Vector Calculus MAT226 Fall 2021
Professor Sormani
Lesson 5: Hyperboloids, Paraboloids 11.6

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT226F21-lesson5-lastname-firstname

and share editing of that document with me sormanic@gmail.com and with our graders. If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT226 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

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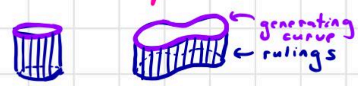
This lesson teaches hyperboloids and paraboloids. It assumes you already know what a hyperbola and a parabola are. Please review these notions before watching the Playlist 226F21-5-1to14.

Vector Calculus Lesson 5

Hyperboloids and Paraboloids

Surfaces in Space

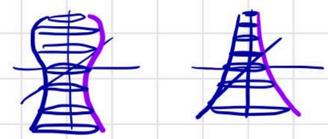
- Cylindrical Surfaces



- Quadric Surfaces (Hyperboloids + Paraboloids)

$$ax^2 + by^2 + cz^2 = d$$

- Surfaces of Revolution



In this lesson we will learn to graph with pencils to give a deeper understanding

If you wish you may also learn how to graph with MATLAB or MAPLE (extra credit) or another graphics program.

Cylindrical Surfaces (also called cylinders)

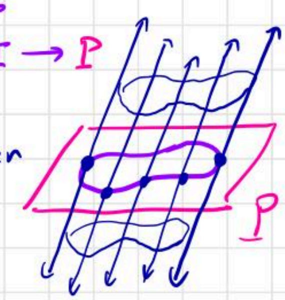
Start with a plane P in $\mathbb{R}^3$

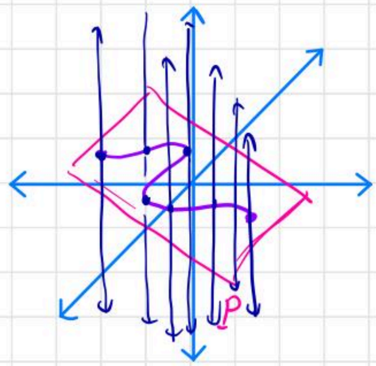
draw any curve in the plane

"generating curve" $\quad C : I \to P$

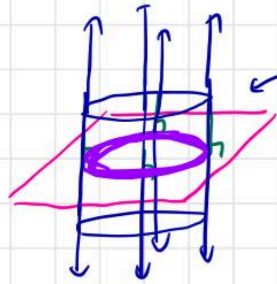
draw lines through the curve, C, that are parallel to each other and do not lie inside P

These lines are called "rulings"



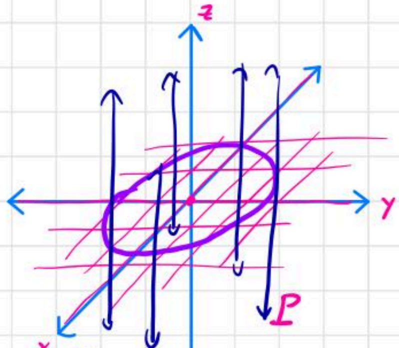


If the rulings are $\perp$ to P at 90° right angle then we say it is a <u>right</u> cylinder



right <u>circular</u> cylinder

Example: A right <u>circular</u> cylinder with P = xy plane = $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, z = 0 \right\}$



and radius R

generating curve is a circle of radius R

$\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| \, x^2 + y^2 = R^2 \right\}$

rulings $\perp$ to the plane

this run through all values of z

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x^2 + y^2 = R^2 \text{ and } z \in \mathbb{R} \right\}$$

free

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

position lies on circle

$$\left\{ \begin{pmatrix} x \\ y \\ t \end{pmatrix} \middle| \, x^2 + y^2 = R^2 \text{ and } z = t \in \mathbb{R} \right\}$$

free parameter.

A cylindrical Surface with

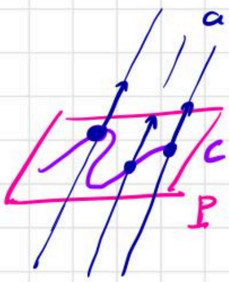
Plane $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid ax + by + cz = d \right\}$

generating curve $C : I \to P$

$C(s) = (c_1(s), c_2(s), c_3(s))$ where $s \in I$

and $a\,c_1(s) + b\,c_2(s) + c\,c_3(s) = d$

and rulings in direction $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.



