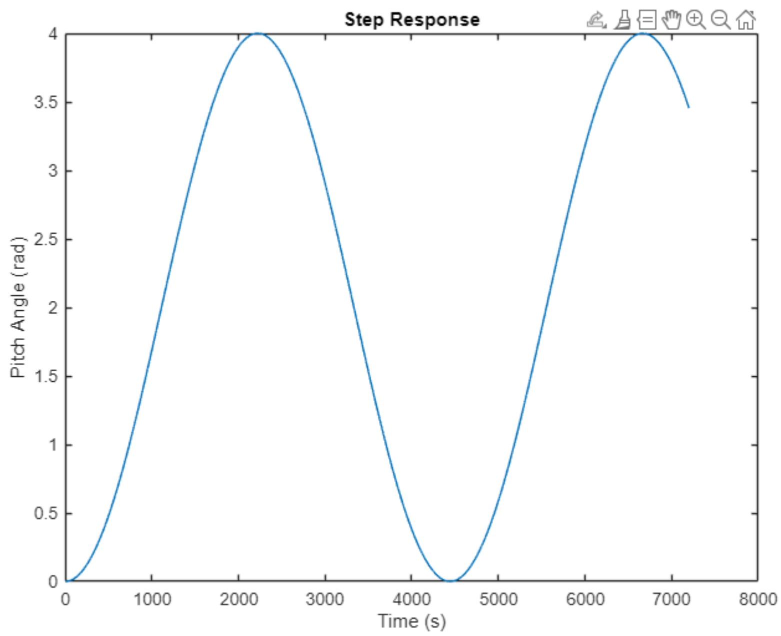
4. **Analyze the system's response** to a step input in elevator deflection.

```
% Aircraft parameters
C_malpha = -0.01; % Pitching moment coefficient due to angle of attack
C_mdeltae = 0.02; % Pitching moment coefficient due to elevator deflection
C_Lalpha = 4.5; % Lift coefficient due to angle of attack
C_Ldeltae = 0.5; % Lift coefficient due to elevator deflection
I_y = 5000; % Pitch moment of inertia (kg*m^2)
m = 1000; % Mass (kg)
V_eq = 50; % Equilibrium airspeed (m/s)
A = [0 1 0 0;
    C_malpha/I_y 0 0 0;
    0 1 -C_Lalpha/(m*V_eq) 0;
    0 0 -C_Lalpha/m 0];
B = [0; C_mdeltae/I_y; -C_Ldeltae/(m*V_eq); 0];
C = [1 0 0 0];
D = 0;
sys = ss(A, B, C, D);
eig_values = eig(A);
disp(eig_values);
t = 0:0.1:3600; % Time vector
u = 2; % Step input magnitude
%[y, t, x] = step(sys, t, u);
[y, t, x] = step(sys, t);
% Plot the response
plot(t, y);
xlabel('Time (s)');
ylabel('Pitch Angle (rad)');
title('Step Response');
```



Analyzing the response, the aircraft is neurally stable. Why it's neutrally stable and what can be done to decrease the response time.

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