Exam 2 Review

Sec 2.11, 3.1-3.4, Chapter 4 and 5

NOTE: On the exam you may be asked to write and explain your thinking (not just solve problems).

1. Find the negation of each sentence. For a) and b), give your answer in words. For c) and d), give your answer first in symbols, then in words.
    a. a and b are both positive
    b. every integer is either prime or composite
    c. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ y^2 = x$
    d. $(p\ is\ even) \Rightarrow (q\ is\ odd)$
2. How many lists of length 5 are there, taken from the set $\{A, B, C, D, E, F, G, H\}$, if
    a. repetition is not allowed
    b. repetition is not allowed and the last two entries must be vowels
    c. repetition is allowed
    d. repetition is allowed, and the list must contain at least one repeated letter
3.
    a. You have three vampire books, four steampunk novels, and five post-apocalyptic teen romances. How many ways are there of arranging them on a shelf if books of the same type must be kept together?
    b. How many 5 element subsets are there from a set with 8 elements?
    c. Find $|\{X \in P(A) : A = \{0, 1, 2, 3, 4, 5, 6\}\ and\ |X| = 3\}|$
    d. After expanding $(a + b)^5$, what is the coefficient of the term a^3b^2?
    e. In a class consisting of 12 men and 15 women, how many different ways are there of choosing a group consisting of 3 men and 6 women?
    f. What formula do we use to determine the number of subsets of size k of a set of size n? Write the formula, and describe what the different parts of the formula mean (where does the formula come from?).

Prove each proposition. Clearly state whether you are using a direct or contrapositive proof.

4. Proposition. For $a \in \mathbb{Z}$, if $a^2 + 2a + 6$ is odd, then a is odd.
5. Proposition. For $x, y \in \mathbb{Z}$, if $x + y$ is even, then x and y have the same parity.
6. Proposition. Suppose $a, b \in \mathbb{Z}$. If $a|b$ then $a^2|b^2$.
7. Proposition. If $n \in \mathbb{Z}$ then $5n^2 + 3n$ is even.
8. Proposition. For any integer n, if n^2 is odd then n is odd.

Exam 2 Review ANSWER KEY

If you discover an error please let me know, either in class, on the OpenLab, or by email to jreitz@citytech.cuny.edu. Corrections will be posted on the "Exam Reviews" page.

1. a. a is negative or b is negative.
   b. There is an integer that is neither prime nor composite.
   c. $\exists x \in \mathbb{R}\, \forall y \in \mathbb{R},\ y^2 \neq x$. There is a real number x such that for all real numbers y, we have $y^2 \neq x$.
   d. $(p\ is\ even) \wedge (q\ is\ even)$. Both p and q are even.
2. a. $\frac{8!}{3!} = 6720$ b. $2 \cdot 1 \cdot 6 \cdot 5 \cdot 4 = 240$ c. $8^5 = 32768$ d) $8^5 - \frac{8!}{3!} = 26048$
3. a. $3! \cdot (3! \cdot 4! \cdot 5!) = 103680$ b. $\binom{8}{5} = 56$ c. $\binom{7}{3} = 35$ d. $\binom{5}{3} = 10$
   e. $\binom{12}{3} \cdot \binom{15}{6} = 1101100$
   f. $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

NOTE: For problems requiring you to prove something, there is usually more than one correct answer, and it is often possible to use more than one type of proof correctly. The following are examples of correct solutions, yours may be different.

4. Proposition. For $a \in \mathbb{Z}$, if $a^2 + 2a + 6$ is odd, then a is odd.
   I will use contrapositive proof.
   Proof. Suppose a is even. Then $a = 2b$ for some integer b, by the definition of even number. Thus $a^2 + 2a + 6 = (2b)^2 + 2(2b) + 6 = 4b^2 + 4b + 6 = 2(2b^2 + 2b + 3)$. So $a^2 + 2a + 6 = 2c$, where c is the integer $c = 2b^2 + 2b + 3$(note that c is an integer by closure of the integers under addition and multiplication). Therefore $a^2 + 2a + 6$ is an even number, by the definition of even number. ☐
5. Proposition. For $x, y \in \mathbb{Z}$, if $x + y$ is even, then x and y have the same parity.
   I will use contrapositive proof.
   Proof. Suppose x and y have opposite parity. There are two cases, first where x is odd and y is even, and second where y is odd and x is even.
   (NOTE: The proofs of these two cases will look ***exactly*** *the same, except that 'x' will be switched with 'y'. Because of this, I do NOT need to write out both cases -- instead, I simply use the phrase "without loss of generality" or "WLOG" and choose one of them. This blue paragraph is not part of the proof).*
   Without loss of generality, assume x is odd and y is even. Then $x = 2a + 1$ and $y = 2b$, by the definitions of 'odd' and 'even' respectively. Therefore

$x + y = 2a + 1 + 2b = 2(a + b) + 1$, and so if we let $c = a + b$ then $x + y = 2c + 1$. c is an integer by closure of the integers under addition and multiplication, and thus $x + y$ is odd, by the definition of odd number. ☐

6. Proposition. Suppose $a, b \in \mathbb{Z}$. If $a|b$ then $a^2|b^2$.
   I will use direct proof.
   Proof. Suppose $a|b$. Then $b = ac$ for some integer *c*, by the definition of divides, and so $b^2 = (ac)^2 = a^2c^2$. Let$k = c^2$. Then *k* is an integer by closure of the integers under multiplication, and $b^2 = a^2k$ by substitution. Therefore $a^2|b^2$ by the definition of divides. ☐
7. Proposition. If $n \in \mathbb{Z}$ then $5n^2 + 3n$ is even.
   I will use direct proof.
   Proof. Suppose $n \in \mathbb{Z}$. There are two cases, *n* is odd or *n* is even.
   Case 1. Suppose *n* is odd. Then $n = 2a + 1$ for some integer *a* (by the definition of odd number), and $5n^2 + 3n = 5(2a + 1)^2 + 3(2a + 1) = 2(10a^2 + 13a + 4)$. Thus $5n^2 + 3n = 2k$ for $k = 10a^2 + 13a + 4$. Since *k* is an integer by closure of the integers under addition and multiplication, we conclude that $5n^2 + 3n$ is even, by the definition of even number.
   Case 2. Suppose *n* is even. Then $n = 2a$ for some integer *a* (by the definition of even number), and $5n^2 + 3n = 5(2a)^2 + 3(2a) = 2(10a^2 + 3a)$, and so $5n^2 + 3n$ is even by the definition of even number. ☐
8. Proposition. For any integer *n*, if n^2 is odd then *n* is odd.
   I will use contrapositive proof.
   Proof. Suppose *n* is even. Then $n = 2b$ for some integer *b*, by the definition of even number. Therefore $n^2 = (2b)^2 = 2(2b^2)$, and so $n^2 = 2k$ for the integer $k = 2b^2$ (note that *k* is an integer by closure of the integers under multiplication). Thus n^2 is even, by the definition of even number. ☐