$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1(s) \\ c_2(s) \\ c_3(s) \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

position changes $s \in I$

direction is the same because all rulings are parallel

The cylindrical surface includes all the lines

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1(s) \\ c_2(s) \\ c_3(s) \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \;\middle|\; s \in I \quad t \in \mathbb{R} \right\}$$

"parametrized surface"

$$x = c_1(s) + t\,v_1$$
$$y = c_2(s) + t\,v_2$$
$$z = c_3(s) + t\,v_3$$

Example Right Circular Cylinder $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x^2 + y^2 = R^2 \right\}$

What is the generating curve? $\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \mid x^2 + y^2 = R^2 \right\}$

we can parametrize it

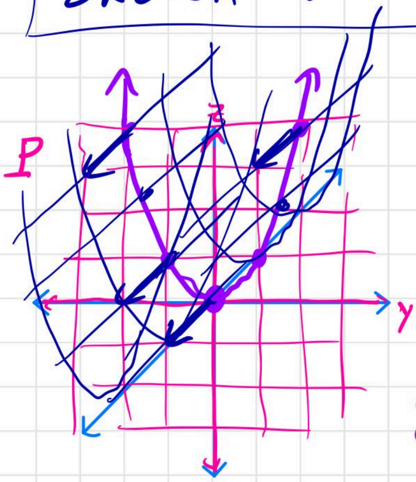
$C(s) = (R\cos(s), R\sin(s), 0)$ where $s \in I = [0, 2\pi]$

Right cylinder $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \perp$ to xy plane.

note $C(s)$ is in this plane because $z = 0$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\cos(s) \\ R\sin(s) \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s \in [0, 2\pi] \quad t \in \mathbb{R} \right\}$$

Sketch the cylinder $z = y^2$



$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; z = y^2 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ y^2 \end{pmatrix} \;\middle|\; y \in \mathbb{R},\ \underline{x \in \mathbb{R}} \right\}$$

free

only $z = y^2$

generating curve involves z and y (not x)

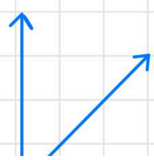
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ y^2 \end{pmatrix} \;\middle|\; y \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \;\middle|\; z = y^2 \right\}$$ in yz plane

a parabola

the cylinder is:

$t = x$

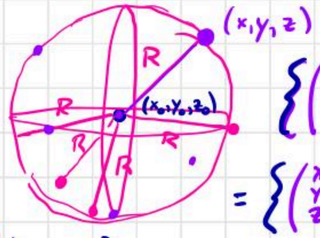
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ y^2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; \begin{matrix} t \in \mathbb{R} \\ y \in \mathbb{R} \end{matrix} \right\}$$



position on parabola

direction is $\hat{\imath}$ $\hat{x}$ in direction of x axis.

A sphere about (x_0, y_0, z_0) of radius R
center



many (x,y,z)
one fixed center (x_0, y_0, z_0)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right\| = R \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = R \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2 \right\}$$

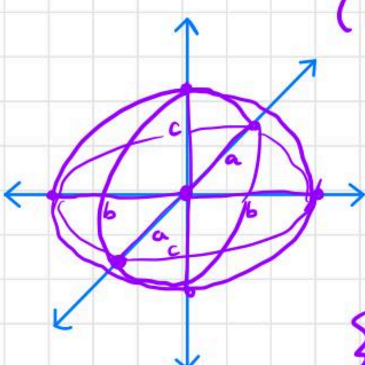
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2$$

$$x^2 + 2x x_0 + x_0^2 + y^2 + 2y y_0 + y_0^2 + z^2 + 2z z_0 + z_0^2 = R^2$$

Stretch our sphere by a in x direction
b in y direction
c in z direction

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) \in \text{the sphere of radius 1 about center } \underline{(0,0,0)} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \right\}$$



ellipsoid.

