Lesson 7:
Volume of Sphere, Cone, Cylinder

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Standards

By the nature of starting with 3D printed models instead of pencil and paper, these lesson plans take on topics in a somewhat different order than is typical. We also only select those topics that benefit from 3D printed model support and may add material not covered explicitly by any standard. However the concepts are central to just about any overall program. We do not claim perfect or complete alignment with any state or federal standard, but for teacher convenience here are Common Core topics that we feel are addressed at least in part by the material in this lesson.

CCSS.Math.Content.8.G.C.9 8th grade math. “Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.”

Objectives

- Know the formulas for the volume of a cone, cylinder and sphere
- Understand the relationships among the volumes of a cone, hemisphere, and cylinder of the same radius and height are in the ratio 1 : 2 : 3.

Models Used

You can find the models for this Lesson at https://github.com/whosawhatsis/Geometry. See the Teacher’s Guide for how to use Github.

- sphere_cone_volume.scad
    - Prints a cone, half-sphere and cylinder in volume ratio 1 : 2 : 3, and hollow molds of the same volumes

Other Supplies

- A scale (a postal scale or lab scale that can measure to fractions of a gram would be ideal)
- About 1/4 cup of table salt, sand, or other fine granular substance that does not pack down when compressed.
- A calculator

Volume

First, let's talk about some math that is pretty old—some of it was known thousands of years ago, and most of the rest of it was developed in the 1600s. There are a lot of practical applications of knowing the volume of 3D objects (specifically, cones, spheres and cylinders).

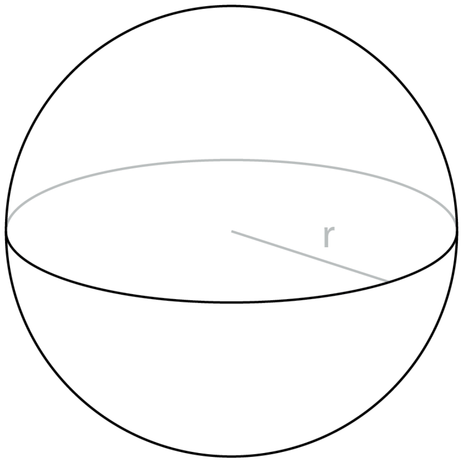
The volume of an object is how much space it encloses, usually quoted in a cubic length unit, like cubic meters (or m^3) or cubic centimeters (cm^3, or cc). You can imagine it by thinking how many little cubes could be stuffed into the object. Another standard unit of volume is the liter. A thousandth of a liter, a milliliter (mL) is also equal to 1 cm^3.

It turns out that there are simple relationships among the volumes of a sphere, cylinder and cone. In this Lesson, we will just give you the formulas. In Lesson 8, we'll talk about where these relationships come from, using a rule called Cavalieri's Principle.

Volume of a Sphere

You can think of a *sphere* as the 3D equivalent of a circle. A ping-pong ball is a sphere. The Earth is more or less spherical. Technically, a sphere is defined as the set of points that are all the same distance (the radius) from its center. Given that, it's not too surprising that all you need to get the volume of a sphere is the radius, which we will call *r*.

Volume of a sphere $= \frac{4}{3}\pi r^3$



For example, if we had a sphere that had a radius of 2 cm, the volume would be

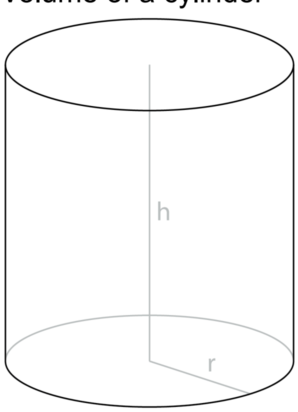
$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi * (2cm)^3 = \frac{4}{3}\pi * 8cm^3 = 33.51cm^3$$

If you wanted to measure all the way around a sphere (making a circle like the equator on Earth) just like a 2D circle of radius *r* the circumference will be $2\pi r$. Look at Lesson 6 for a discussion of what happens if you slice through a sphere in a way that does not go through its center.

