TAKAPUNA GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

SCHOLARSHIP CALCULUS

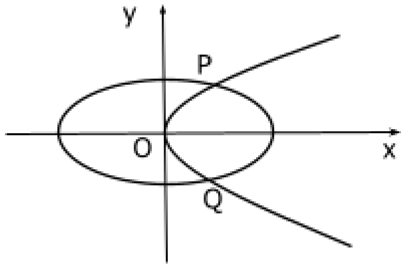
CONICS QUESTIONS

COMPLETE BY WEEK 3 TERM 4

Remember that to study a subject at a more advanced level is to read for the subject. The Scholarship booklets do not necessarily follow the Level 3 syllabus. There may be a lunchtime class or a tutorial to help you but you are expected to investigate any topic with which you are unfamiliar. Read *Delta*, check in the write-on book, google the topic, ask questions …….

Exercise 1 : Conic Sections

1 The parabola with equation $y^2 = 4ax$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > 0 and b > 0, meet at points P and Q as shown.



At P, the tangent to the parabola is perpendicular to the tangent to the ellipse.

(a) (i) Show that $b^2 = 2a^2$.

(ii) Hence find, in terms of a, the length OP.

(b) The tangent to the parabola at P meets the x-axis at M and the tangent to the ellipse at P meets the x-axis at N. Show that $MN = 2\sqrt{2}a$.

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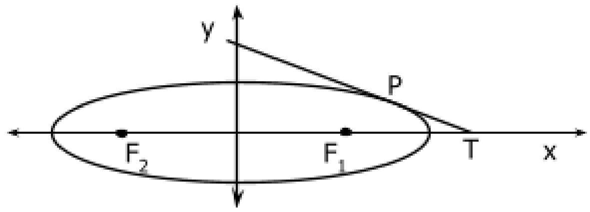


Whispering galleries are enclosures that can be made in an elliptical shape. A bird's-eye view of a whispering gallery that is 2a metres long and 2b metres wide is shown in the diagram.

The equation for this ellipse is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

One person stands at F_1 (c, 0) where $c^2 = a^2 - b^2$. Another person stands at F_2 (-c, 0). P is a point on the edge of the whispering gallery with x-coordinate x = c. In this question you will show that a sound made at F_1 is reflected at P to the point F_2.

(a) Show that the coordinates of P are $\left(c, \frac{b^2}{a}\right)$.

(b) QPT is tangent to the ellipse at P. Show that the gradient of the tangent QPT to the ellipse at P is $\frac{-c}{a}$.

(c) Using the results from (i) and (ii) prove that the acute angle between F_1P and the tangent QPT is the Same as the acute angle between F_2P and the tangent QPT.

3 (*Scholarship 2000*) (a) An ellipse has the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where a > b > 0. The point C, (c, 0), is such that c>a and from the point C tangents to the ellipse are drawn. One tangent meets the ellipse at the point (m, n) where n>0. Prove that the gradient of the tangent at the point (m, n) is $\frac{-b^2m}{a^2n}$.

(b) Prove that $m = \frac{a^2}{c}$.

(c) Prove that $n = \frac{b}{c}\sqrt{(c^2 - a^2)}$.

(d) Discuss what happens when $c \to a$.

4 (*Scholarship 1999*) A comet approaches two planets in such way that the difference between the comet's distances from the planets is always equal to d, a constant. Assume the planets can be taken as stationary over the time interval we are interested in.

(a) Let the planets be at points (-c, 0) and (c, 0) in some Cartesian coordinate system. Let the planet at (c, 0) be the planet the comet passes closest to. The comet passes between the two planets.

(i) Prove d is less than 2c.

[*Hint: Let the comet's path cut the line joining the two planets at a distance of k from the origin.]*

(ii) Write an expression connecting the coordinates (x, y) of the comet with d, the difference between the distances of the comet from the planets.

(iii) Obtain the equation of the comet's path, $\frac{4x^2}{d^2}-\frac{4y^2}{4c^2-d^2}=1$, from the expression in (ii).

