Linear Algebra
Lesson 15
Proofs with Matrices

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313F21-lesson15-lastname-firstname

and share editing of that document with me sormanic@gmail.com and with our graders. If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

Part I teaches basic proofs and has five required HW problems

Part II uses sum notation for proofs and has extra credit problems

****
Part I: Watch Playlist 313F20-15-PartI
****

1:23 AM Sat Sep 26 — Linear Algebra

Lesson 15 Proofs with Matrices

In this lesson we will prove:

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associativity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

Defn: The **zero matrix**, O, is a matrix which has zeroes everywhere.

Thm $A \times O = ?$ and $O \times A = ?$

Defn The **identity matrix**, I, is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm $A \times I = ?$ and $I \times A = ?$

What about

$$A \times B \overset{?}{=} B \times A \,?$$

Linear Algebra

Defn of Matrix Mult

$A \in \mathcal{M}_{n \times m}$ — m columns, n rows

$B \in \mathcal{M}_{m \times \ell}$ $\quad A \times B \in \mathcal{M}_{n \times \ell}$

$$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & b_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \\ b_{m1} & & b_{m3} & & b_{m\ell} \end{pmatrix} = ?$$

j counter goes across the i^{th} row of A →

↓ j counter goes down in the k^{th} column of B

$[A \times B]_{ik}$ = dot product of i^{th} row of A and k^{th} column of B

Defn of Matrix Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+0 \\ 3+0 & 4+1 \end{bmatrix}$$

$A + B \in \mathcal{M}_{n \times m}$

when $A, B \in \mathcal{M}_{n \times m}$.

Homework Multiply the Following Matrices:

Thm Distribution of Matrix Mult over Add.

$$A\times(B+C) = A\times B + A\times C$$

Proof for $A, B, C \in M_{2\times 2}$:

Statements	Justifications
① $\underbrace{A\times(B+C)}_{LHS} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right)$	① Given $A, B, C \in M_{2\times 2}$ Defn of $M_{2\times 2}$
② $= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} (b_{11}+c_{11}) & (b_{12}+c_{12}) \\ (b_{21}+c_{21}) & (b_{22}+c_{22}) \end{bmatrix}$	② Defn of Matrix Addition
③ $= \begin{bmatrix} a_{11}(b_{11}+c_{11}) + a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12}) + a_{12}(b_{22}+c_{22}) \\ a_{21}(b_{11}+c_{11}) + a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12}) + a_{22}(b_{22}+c_{22}) \end{bmatrix}$	③ Defn of Matrix Mult
④ $= \begin{bmatrix} a_{11}b_{11}+a_{11}c_{11} + a_{12}b_{21}+a_{12}c_{21} & a_{11}b_{12}+a_{11}c_{12} + a_{12}b_{22}+a_{12}c_{22} \\ a_{21}b_{11}+a_{21}c_{11} + a_{22}b_{21}+a_{22}c_{21} & a_{21}b_{12}+a_{21}c_{12} + a_{22}b_{22}+a_{22}c_{22} \end{bmatrix}$	④ distribution of mult over add for $\mathbb{R}$

Next do right hand side until they match.

(5) RHS $A \times B + A \times C =$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

(6)
$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

(7)
$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{11}c_{11} + a_{12}c_{21} & a_{11}b_{12} + a_{12}b_{22} + a_{11}c_{12} + a_{12}c_{22} \\ a_{21}b_{11} + a_{22}b_{21} + a_{21}c_{11} + a_{22}c_{21} & a_{21}b_{12} + a_{22}b_{22} + a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

Now check this matrix

(5) Given $A, B, C \in M_{2\times2}$ and defn of $M_{2\times2}$.

(6) Defn of Matrix Mult and Order of Operations

(7) Defn of Matrix Addition

matches step 4

QED

Thm Associativity of Matrix Mult

$$A\times(B\times C) = (A\times B)\times C$$

Proof (for $A,B,C \in M_{2\times 2}$):

① LHS $A\times(B\times C) =$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right)$$

② $$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} (b_{11}c_{11}+b_{12}c_{21}) & (\quad) \\ (\quad) & (\quad) \end{bmatrix}$$

③ $$= \begin{bmatrix} a_{11}(b_{11}c_{11}+b_{12}c_{21}) + a_{12}(\quad) \\ a_{21}(b_{11}c_{11}+b_{12}c_{21}) + a_{22}(\quad) \end{bmatrix}$$

④ Simplify

Linear Algebra

$\boxed{\text{HW!}}$ Fill in rest of proof

① Given $A,B,C \in M_{2\times 2}$
Defn of $M_{2\times 2}$

② Defn of Matrix Mult.

$$\begin{bmatrix} a_{11}(\quad) + a_{12}(\quad) \\ a_{21}(\quad) + a_{22}(\quad) \end{bmatrix}$$

③ Defn of Matrix Mult.