centered at (x_0, y_0, z_0)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 + \left(\frac{z-z_0}{c}\right)^2 = 1 \right\}$$

both quadric surfaces

$$\frac{x^2}{a^2} + 2\frac{x}{a}\frac{x_0}{a} + \frac{x_0^2}{a^2} + \frac{y^2}{\ldots} \; y \ldots - z^2 \ldots z \ldots = 1$$

classwork write out

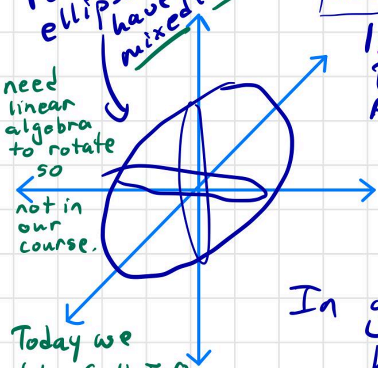
Quadric Surfaces

include spheres + ellipsoids

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + Gxy + Hyz + Ixz + J = 0 \right\}$$

rotated ellipsoids have mixed terms

need linear algebra to rotate so not in our course.



Today we take $G=H=I=0$ no mixed terms.

$\boxed{\text{includes spheres:}}$

$$1x^2 + 2x x_0 + x_0^2 + 1y^2 + 2y y_0 + y_0^2 + 1z^2 + 2z z_0 + z_0^2 = R^2$$

$A=1$ $\quad D=2x_0$ $\quad B=1$ $\quad E=2y_0$ $\quad C=1$ $\quad F=2z_0$

$G=0 \quad H=0 \quad I=0 \quad J = x_0^2 + y_0^2 + z_0^2 - R^2$

Extra Credit do this for an ellipsoid.

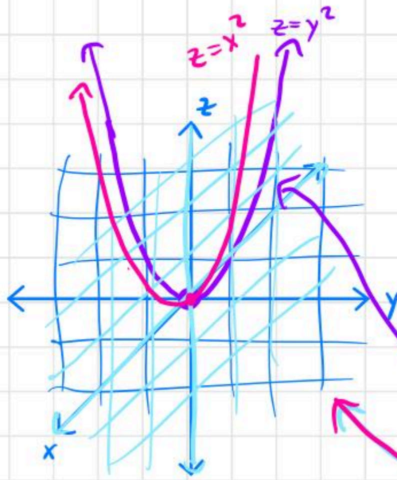
In general quadic surfaces can have negative values for some of the constants

Paraboloid $z = x^2 + y^2$

This is a quadric surface

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; 1x^2 + 0x + 1y^2 + 0y + 0z^2 - 1z = 0 \right\}$$

no mixed terms



How do we graph this paraboloid?

Since there are no mixed terms there is no rotating

So only need to check intersections of the surface with

xy plane has $z=0$: $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; \begin{matrix} z=x^2+y^2 \\ z=0 \end{matrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

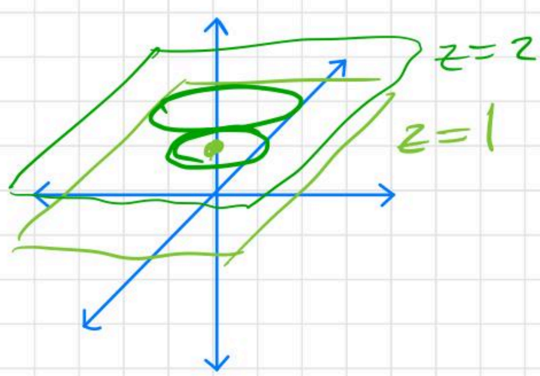
yz plane: has $x=0$: $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; \begin{matrix} z=x^2+y^2 \\ x=0 \end{matrix} \right\} = \left\{ \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \middle| \; z=y^2 \right\}$ a parabola