[*Hint: Rearrange the expression so that there is only one square root on the left-hand side*]

(b) This particular comet has locus $\frac{x^2}{25}-\frac{y^2}{200}=1$.

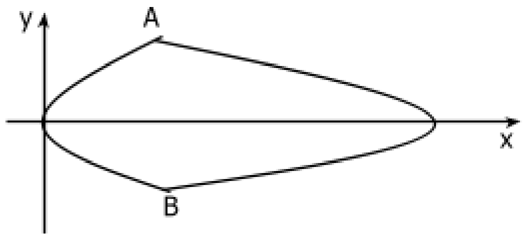
(i) Find the value of d if the equation of the comet is to be written in the form of (a) (iii).

(ii) Find the coordinates of the positions of the two planets.

5 The angle between the two tangents to the parabola $y^2 = 4ax$ from the point (h, k) is θ.

Prove that $\tan\theta = \sqrt{\dfrac{k^2 - 4ah}{(h+a)^2}}$.

Exercise 2 : Using the parametric form



1 Two curves defined parametrically as $x = at^2 + b$, $y = 2at$ and $x = 3b - as^2$, $y = \sqrt{2}as$ (where a and b are constants) meet to form a shape as shown.
Find, in terms of a and b, the co-ordinates (x, y) of the two points of intersection A and B.

2 (*Scholarship 1992*) A relation P is given by the parametric equations : $x = 4t^2 - t4$ and $y = t^3 - 9t$, where t is a real number.

(a) (i) Prove x is an even function.

(ii) Prove y is an odd function.

(iii) Hence, or otherwise, prove the graph of P is symmetrical about the x axis.

(b) Find values of t for which x0.

(c) Find the greatest value of x, giving a reason.

(d) Write down the co-ordinates of points corresponding to t = 0, 0.5, 1, 1.5, 2, 3.

(e) Find the highest point on the curve for x > 0.

(f) Find an expression for the gradient of the tangent to the graph of P at t = a.

(g) Sketch the graph of P.

3 A curve is defined parametrically as $x = a\sec\theta - 1$ and $y = 2\tan\theta + 2$ where $-\pi < \theta < \pi$ and $\theta \neq \frac{\pi}{2}$.

(a) Find the Cartesian equation of the curve.

(b) The normal at the point P where $\theta = \frac{\pi}{4}$ meets the curve again at the point Q in the region where x < 0.
Find the y-ordinate of Q in the form $\frac{a}{b}$ where a and b are integers.

(c) The tangent at the point R is parallel to the tangent at P. Find the y-ordinate of a point S where the normal at R meets the curve again in the region where x > 0.

ANSWERS

Exercise 1 : Conic Sections

1(a)(ii) $a\sqrt{2\sqrt{2} - 1}$

3(d) As $c \to a$, $n \to 0$ and $m \to a$

4(a)(ii) $d = \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}$

4(a)(iii) $\frac{4x^2}{d^2} - \frac{4y^2}{4c^2 - d^2} = 1$

4(b)(i) d = 10

4(b)(ii) (-15, 0) and (15, 0)

Exercise 2 : Using the parametric form

1 A $(5b/3, 2\sqrt{2ab/3})$ and B $(5b/3, -2\sqrt{2ab/3})$

2(b) $-2 \leq t \leq 2$

t	0	0.5	1	1.5	2	3
x	0	0.9	3	3.9	0	-45
y	0	-4.4	-8	-10.1	-10	0

2(c) $x \leq 4$.

2(d)

2(e) local maximum at $(3, 6\sqrt{3})$ This is the highest point on the graph.

2(f) $\frac{dy}{dx} = \frac{3a^2 - 9}{8a - 4a^3}$

2(g)

3(a) $(x+1)^2 - \frac{(y-2)^2}{4} = 1$

which is a hyperbola centre (-1, 2).

3(b) y = $\frac{160}{31}$.

3(c) y = $-\frac{36}{31}$.