④ algebra

2:10 AM Sat Sep 26 | Linear Algebra | Linear Algebra 71%

Defn The zero matrix, $\mathbb{O}$, is a matrix which has zeroes everywhere.

Thm $A \times \mathbb{O} = \mathbb{O}$

Proof: $(A \in M_{2\times 3} \quad \mathbb{O} \in M_{3\times 4})$

① $A \times \mathbb{O} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

② $= \begin{bmatrix} a_{11}\cdot 0 + a_{12}\cdot 0 + a_{13} 0 & a_{11}0 + \dots & 0+0+0 & 0 \\ a_{21}\cdot 0 + a_{22}\cdot 0 + a_{23} 0 & 0+0+0 & 0+0+0 & 0 \end{bmatrix}$

③ $= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_{2\times 4}$

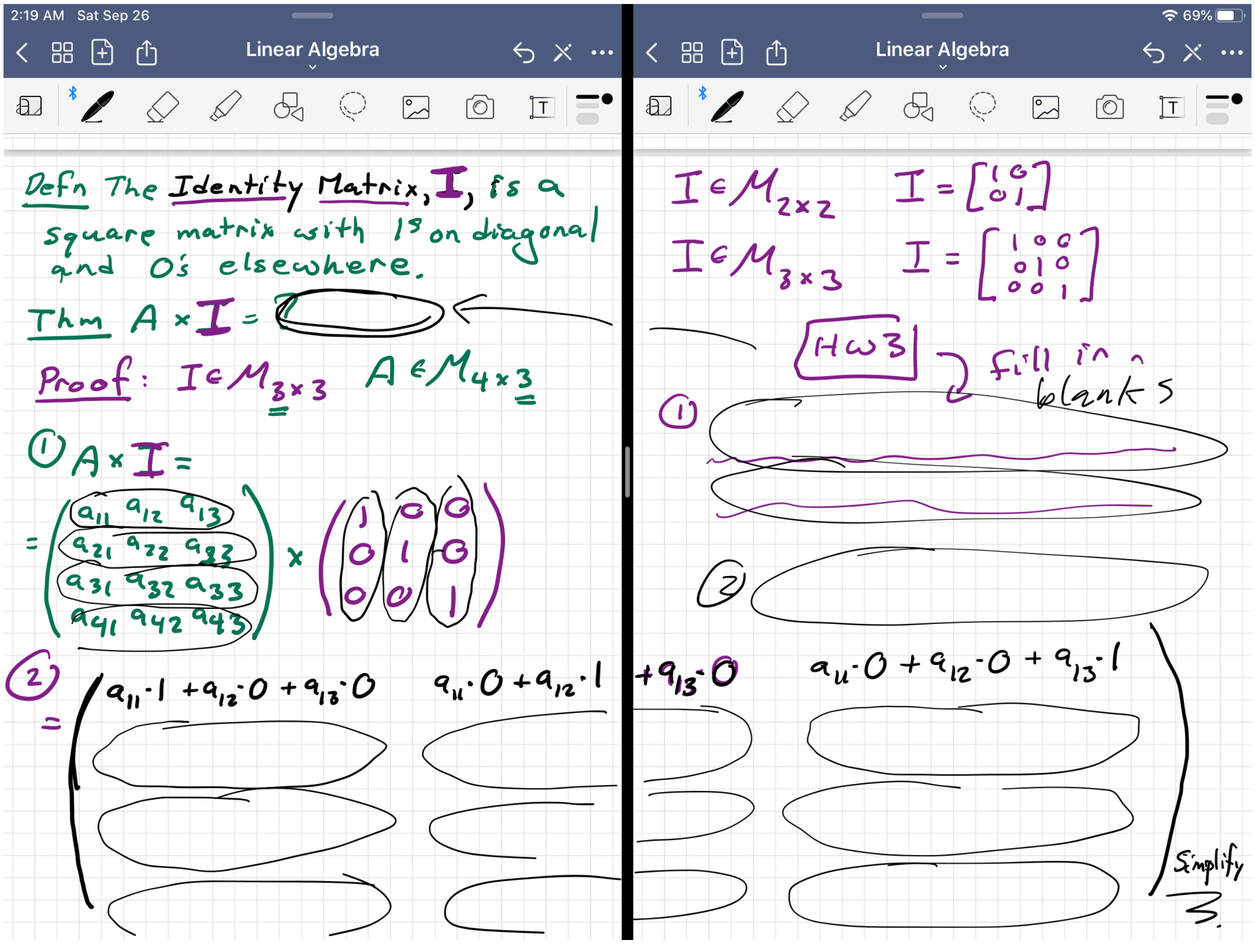
we will figure out the answer.