Volume of a Cylinder

A cylinder is an object that has a constant circular cross section, like a soup can. The sides are straight and parallel to each other, at right angles to the circular cross-section. (There are skewed cylinders with sides that are not at right angles, and we'll talk about those in Lesson 8.) To get the volume of a cylinder, we multiply that cross-section times the height. Since the cross-section is circular, its area is πr^2. If we call the height *h,* then

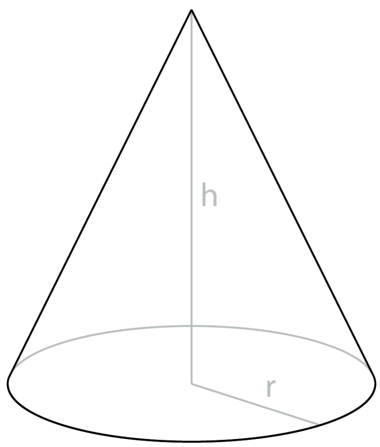
Volume of a cylinder = base area * height = $\pi r^2 h$.



Volume of a Cone

A cone, like a cylinder, has a circular cross-section. However, this cross-section steadily gets smaller as we go from the base of the cone up to its point. An ice-cream cone or a party hat are cones. You might expect that the area of the base, πr^2 would be involved somehow. As it turns out, If a cone has height *h*, then

Volume of a cone = $\frac{1}{3}$* area of the base * height =$\frac{1}{3}\pi r^2 h$.



Notice that this is exactly $\frac{1}{3}$ the volume of a cylinder of the same radius and height. Democritus of Greece, who lived about 2400 years ago, is credited with being the first to realize this, although why it was true had to wait about 2000 years for Cavalieri (who we will talk about in Lesson 8). Archimedes (about 100 years after Democritus) expanded his ideas to get the relationships among the volumes of spheres, cones, and cylinders.

Volume Experiments

Let’s review the volume formulas.

Shape	Volume
Sphere	$\frac{4}{3}\pi r^3$
Cylinder	$\pi r^2 h$
Cone	$\frac{1}{3}\pi r^2 h$

Now, we will explore a special case, when the height (*h*) of the cylinder and cone are equal to the radius (*r*). What are the volumes then, if we make *h* = *r*?

	Volume if height (h) = radius (r)
Sphere	$\frac{4}{3}\pi r^3$
Cylinder	$\pi r^2 r = \pi r^3$
Cone	$\frac{1}{3}\pi r^2 r = \frac{1}{3}\pi r^3$

We've created a model that has a half-sphere, cylinder, and cone for this special case, where all three volumes are a multiple of $\frac{1}{3}\pi r^3$. Since the cone is the smallest, let's think about all the volumes as multiples of the volume of a cone.

	Volume compared to cone
Half Sphere	2 (full sphere would be 4)
Cylinder	3
Cone	1

The model **sphere_cone_volume.scad** generates six pieces: a half-sphere (better known as *hemisphere*), a cone, and a cylinder that all have the same radius, *r* (in mm), defaulted to 25mm, and molds of each of those objects. The half-sphere, cone, and cylinder all fit into their respective molds. (We didn't make a full sphere because it is too hard to 3D print and use a hollow version.)

You can adjust the value of *r* to get a set of bigger or smaller pieces. It is better to adjust the value of r in OpenSCAD versus scaling in your slicer, since some detailed features of the molds, and clearances, might not scale well if you scale the whole model. If the fit of the pieces into their molds is too tight, increase the value of the clearance variable (defaulted to 0.3mm). You'll need it for the density assessment later in this Lesson, so be sure to write down what value you used.

Here we can see all six pieces. Notice that the cylinder has a little handle on one end. That's to help you get it out of the cylinder mold, which tends to get stuck otherwise.

Now you can put each model into its corresponding mold to see that they fit together and that the hollow molds have the same volume (within the clearances) as do the original shapes.

Assessment

Let’s try an experiment to see if it is really true that the ratio of the volumes of a cone, half-sphere and cylinder is 1 : 2 : 3 (when the radius of the cone and cylinder are equal to their height).

First, fill the cone with something that won’t pack down. Sand, sugar or salt will work. We’re using Himalayan (pink) salt so that it shows up well.

It can be difficult to get 3D prints to hold water around seams. Water's surface tension also makes it difficult to fill to a precise level, and to pour without spilling, so we suggest something granular instead. Flour packs down, so it is less precise for this purpose. First, fill the cone mold. Then pour the contents of the cone mold into the half-sphere. Then fill the cone model again.

Pour the contents of the cone into the half-sphere too. It should just exactly fill.

Now fill the cone a third time. Pour it and the half-sphere into the cylinder (1 + 2 = 3). And you will see that this works!

