Linear Algebra MAT313 Fall 2022
Professor Sormani
Lesson 28
Part I Vector Subspace
Part II Null Space
Part III Span
Part IV Basis
Part V Dimension
Part VI Hilbert Space: Infinite Dimensional Space
Part VII Fourier Series: Analog to Digital

Dec 18: Congratulations! You have completed all the lessons for the course! Let me know if you have any questions.

Be sure to complete the review lesson and sample finals before taking the final.

You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:
MAT313F22-lesson28-last-first
and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

Parts I-V are required as they review theorems and definitions we learned before for the final and analyze how they work on vector spaces in general. We have a different playlist for each of the parts. Parts VI and VII are important for math majors, physics majors, and engineers as they concern Hilbert Space and Fourier Series and the conversion of analog to digital sound. Each Part has its own playlist.

Please do the homework like classwork: Immediately when you get to it. It is much easier this way.

Lesson 27

Part I Vector Subspaces

Part II Null Spaces

Part III Spans

Part IV Basis

Part V Dimension

Part VI Hilbert Space

Part VII Fourier Series

Linear Algebra III

Defn: A Vector Space $\mathbb{V}$ is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{w} \in \mathbb{V} \quad \vec{v} + \vec{w} \in \mathbb{V}}$
- Associatity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in \mathbb{V} \quad (\vec{v}+\vec{w})+\vec{u} = \vec{v} + (\vec{u}+\vec{w})$
- Commutativity Property: $\forall \vec{v}, \vec{w} \in \mathbb{V} \quad \vec{v}+\vec{w} = \vec{w}+\vec{v}$
- Additive Identity Property: $\exists \vec{0} \in \mathbb{V}$ such that $\forall \vec{v} \in \mathbb{V} \quad \vec{0}+\vec{v} = \vec{v} = \vec{v}+\vec{0}$
- Additive Inverses Property: $\forall \vec{v} \in \mathbb{V} \quad \exists -\vec{v} \in \mathbb{V}$ s.t $\vec{v}+(-\vec{v}) = (-\vec{v})+\vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in \mathbb{V} \quad t\vec{v} \in \mathbb{V}}$
- Compatibility Property: $\forall s,t \in \mathbb{R} \quad \forall \vec{v} \in \mathbb{V} \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in \mathbb{V} \quad 1\vec{v} = \vec{v}$
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in \mathbb{V} \quad t(\vec{v}+\vec{w}) = t\vec{v} + t\vec{w}$
- Distribution over Scalar Addition Property: $\forall s,t \in \mathbb{R} \quad \forall \vec{v} \in \mathbb{V} \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$

Defn: A Linear Map $F: \mathbb{V} \to \mathbb{W}$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in \mathbb{V} \quad F(v+u) = F(v) + F(u)$
- Preserves Scalar Mult: $\forall v \in \mathbb{V} \ \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

Lesson 27 (not 29)

Part I Vector Subspaces

Part II Null Spaces

Part III Spans

Part IV Basis

Part V Dimension

Part VI Hilbert Space

Part VII Fourier Series

Review of Concepts we learned about vectors in $\mathbb{R}^n$ and see now that they are interesting to apply to vector spaces. (function spaces)

Analog $\leftrightarrow$ Digital
Sound

Watch **Playlist 313F20-27-Part1**

Linear Algebra III

Part I

Vector Subspaces

Defn V is a vector subspace of a vector space W if $V \subset W$ (it a subset of W) and has the same addition and scalar multiplication V is a vector space itself.

Thm: We need only check
- Closed under Vector Addition

  and
- Closed under Scalar Mult.