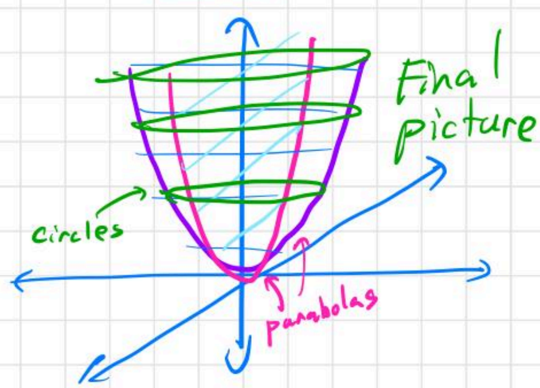
xz plane: has $y=0$: $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; \begin{matrix} z=x^2+y^2 \\ y=0 \end{matrix} \right\} = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \middle| \; z=x^2 \right\}$ a parabola

Next look at levels of the surface where z = constant

$z=k$ plane $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; \begin{matrix} z=x^2+y^2 \\ z=k \end{matrix} \right\} = \left\{ \begin{pmatrix} x \\ y \\ k \end{pmatrix} \middle| \; k = x^2+y^2 \right\}$

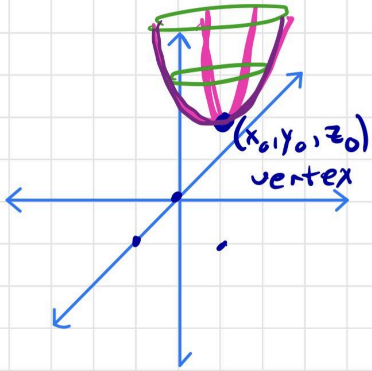
take $k = R^2$ these are circles





Also an elliptic paraboloid

$$z - z_0 = \left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2$$



parabolas

$$z = \left(\frac{x - x_0}{a}\right)^2 + z_0 \quad \text{in } y = y_0 \text{ plane}$$

$$z = \left(\frac{y - y_0}{b}\right)^2 + z_0 \quad \text{in } x = x_0 \text{ plane}$$

ellipses $z = z_0 + k$ planes

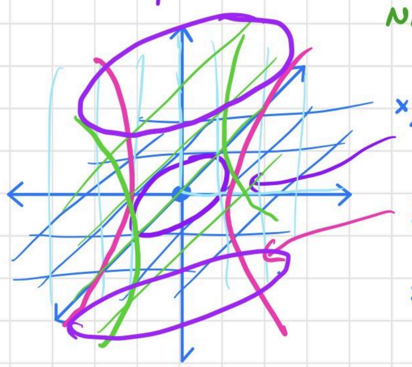
$$k = \left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2$$

Hyperboloids

one sheeted $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 - \left(\frac{z-z_0}{c}\right)^2 = 1 \right\}$

two sheeted $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \left(\frac{x-x_0}{a}\right)^2 - \left(\frac{y-y_0}{b}\right)^2 - \left(\frac{z-z_0}{c}\right)^2 = 1 \right\}$

Example: $x^2 + y^2 - z^2 = 1$



No mixed terms consider intersections with

xy plane: $z=0$ $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{matrix} x^2+y^2-z^2=1 \\ z=0 \end{matrix} \right\} = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x^2+y^2=1 \right\}$ circle

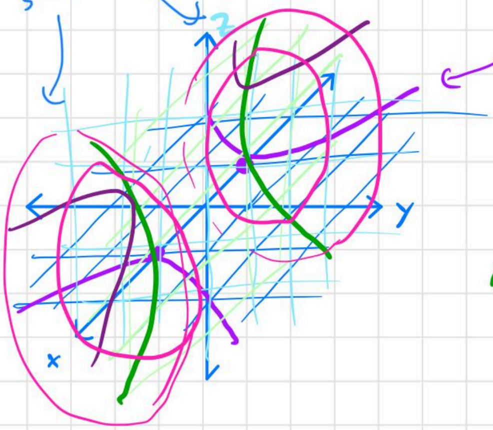
yz plane $x=0$ $\left\{ \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \;\middle|\; 0^2+y^2-z^2=1 \right\}$ hyperbola with $y=\pm 1$ when $z=0$

xz plane $y=0$ $\left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \;\middle|\; x^2+0^2-z^2=1 \right\}$ also a hyperbola where $x=\pm 1$ when $z=0$

also check $z=k$ planes $\left\{ \begin{pmatrix} x \\ y \\ k \end{pmatrix} \;\middle|\; x^2+y^2=1+k^2 \right\}$ a circle of radius $\sqrt{1+k^2}$