SOLUTIONS

Exercise 1 : Conic Sections

1(a)(i) Gradient of parabola $y^2 = 4ax$ is given by : $2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

Gradient of the ellipse : $\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-b^2x}{a^2y}$

$\Rightarrow$ At P these are perpendicular $\Rightarrow \frac{2a}{y} = \frac{a^2y}{b^2x} \Rightarrow 2ab^2x = a^2y^2 \Rightarrow 2ab^2x = a^2 4ax \Rightarrow b^2 = 2a^2$

1(a)(ii) Since $b^2 = 2a^2$ and $y^2 = 4ax$ then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes $\frac{x^2}{a^2} + \frac{4ax}{2a^2} = 1 \Rightarrow x^2 + 2ax - a^2 = 0$

$\Rightarrow$ since x is positive, $x = a(\sqrt{2} - 1) \Rightarrow x^2 = a^2(\sqrt{2} - 1)^2 \Rightarrow y^2 = 4a^2(\sqrt{2} - 1) \Rightarrow$ the distance from the origin to P is $\sqrt{(x^2 + y^2)}$ which is $\sqrt{\{a^2(\sqrt{2} - 1)^2 + 4a^2(\sqrt{2} - 1)\}} = a\sqrt{\{2\sqrt{2} - 1\}}$

1(b) The tangent to the parabola at P has gradient $2a/y_1$ and goes through P with co-ordinates (x_1, y_1).

At M, the co-ordinates are $(x_M, 0) \Rightarrow$ gradient is $\frac{y_1}{x_1 - x_M} = \frac{2a}{y_1} \Rightarrow y_1^2 = 2a(x_1 - x_M)$

But since $y_1^2 = 4ax_1 \Rightarrow 2x_1 = x_1 - x_M \Rightarrow x_M = -x_1 \Rightarrow$ x co-ordinate at M is $a(1 - \sqrt{2})$

The tangent to the ellipse at P has gradient $-b^2x_1/a^2y_1$ and goes through P with co-ordinates (x_1, y_1).

At N, the co-ordinates are $(x_N, 0) \Rightarrow$ gradient $= \frac{y_1}{x_1 - x_M} = \frac{-b^2x_1}{a^2y_1} \Rightarrow a^2y_1^2 = -b^2x_1(x_1 - x_N)$

But $y_1^2 = 4ax_1 \Rightarrow 4a^3x_1 = -b^2x_1(x_1 - x_N) \Rightarrow x_N = x_1 + \frac{4a^3}{b^2} \Rightarrow$ since $b^2 = 2a^2$, $x_N = x_1 + 2a = \sqrt{2}\,a - a + 2a$

$\Rightarrow$ x co-ordinate at N is $a(1 + \sqrt{2}) \Rightarrow$ distance MN $=\{a(1 + \sqrt{2})\} - \{a(1 - \sqrt{2})\} = a + a\sqrt{2} - a + a\sqrt{2} = 2\sqrt{2}\,a$

2(a) At P, $x = c \Rightarrow \frac{c^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2\left(1 - \frac{c^2}{a^2}\right) = b^2\left(\frac{a^2 - (a^2 - b^2)}{a^2}\right) \Rightarrow y = \frac{b^2}{a} \Rightarrow P\left(c, \frac{b^2}{a}\right)$

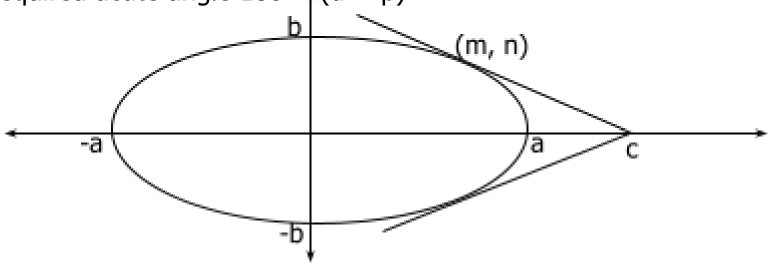
2(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}.\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-xb^2}{ya^2} \Rightarrow$ at P, $m = \frac{-c}{a}$