① by given $A \in M_{2\times 3}$ & defn $M_{2\times 3}$
by defn $\mathbb{O}$ matrix in $M_{3\times 4}$

② by defn of matrix mult

③ by arithmetic

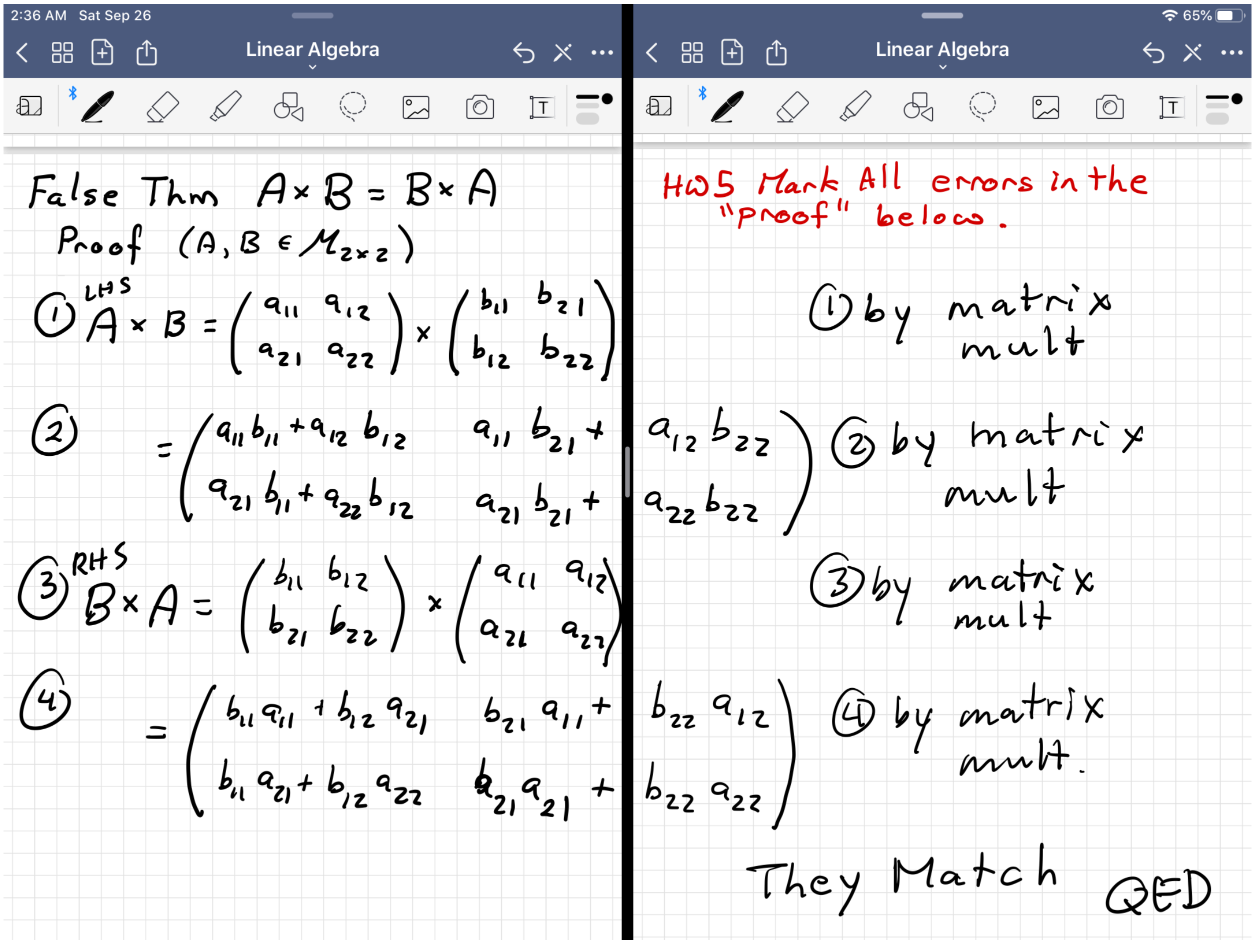
$\boxed{HW2}$ $\mathbb{O} \times A = \mathbb{O}$ when $A \in M_{2\times 3}$ $\quad \mathbb{O} \in M_{4\times 2}$