Classwork do $x^2-y^2-z^2=1$ pause + try

two sheeted



intersection with xy plane $z=0$ $\left\{\begin{pmatrix}x\\y\\0\end{pmatrix} \middle| \ x^2-y^2=1\right\}$

hyperbola

precalc

$x=0$ $0^2-y^2=1$ no y values

$y=0$ $x^2=1$ so $x=\pm 1$

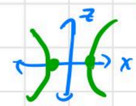
intersection with yz plane $x=0$ $\left\{\begin{pmatrix}0\\y\\z\end{pmatrix} \middle| \ -y^2-z^2=1\right\}=\emptyset$

empty because $-y^2-z^2\leq 0$ cannot be 1

intersection with xz plane $y=0$ $\left\{\begin{pmatrix}x\\0\\z\end{pmatrix} \middle| \ x^2-z^2=1\right\}$

hyperbola $x=0$ has no z

$z=0$ has $x=\pm 1$



levels of $z=k$ $\left\{\begin{pmatrix}x\\y\\k\end{pmatrix} \middle| \ x^2-y^2=1+k^2\right\}$

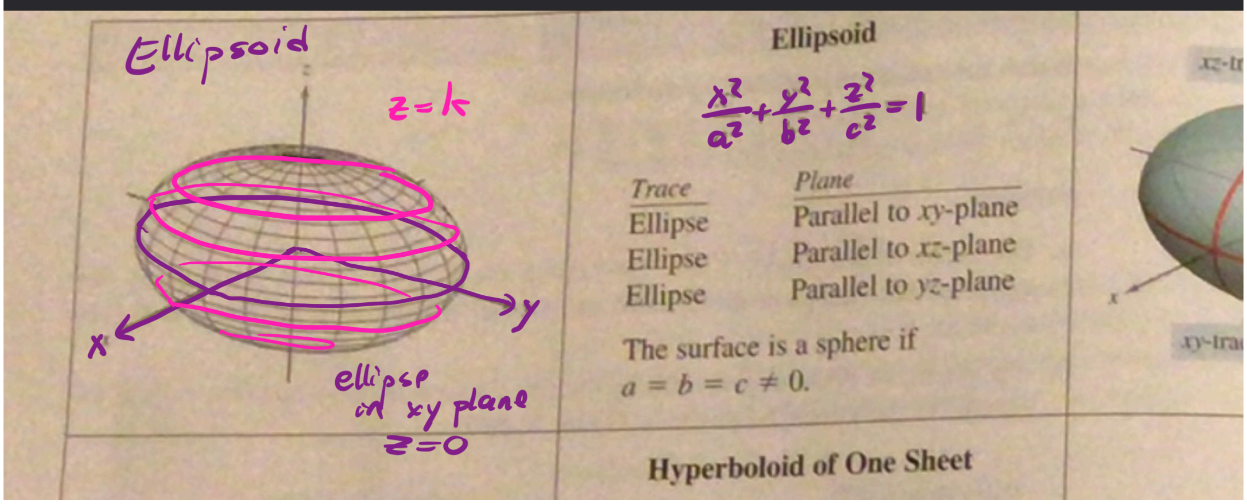
also check $x=k$ $\left\{\begin{pmatrix}k\\y\\z\end{pmatrix} \middle| \ k^2-y^2-z^2=1\right\}$