2(c) $<F_1PT = \alpha = \tan^{-1}\left(\frac{-c}{a}\right) - 90^\circ = \tan^{-1}\left(\frac{c}{a}\right)$ $F_2F_1 = 2c$ and $F_1P = \frac{b^2}{a} \Rightarrow <F_2PT = \beta = \tan^{-1}\left(\frac{b^2}{2ac}\right)$

$\Rightarrow <F_2PF_1 = \alpha + \beta = \tan^{-1}\left(\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\right) = \tan^{-1}\left(\frac{a}{c}\right)$

$\Rightarrow$ required acute angle $180^\circ - (\alpha + \beta) = \tan^{-1}\left(\frac{\tan180 - \frac{a}{c}}{1 - \tan180.\frac{a}{c}}\right) = \tan^{-1}\left(\frac{-a}{c}\right)$

3(a)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}.\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-b^2x}{a^2y} \Rightarrow$ at the point (m, n) is $\frac{-b^2m}{a^2n}$

3(b) The gradient of the line joining (m, n) to (c, 0) is $\frac{n}{m-c}$ $\Rightarrow$ $\frac{n}{m-c} = \frac{-b^2m}{a^2n}$ $\Rightarrow$ $a^2 n^2 + b^2 m^2 = b^2 mc$

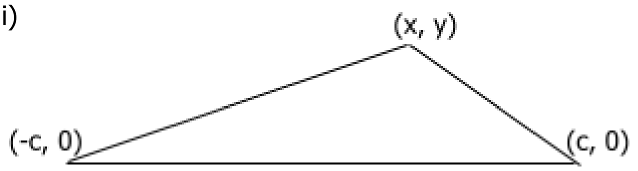
$\Rightarrow$ $\frac{n^2}{b^2}+\frac{m^2}{a^2}=\frac{mc}{a^2}$ $\Rightarrow$ since (m, n) lies on the ellipse, $\frac{mc}{a^2}=1$ $\Rightarrow$ $m=\frac{a^2}{c}$

3(c) $\frac{m^2}{a^2}+\frac{n^2}{b^2}=1$ $\Rightarrow$ $\frac{1}{a^2}\frac{a^4}{c^2}+\frac{n^2}{b^2}=1$ $\Rightarrow$ $\frac{n^2}{b^2}=1-\frac{a^2}{c^2}$ $\Rightarrow$ $n^2=\frac{b^2}{c^2}\left(c^2-a^2\right)$ $\Rightarrow$ $n=\frac{b}{c}\sqrt{\left(c^2-a^2\right)}$

3(d) As $c \to a$, $n \to 0$ and $m \to a$

4(a)(i) When the comet meets the line connecting the two planets, the distance from the furthest point is c + k and the distance from the closest is c + k $\Rightarrow$ (c + k) + (c – k) = d so d = 2k. Since x < c, d < 2c.

(ii)



$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=d$$

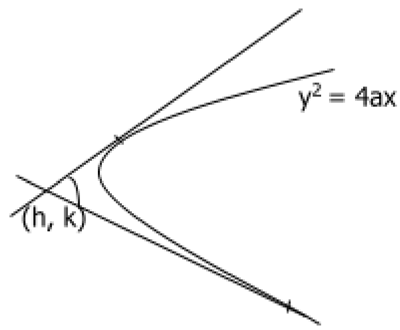
(iii) $\sqrt{(x+c)^2+y^2}=d+\sqrt{(x-c)^2+y^2}$ $\Rightarrow$ $4xc-d^2=2d\sqrt{(x-c)^2+y^2}$

$\Rightarrow$ $\left(16c^2-4d^2\right)x^2-4d^2y^2=d^2\left(4c^2-d^2\right)$ $\Rightarrow$ $\frac{4x^2}{d^2}-\frac{4y^2}{4c^2-d^2}=1$