Proof We must check $\boxed{\text{Add Identity} \in V}$
We must check $\boxed{\text{inverses in } V}$

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{\omega} \in V \quad \vec{v} + \vec{\omega} \in V}$ *
- Associatity Property: $\forall \vec{v}, \vec{\omega}, \vec{u} \in V \quad (\vec{v}+\vec{\omega})+\vec{u} = \vec{v}+(\vec{u}+\vec{\omega})$ ✓
- Commutativity Property: $\forall \vec{v}, \vec{\omega} \in V \quad \vec{v}+\vec{\omega} = \vec{\omega}+\vec{v}$ ✓
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0}+\vec{v} = \vec{v} = \vec{v}+\vec{0}$ ←
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t $\vec{v}+(-\vec{v}) = (-\vec{v})+\vec{v} = \vec{0}$ ←

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$ *
- Compatibility Property: $\forall s,t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$ ✓
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$ ✓
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{\omega} \in V \quad t(\vec{v}+\vec{\omega}) = t\vec{v}+t\vec{\omega}$ ✓
- Distribution over Scalar Addition Property: $\forall s,t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v}+t\vec{v}$ ✓

Defn: A Linear Map $F: V \to W$
- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+u) = F(v)+F(u)$
- Preserves Scalar Mult: $\forall v \in V \ \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

Part I

Vector Subspaces

<u>Defn</u> V is a vector subspace of a vector space W if

$V \subset W$ (it a subset of W)

and has the same addition and scalar multiplication

V is a vector space itself.

Vector Subspace <u>Thm</u>: We need only check

- Closed under Vector Addition

and

- Closed under Scalar Mult.

<u>Proof</u> We must check Add Identity $\in V$

We must check inverses in V

Linear Algebra III

Some Useful Lemmas:

are true on any vector space

<u>Lemma</u>: $\vec{0}$ is unique

There is only one vector $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$ $\forall \vec{v} \in V$.

Easy to see in $\mathbb{R}^2$ $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

in function spaces $O(x) = 0$ constant

Skip the proof in general.

<u>Lemma</u>: $0 \in \mathbb{R}$ and any $\vec{v} \in V$, $0\vec{v} = \vec{0}$.

Easy to see in $\mathbb{R}^2$ $0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0v_1 \\ 0v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$

in function spaces $(0f)(x) = 0 \cdot f(x) = 0$ const

Skip the proof in general.

<u>Lemma</u>: Inverses are Unique

Easy to see in $\mathbb{R}^2$ inv of $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ was $\begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$

in function spaces took $-f(x)$.

<u>Lemma</u>: $-1 \in \mathbb{R}$ and $\vec{v} \in V$ then $-1\vec{v} = -\vec{v}$ the inverse of $\vec{v}$.

Easy to see in $\mathbb{R}^2$ and function spaces.

V is a vector space itself.

Vector Subspace

Thm: If V nonempty subset of W which is

- Closed under Vector Addition

and

- Closed under Scalar Mult of W,

then V is a vector subspace of W.

Proof We must check $\boxed{\text{Add Identity} \in V}$

We must check $\boxed{\text{inverses in } V}$

Show Add Identity is in V

① Take $0 \in \mathbb{R}$ and any $\vec{v} \in V$ ① Lemma →

$0\vec{v} = \vec{0}$

② $\vec{0} = 0\vec{v} \in V$ ② Given Closed under Scalar Mult

Thus $\vec{0} \in V$

& Show inverses are in V

$\boxed{\text{HW1}}$ Use our given properties and one of our lemmas to prove inverses are in V (2 lines)

QED

Some Useful Lemmas:

are true on any vector space

Lemma: $\vec{0}$ is unique

There is only one vector $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$ $\forall \vec{v} \in V$.

Easy to see in $\mathbb{R}^2$ $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

in function spaces $0(x) = 0$ constant

Skip the proof in general.

* **Lemma:** $0 \in \mathbb{R}$ and any $\vec{v} \in V$, $0\vec{v} = \vec{0}$.

Easy to see in $\mathbb{R}^2$ $0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0v_1 \\ 0v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$

in function spaces $(0f)(x) = 0 \cdot f(x) = 0$ const

Skip the proof in general.

Lemma: Inverses are Unique

Easy to see in $\mathbb{R}^2$ inv of $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ was $\begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$

in function spaces took $-f(x)$.

Lemma: $-1 \in \mathbb{R}$ and $\vec{v} \in V$ then $-1\vec{v} = -\vec{v}$ the inverse of $\vec{v}$.

Easy to see in $\mathbb{R}^2$ and function spaces.