HW1-HW3 are described above.

HW4 Prove IxA=A when A is 3x5.

HW5 is to **find the errors** in the incorrect proof below which has many errors so find all of the errors. Then either fix the proof or find a pair of specific 2x2 matrices A and B for which this fails.



HW6: Prove that if A and B are 2x2 matrices and v is a vector then
A(Bv)=(AxB)v
Hint: Bv is a vector so on the LHS you first use matrix times vector and then matrix times vector again. On the RHS you need to use matrix times matrix to find (AxB) and then matrix times vector to multiply the answer times v.

****

Part II: Watch [Playlist 313F20-15-PartII] which is Extra Credit highly recommended for math majors and has three extra credit problems.
****

Lesson 15 Proofs with Matrices

In this lesson we will prove:

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associativity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

Defn: The **zero matrix**, O, is a matrix which has zeroes everywhere.

Thm $A \times O = ?$ and $O \times A = ?$

Defn The **identity matrix**, I, is a is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm $A \times I = ?$ and $I \times A = ?$

What about

$$A \times B \overset{?}{=} B \times A \, ?$$

Part II Proofs using Σ notations

Defn of Matrix Mult

$$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

i^{th} row of A: $(a_{i1} \; a_{i2} \cdots a_{im}) \cdot \begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{pmatrix}$ k^{th} column of B

Defn of Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$

$$= a_{ik} + b_{ik}$$

Thm Distribution of Matrix Mult over Add

$$A\times(B+C) = A\times B + A\times C$$

Proof using Σ notation

for $A\in M_{n\times m}$ $B,C\in M_{m\times \ell}$:

① LHS $[A\times(B+C)]_{ik} = \sum_{j=1}^{m}[A]_{ij}\,[B+C]_{jk}$

① by defn of matrix mult

② $=\sum_{j=1}^{m}[A]_{ij}\left([B]_{jk}+[C]_{jk}\right)$

② defn of matrix addition

③ $=\sum_{j=1}^{m}\left([A]_{ij}[B]_{jk}+[A]_{ij}[C]_{jk}\right)$

③ distribution of mult over addition of reals.

④ $=\sum_{j=1}^{m}\left([A]_{ij}[B]_{jk}\right)+\sum_{j=1}^{m}[A_{ij}][C_{jk}]$ ④ req/ Σ for

Linear Algebra

Part II Proofs using Σ notations

Defn of Matrix Mult

$$[A\times B]_{ik} = \sum_{j=1}^{m}[A]_{ij}[B]_{jk}$$

ith row of A: $a_{i1}\ a_{i2}\cdots a_{im}$ $\cdot$ $\begin{matrix} b_{1k}\\ b_{2k}\\ \vdots\\ b_{mk}\end{matrix}$ kth column of B

Defn of Addition

$$[A+B]_{ik} = [A]_{ik}+[B]_{ik}$$
$$= a_{ik}+b_{ik}$$

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Proof using Σ notation

for $A \in M_{n \times m}$ $B, C \in M_{m \times \ell}$:

① LHS $$[A \times (B+C)]_{ik} = \sum_{j=1}^{m} [A]_{ij} [B+C]_{jk}$$

① by defn of matrix mult

② $$= \sum_{j=1}^{m} [A]_{ij} \left([B]_{jk} + [C]_{jk} \right)$$

② defn of matrix addition

③ $$= \sum_{j=1}^{m} \left([A]_{ij} [B]_{jk} + [A]_{ij} [C]_{jk} \right)$$

③ distribution of mult over addition of reals.