4(b)(i) $\frac{4}{d^2}=\frac{1}{25}$ $\Rightarrow$ d = 10

(ii) $\frac{4}{4c^2-100}=\frac{1}{200}$ gets $c^2 = 225$ $\Rightarrow$ Coordinates of planets are (-15, 0) and (15, 0)

5 Let a tangent to the curve be y = mx + c



Solving simultaneously with the curve $\Rightarrow$ $(mx + c)^2 = 4ax$

$\Rightarrow$ $m^2x^2 + (2mc - 4a)x + c^2 = 0$

$\Rightarrow$ $\Delta = (2mc - 4a)^2 - 4m^2c^2 = -16amc + 16a^2 = 0$ if the line is a tangent

$\Rightarrow$ a = mc

But y = mx + c passes through (h, k) $\Rightarrow$ c = k – mh

$\Rightarrow$ a = m(k – mh) $\Rightarrow$ $hm^2 - km + a = 0$ $\Rightarrow$ $m = \frac{k \pm \sqrt{k^2-4ah}}{2h}$

Let $m_1=\frac{k+\sqrt{k^2-4ah}}{2h}$ and $m_2=\frac{k-\sqrt{k^2-4ah}}{2h}$

$\Rightarrow$ tan θ = tan (difference between the angles the tangents make with OX) = $\frac{m_1-m_2}{1+m_1m_2}$

$$=\frac{\left(\frac{k+\sqrt{k^2-4ah}}{2h}\right)-\left(\frac{k-\sqrt{k^2-4ah}}{2h}\right)}{1+\left(\frac{k+\sqrt{k^2-4ah}}{2h}\right)\left(\frac{k-\sqrt{k^2-4ah}}{2h}\right)}\times\frac{4h^2}{4h^2}=\frac{\left[\left(k+\sqrt{k^2-4ah}\right)-\left(k-\sqrt{k^2-4ah}\right)\right]\times 2h}{4h^2+\left(k+\sqrt{k^2-4ah}\right)\left(k-\sqrt{k^2 4ah}\right)}$$

$$=\frac{2\sqrt{k^2-4ah}\times 2h}{4h^2+\left(k^2-\left(k^2-4ah\right)\right)}=\frac{4h\sqrt{k^2-4ah}}{4h^2+4ah}=\frac{\sqrt{k^2-4ah}}{h+a}=\sqrt{\frac{k^2-4ah}{(h+a)^2}}$$

<u>Exercise 2 : Using the parametric form</u>

1 $at^2 + b = 3b - as^2$ and $2at = \sqrt{2}as$ $\Rightarrow$ $\sqrt{2}t = s$ $\Rightarrow$ $at^2 + b = 3b - a(\sqrt{2}t)^2$ $\Rightarrow$ $at^2 + b = 3b - 2at^2$

$\Rightarrow$ $3at^2 = 2b$ $\Rightarrow$ $t = \pm\sqrt{(2b/3a)}$ $\Rightarrow$ $s = \pm 2\sqrt{(b/3a)}$ $\Rightarrow$ $x = {}^5/_3\, b$ and $y = \pm\sqrt{(8ab/3)}$

$\Rightarrow$ A(5b/3 , $2\sqrt{(2ab/3)}$) and B (5b/3 , $-2\sqrt{(2ab/3)}$)

2(a) (i) $x(-t) = 4(-t)^2 - (-t)^4 = 4t^2 - t^4 = x(t)$ $\Rightarrow$ hence x is even.

(ii) $y(-t) = (-t)^3 + 9t = -t^3 + 9t = -(t^3 - 9t) = -y(t)$ ⇒ hence y is odd.

(iii) Suppose (x_o, y_o) is a point of P corresponding to $t=t_o$. Then $x_o = 4t_o^2 - t_o^4$ and $y_o = t_o^3 - 9t_o$.
The value $t = -t_o$ gives the point $(x_o, -y_o)$ by the results above ⇒ graph is symmetric about the x-axis.