Writing again more neatly:

Vector Subspace Thm:

If V is a nonempty subset of W which is

- *Closed Under Addition
- *Closed Under Scalar Mult.

then V is a vector subspace of W

Proof: Must check all properties of a vector space hold.

- * Closed Under vector addition (given)
- * Closed Under scalar mult (given)

Check off 6 properties using that they are true for vectors in W and all vectors in V are in W ✓✓✓✓✓✓

→ Additive Id Property (proved using a lemma)

→ Additive Inverses Prop (HW also proven with a lemma)

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{w} \in V \quad \vec{v}+\vec{w} \in V}$ *
- Associatity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v}+\vec{w})+\vec{u} = \vec{v}+(\vec{u}+\vec{w})$ ✓
- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v}+\vec{w} = \vec{w}+\vec{v}$ ✓
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0}+\vec{v} = \vec{v} = \vec{v}+\vec{0}$ ←
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t $\vec{v}+(-\vec{v}) = (-\vec{v})+\vec{v} = \vec{0}$ ←

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \ \vec{v} \in V \quad t\vec{v} \in V}$ *
- Compatibility Property: $\forall s,t \in \mathbb{R} \ \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$ ✓
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$ ✓
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \ \forall \vec{v}, \vec{w} \in V \quad t(\vec{v}+\vec{w}) = t\vec{v}+t\vec{w}$ ✓
- Distribution over Scalar Addition Property: $\forall s,t \in \mathbb{R} \ \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v}+t\vec{v}$ ✓

QED

Vector Subspace Thm:

If V is a nonempty subset of W which is

* Closed Under Addition
* Closed Under Scalar Mult.

then V is a vector subspace of W

Proof: Must check all properties of a vector space hold.

* Closed Under vector addition (given)
* Closed Under scalar mult (given)

Check off 6 properties using that they are true for vectors in W and all vectors in V are in W //////

→ Additive Id Property (proved using a Lemma)

→ Additive Inverses Prop (HW1 also proven with a lemma)

QED

$\boxed{HW2}$ $\left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \middle| \ 2v_1 - 4v_2 = 0 \right\}$ is a vector subspace of $\mathbb{R}^3$

$\boxed{HW3}$ P_3 is a vector subspace $C([0,1])$

2:55 PM Sat Dec 5

Linear A… | …gebra III

SCREEN RECORDING now

Screen Recording video saved to Photos

$\boxed{HW2}$ $\left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \middle| \; 2v_1 - 4v_2 = 0 \right\} = V$

is a vector subspace of $\mathbb{R}^3$

- Closed Under Vector Addition

① Given $\vec{v}, \vec{w} \in V$ — ① defn of V

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ with $2v_1 - 4v_2 = 0$

$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ with $2w_1 - 4w_2 = 0$

② $\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$ — ② defn of $+$ on V

③ $2(v_1 + w_1) - 4(v_2 + w_2) =$ — ③ by algebra and step 1

$= 2v_1 + 2w_1 - 4v_2 - 4w_2$

$= \underbrace{2v_1 - 4v_2}_{} + \underbrace{2w_1 - 4w_2}_{}$

$= \quad 0 \quad + \quad 0 \quad = 0$

④ $\vec{v} + \vec{w} \in V$ — ④ by defn of V

Also check closed under scalar

$\boxed{HW3}$ P_3 is a vector subspace $C([0,1])$

- $P_3 \subset C([0,1])$ because polynomials are Continuous.

- Closed under addition

① Given $p, q \in P_3$ — ① by defn of P_3

$p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

$q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3$

② $p(x) + q(x) = (p_0 + q_0) +$ — ② by add in P_3

$+ (p_1 + q_1)x + (p_2 + q_2)x^2$

$+ (p_3 + q_3)x^3$

③ $p(x) + q(x) \in P_3$ — ③ defn of P_3

- Closed under scalar mult

$\boxed{\text{You must do}}$

for HW3.

Watch [Playlist 313F20-27-Part2]

Part II Null Spaces

Defn: Given a linear map $F: V \to W$ we have

$$Null(F) = \{\vec{v} \in V \mid F(\vec{v}) = \vec{0}\}$$

Recall a linear Map preserves vector addition and scalar multiplication.