④ $$= \sum_{j=1}^{m} \left([A]_{ij} [B]_{jk} \right) + \sum_{j=1}^{m} [A_{ij}][C_{jk}]$$

④ reorder Σ

⑤ RHS $$[A \times B + A \times C]_{ik} =$$

$$= [A \times B]_{ik} + [A \times C]_{ik}$$

⑤ defn of matrix add.

⑥ $$= \sum_{j=1}^{m} [A]_{ij} [B]_{jk} + \sum_{j=1}^{m} [A]_{ij} [C]_{jk}$$

⑥ by def matrix Mult.

Steps 4 + 6 Match

QED

$\boxed{EC}$ $A, B \in M_{n \times m}$ $C \in M_{m \times \ell}$

$$(A+B) \times C = A \times C + B \times C$$

Lesson 15 Proofs with Matrices

In this lesson we will prove:

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associativity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

Defn: The **zero matrix**, O, is a matrix which has zeroes everywhere.

Thm $A \times O = O$ and $O \times A = O$

Defn The **identity matrix**, I, is a is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm $A \times I = A$ and $I \times A = A$

What about

$$A \times B \overset{?}{=} B \times A \; ? \quad \text{FALSE}$$

Thm: If $A \in M_{n \times m}$ and $O \in M_{m \times \ell}$ then $A \times O = O \in M_{n \times \ell}$

Proof:

(1) $[A \times O]_{ik} = \sum_{j=1}^{m} [A]_{ij} [O]_{jk}$ — (1) by defn matrix mult

(2) $= \sum_{j=1}^{m} [A]_{ij} \cdot 0$ — (2) by defn of zero matrix

(3) $= \sum_{j=1}^{m} 0$ — (3) by $a \cdot 0 = 0$ for any $a \in \mathbb{R}$

(4) $= 0$ — (4) $0 + 0 = 0$

(5) $= [O]_{ik}$. — (5) Defn of zero matrix QED

$\boxed{EC}$ $\underset{M_{n \times m}}{O} \times \underset{M_{m \times \ell}}{A} = \underset{M_{n \times \ell}}{O}$

Lesson 15 Proofs with Matrices

In this lesson we will prove:

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associativity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

Defn: The **zero matrix**, O, is a matrix which has zeroes everywhere.

Thm $A \times O = O$ and $O \times A = O$

Defn The **identity matrix**, I, is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm $A \times I = A$ and $I \times A = A$

What about

$$A \times B \overset{?}{=} B \times A \, ? \quad \text{FALSE}$$

Thm Given $A \in M_{n \times m}$, $I \in M_{m \times m}$ then $A \times I = A$

$\boxed{EC3}$ $I \times A = A$ $\quad I \in M_{m \times m}$, $A \in M_{m \times \ell}$

Thoughts

$$[I]_{ij} = \begin{cases} 1 & \text{if } i = j \text{ (diagonal)} \\ 0 & \text{if } i \neq j \text{ (elsewhere)} \end{cases}$$

Defn of Identity Matrix

Proof:

① $[A \times I]_{ik} = \sum_{j=1}^{m} [A]_{ij} [I]_{jk}$ — ① by defn matrix mult

② $= [A]_{i1} I_{1k} + [A]_{i2} I_{2k} + \dots + [A]_{ik} I_{kk} + \dots + [A]_{im} I_{mk}$ — ② defn of Σ

③ $= [A]_{i1} \cdot 0 + [A]_{i2} \cdot 0 + \dots + [A]_{ik} \cdot 1 + \dots + A_{im} \cdot 0$ — ③ by defn of identity

④ $= 0 + 0 + \dots + 0 + [A]_{ik} + 0 \dots + 0$ — ④ algebra

⑤ $= [A]_{ik}$ — ⑤ by $0 + a = a$ for $a \in \mathbb{R}$

QED

Extra Credit: Prove for arbitrary matrices A and B and vector v using sum notation that A(Bv)=(AxB)v

Extra Credit: Prove for arbitrary matrices A, B, and C using sum notation that Ax(BxC)=(AxB)xC