If the radius and the height of the models are not equal, the relationships get more complicated. We will see in Lesson 8, though, that the relationship of 1:3 in volume between a cone and a cylinder holds as long as the cone and cylinder have the same radius and height as each other.

Once you finish this experiment, save the granulated material you were measuring so we can find its density in the next section.

Density

The density of a material is the mass of a standard volume (like a cubic centimeter) of that material. For example, one gram of water has a volume of one cubic centimeter, so the density of water is one gram per cubic centimeter, or 1 g/cm^3. That's not accidental — the metric system was originally defined that way, although it was later adapted to more precise standards.

You'll note that we talked about mass and not weight. In everyday use, we use these somewhat interchangeably. Technically, though, the mass of an object (measured in kilograms or grams, in the metric system) is the same everywhere. The *weight*, which is the force exerted by gravity on an object with a particular mass, is measured in Newtons in the metric system, and in pounds in the Imperial system. It would be different depending on the gravity you are experiencing. Since you are (presumably) going to do these experiments on the Earth's surface, the distinction may seem a little fussy. However, density in units of mass per cubic volume (not force per cubic volume), and we'll use the more precise term here.

Measuring Density

We just used salt, sugar or sand to explore volume. Obviously sand in particular will have a different density depending on what types of rocks were ground down to create it. Salt or sugar will vary a bit too, based on any impurities, whether they have absorbed water, and the like. You'll have to measure a known quantity first and then go from there. You could also try Play-doh or similar substances if you can smooth off the top evenly enough. To do this, you'll need to put in more than enough to fill the space, then cut off the excess (dental floss works well for cutting Play-doh).

Here is how you can measure density. First, take the cylinder mold (hollow) model. Density is often quoted in grams per cubic centimeter, so let's first get the radius in centimeters. One centimeter (a hundredth of a meter) is 10mm (a thousandth of a meter). If we used the default of r = 25mm, that's 2.5cm. The volume inside this mold is πr^3, or π * $(2.5cm)^3$ = 49 cm^3.

Next, use a postal scale to measure the mass of the cylinder model with nothing in it. That is called the "tare weight" of the container. If possible, measure the mass in grams. (For some reason no one says "tare mass" even though they should.) Ours, printed in PLA at the default radius of 2.5cm, had a tare weight of 8g. Most scales will have a tare function that allows you to place your empty container on the scale and zero it, so that you are only measuring the contents. We'll show the process without using this function, in case your scale doesn't have one.

Next, fill the cylinder with the sugar, salt, sand or whatever else you used in the earlier experiment (let's say we are working with salt), and use the postal scale to measure the mass of the cylinder model plus the material. When we tried it, we got 63.89 grams for the cylinder plus salt.

To get the mass of the sugar alone, subtract the tare weight of the cylinder from the total.
In our experiment, it was 63.89 - 8.00 = 55.89g.

To get the density of the salt alone, divide the mass by the volume.
In our experiment, 55.89g / 49 cm^3 = 1.14g/cm^3

This is what is called the *bulk density*, which allows for air that gets trapped between the grains of salt. (If you could somehow stuff salt grains into one giant salt crystal, it would be denser than the loose granular salt.)

Assessment

Try the experiment described above. If you were a bit off, consider what your sources of error might be. What results might you expect if you did exactly the same thing with the cone or hemisphere molds?

If You Don't Have a 3D Printer

If a 3D printer isn't available, you can make these parts out of things you have lying around, although it will be challenging to get them precisely in the ratio here.

Cylinders are easy to find (straight-sided cups or glasses, or soup cans will work). Cones are easy to make out of paper and should actually self-straighten into a better cone when filled, since the correct shape is the maximum volume for its surface.

A hemisphere is a little tricker. Ping pong balls cut in half would be good, or the bottom of a plastic easter egg is probably close enough. Craft stores also have holiday ornament balls that split in two.

Of course, in any of these cases, accurately measuring the internal dimensions will be the hard part, and of course you won't get them to all be equal radius and height like the printed ones. Also, for most of the hemisphere options, you will need to actually measure the base radius and height separately, since they may not be equal (which means results will be a bit off).

If you want to have 3D prints created for you, it will be expensive if you do it commercially. You might check to see if a local makerspace or school might be up for doing it for you.

Where to read more

- Discussion of volume,and formulas: *Volume* (Wikipedia)
- For the big picture on geometry: *Geometry* (Wikipedia)