$=\left\{\begin{pmatrix}k\\y\\z\end{pmatrix} \middle| \ k^2-1=y^2+z^2\right\}$

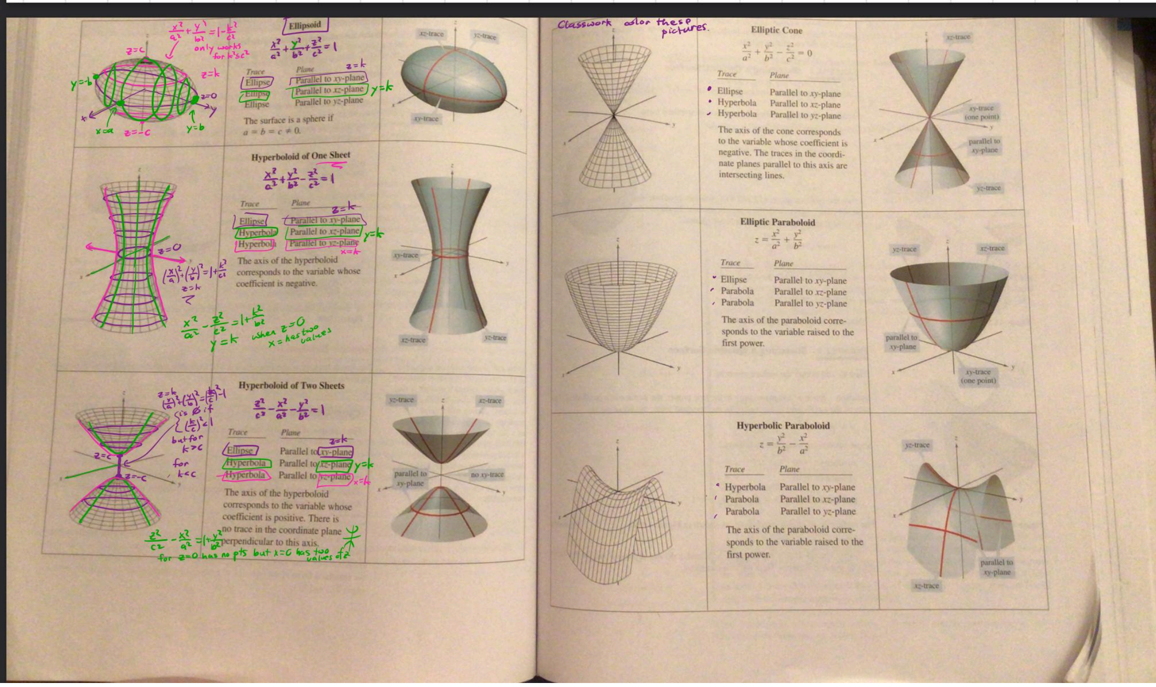
circles when $k^2>1$

See graphics in the book.

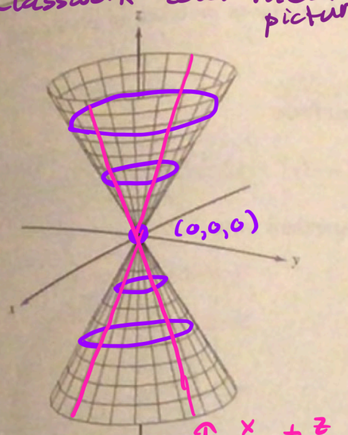
Quadric Surfaces



Quadric Surfaces



Classwork color these pictures.



$\frac{x}{a}=\pm\frac{z}{c}$

when $y=0$

Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Trace	*Plane*	
Ellipse	Parallel to xy-plane	$z=k$
Hyperbola	Parallel to xz-plane	$y=k$
Hyperbola	Parallel to yz-plane	

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$ if $k=0$ $(0,0,0)$

if $k>0$ this is an ellipse

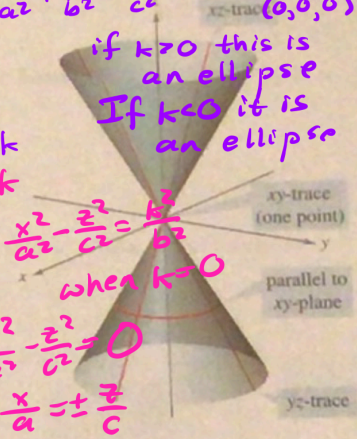
If $k<0$ it is an ellipse

$\frac{x^2}{a^2}-\frac{z^2}{c^2}=\frac{k^2}{b^2}$

when $k=0$

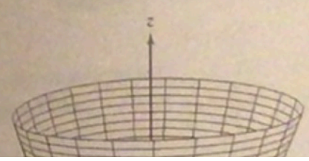
$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$