2(b) $x \geq 0 \Leftrightarrow 4t^2 - t^4 \geq 0 \Rightarrow t^2(4 - t^2) \geq 0 \Rightarrow 4 - t^2 \geq 0$ (since $t^2 \geq 0 \Rightarrow 4 \geq t^2 \Rightarrow -2 \leq t \leq 2$

t	0	0.5	1	1.5	2	3
x	0	0.9	3	3.9	0	-45
y	0	-4.4	-8	-10.1	-10	0

2(c) $x = 4t^2 - t^4 = 4 - (t^4 - 4t^2 + 4) = 4 - (t^2 - 2)^2 \Rightarrow x \leq 4$.

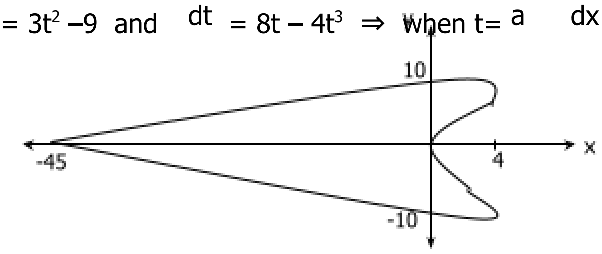
2(d)

2(e) $y = t^3 - 9t \Rightarrow \frac{dy}{dx} = 3t^2 - 9 = 0 \Rightarrow t = \sqrt{3}$ or $-\sqrt{3} \Rightarrow \frac{d^2y}{dx^2} = 6t \Rightarrow$ when $t=\sqrt{3}$, $\frac{d^2y}{dx^2}$ is $6\sqrt{3} \Rightarrow$ minimum

For $t=-\sqrt{3}$, $\frac{d^2y}{dx^2} = -6\sqrt{3}$ ⇒ maximum ⇒ a local maximum at $(3,6\sqrt{3})$ This is the highest point on the graph.

2(f) $\frac{dy}{dt} = 3t^2 - 9$ and $\frac{dx}{dt} = 8t - 4t^3$ ⇒ when $t= a$ $\frac{dy}{dx} = \frac{3a^2 - 9}{8a - 4a^3}$

2(g)



3(a) $(x+1)^2 - \frac{(y-2)^2}{4} = 1$ which is a hyperbola centre (-1, 2).

3(b) $\frac{dx}{d\theta} = \sec\theta \tan\theta$ and $\frac{dy}{d\theta} = 2\sec^2\theta \Rightarrow \frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} = \frac{2}{\sin\theta}$ ⇒ At $\theta = \frac{\pi}{4}$ $\frac{dy}{dx} = 2\sqrt{2}$

⇒ gradient of normal= $-\frac{1}{2\sqrt{2}}$ ⇒ equation of normal $y = -\frac{1}{2\sqrt{2}}x + \frac{9}{2} - \frac{1}{2\sqrt{2}}$ or $x + 1 = 9\sqrt{2} - 2\sqrt{2}y$

⇒ substituting in the equation of the hyperbola gets the quadratic $31y^2 - 284y + 640 = 0$

⇒ $(y - 4)(31y - 160) = 0$ ⇒ since y = 4 is the known solution, $y = \frac{160}{31}$.

3(c) At P gradient of the curve = $\frac{2}{\sin\theta} = 2\sqrt{2}$ ⇒ $\theta = \frac{3\pi}{4}$ ⇒ $x = -\sqrt{2} - 1$ and y=0

⇒ equation of normal $y = -\frac{1}{2\sqrt{2}}x - \frac{1}{2} - \frac{1}{2\sqrt{2}}$ or $x + 1 = -\sqrt{2} - 2\sqrt{2}y$ ⇒ substituting in the equation of the hyperbola gets quadratic $31y^2 + 36y = 0$ ⇒ $y(31y + 36) = 0$ ⇒ since y = 0 is the known solution, $y = -\frac{36}{31}$.