$\frac{x}{a}=\pm\frac{z}{c}$



Elliptic Paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$



Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane

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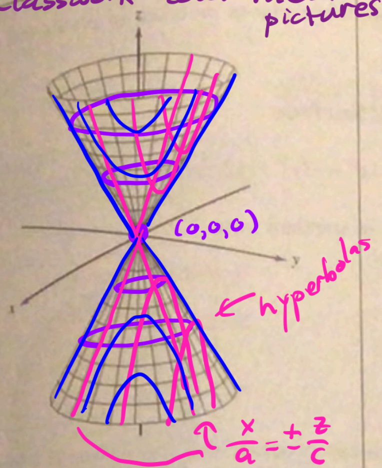
Vector Calculus 226F21

Classwork color these pictures.



$\frac{x}{a} = \pm\frac{z}{c}$ when $y=0$

Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

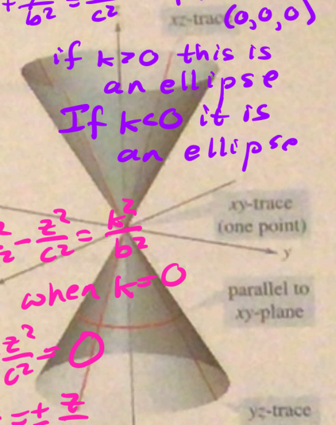
Trace	*Plane*	
Ellipse	$\boxed{\text{Parallel to } xy\text{-plane}}$	$z=k$
Hyperbola	$\boxed{\text{Parallel to } xz\text{-plane}}$	$y=k$
Hyperbola lines	Parallel to yz-plane	

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$ if $k=0$ (0,0,0)

if $k>0$ this is an ellipse

If $k<0$ it is an ellipse



$\frac{x^2}{a^2}-\frac{z^2}{c^2}=\frac{k^2}{b^2}$

when $k=0$

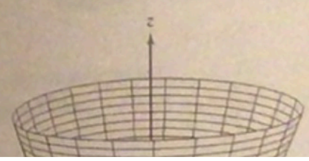
$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$

$\frac{x}{a}=\pm\frac{z}{c}$

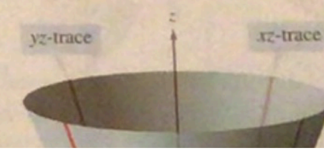
$y=k\neq0$ we get hyperbolas

Elliptic Paraboloid

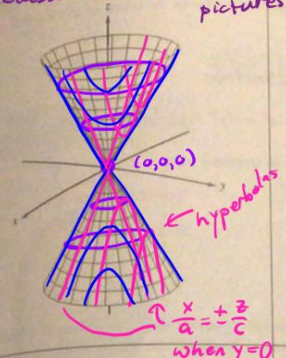
$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$



Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane



Classwork color these pictures.



Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Trace	*Plane*	
Ellipse	Parallel to *xy*-plane	$z=k$
Hyperbola	Parallel to *xz*-plane	$y=k$
Hyperbola (+lines)	Parallel to *yz*-plane	

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$ if $k=0$ $(0,0,0)$

if $k>0$ this is an ellipse

If $k<0$ it is an ellipse

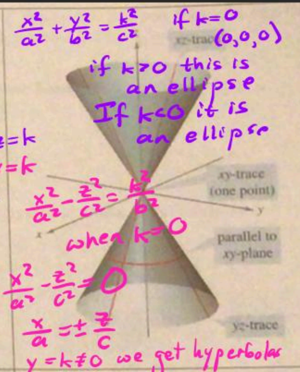
$\frac{x^2}{a^2}-\frac{z^2}{c^2}=\frac{k^2}{b^2}$

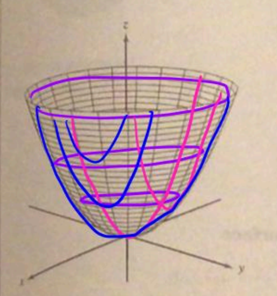
when $k=0$

$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$

$\frac{x}{a}=\pm\frac{z}{c}$

$y=k\neq 0$ we get hyperbolas



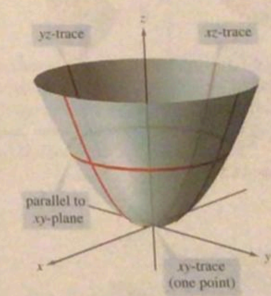


Elliptic Paraboloid

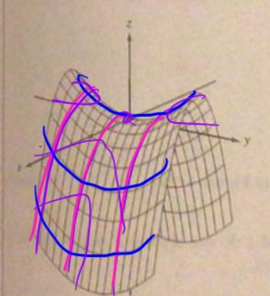
$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



Questions: Email me with the word Question in the subject.

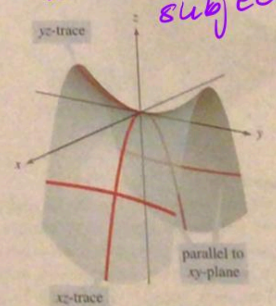


Hyperbolic Paraboloid

$$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



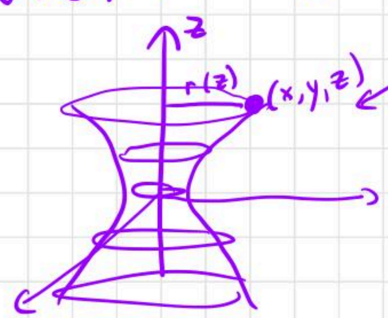
Surfaces of Revolution

start with an axis (example z axis; x axis; y axis)

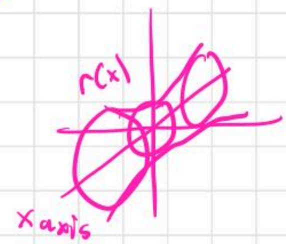
define a generating radius function

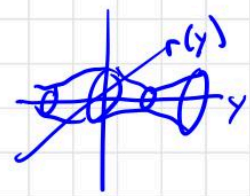
$$r = r(z) \qquad r=r(x) \qquad r=r(y)$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x^2+y^2 = r^2(z) \right\}$$



So that $r(z) =$ dist from $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to z axis

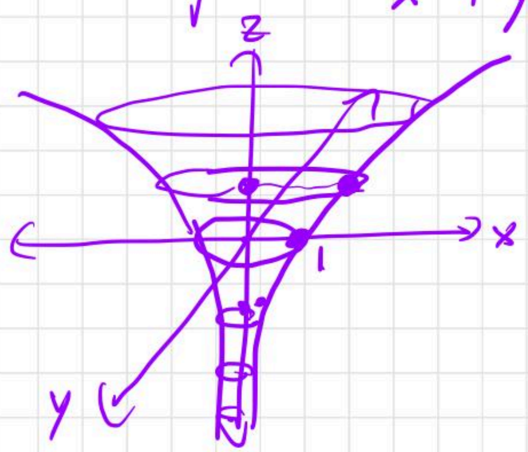




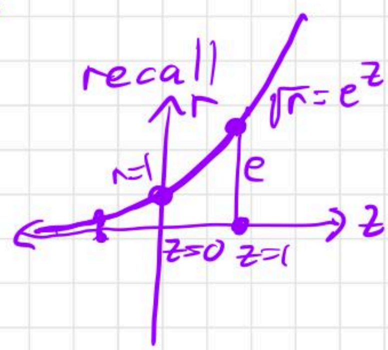
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r(z)\cos\theta \\ r(z)\sin\theta \\ z \end{pmatrix} \;\middle|\; z \in I,\ \theta \in [0, 2\pi] \right\}$$

Example: $x^2+y^2 = e^z$

$e^z = r^2$



recall



HW: 11.6/ 1-6, 9,11,13,15

As always, check your answers in the back of the book and ask questions if you have the wrong answer.   I do not check your work, I just check that you did enough work.

You may skip this homework if you are over a week behind schedule.

Please be sure also to submit Calculus Review Examples from 3.1-3.4 of the textbook.   This has priority over the graphics if it has been a long time since you took Calculus 1.