HW4 Find $Null(F)$ where $F: \mathbb{R}^3 \to \mathbb{R}^2$ $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_3 \end{pmatrix}$

HW5 Find $Null(F)$ where $F: P_3 \to P_2$ $F(p) = p'(x)$.

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V}$
- Associativity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{u} + \vec{w})$
- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V$ $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \ \vec{v} \in V \quad t\vec{v} \in V}$.
- Compatibility Property: $\forall s, t \in \mathbb{R} \ \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \ \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
- Distribution over Scalar Addition Property: $\forall s, t \in \mathbb{R} \ \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$

Defn: A Linear Map $F: V \to W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+u) = F(v) + F(u)$
- Preserves Scalar Mult: $\forall v \in V \ \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

Part II Null Spaces

Defn: Given a linear map $F: \mathbf{V} \to W$ we have

$$Null(F) = \{\vec{v} \in \mathbf{V} \mid F(\vec{v}) = \vec{0}\}$$

Recall a linear Map preserves vector addition and scalar multiplication.

HW4 Find $Null(F)$ where $F: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_3 \end{pmatrix}$

HW5 Find $Null(F)$ where $F: P_3 \to P_2$ $\quad F(p) = p'(x)$.

HW6 Find $Null(F)$ where $F: P_2 \to P_2$ $\quad F(p) = 5p'(x) + 4p(x)$

Linear Algebra III

HW4 $F: \mathbb{R}^3 \to \mathbb{R}^2$

$$\left\{\vec{v} \in \mathbb{R}^3 \;\middle|\; F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} =$$

$$= \left\{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} v_1 - v_2 \\ v_1 + v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$

solving the homogeneous system

$$\boxed{\begin{aligned} v_1 - v_2 &= 0 \\ v_1 + v_3 &= 0 \end{aligned}} \qquad \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_1 + v_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$x_1 + x_3 = 0 \quad x_1 = -x_3$

$x_2 + x_3 = 0 \quad x_2 = -x_3$

$x_3 = x_3$ (free)

$$F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-1 \\ 0+0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R}\right\} = Null(F).$$

Linear Algebra III

$\boxed{\text{HW6}}$ Find Null(F) where

$F: P_2 \to P_2 \quad F(p) = 5p'(x) + 4p(x)$

$$\text{Null}(F) = \{p_2x^2+p_1x+p_0 \mid F(p)=0\}$$

$$= \{p_2x^2+p_1x+p_0 \mid 5(2p_2x+p_1) + 4(p_2x^2+p_1x+p_0) = \underset{\text{const}}{0}\}$$

$$= \{p_2x^2+p_1x+p_0 \mid 4p_2x^2 + (10p_2+4p_1)x + (5p_1+4p_0) = \underset{\text{const}}{0}\}$$

$4p_2=0 \quad 10p_2+4p_1=0 \quad 5p_1+4p_0=0$

Solve this linear system

$$\begin{array}{ccc} p_2 & p_1 & p_0 \end{array}$$
$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right) \longrightarrow \text{row reduction}$$

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right) \xrightarrow{\rho_1 \to \frac{1}{4}\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{\rho_2 \to \rho_2 - 10\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right) \xrightarrow{\rho_2 \to \frac{1}{4}\rho_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right) \xrightarrow{\rho_3 \to \rho_3 - 5\rho_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{\rho_3 \to \frac{1}{4}\rho_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad \begin{array}{l} p_2 = 0 \\ p_1 = 0 \\ p_0 = 0 \end{array}$$

$$\text{Null}(F) = \{0x^2+0x+0\}$$

Null space is trivial

"Kernal is trivial"

If $F: V \to W$ has $\text{Null}(F) = 0$ then F is one to one.

4:20 PM Sat Dec 5

Linear A... | SCREEN RECORDING now Screen Recording video saved to Photos | ...gebra III

Thm: If $F: V \to W$ is a linear map between vector spaces V and W and if $Null(F) = \{\vec{0}\}$ then F is one-to-one.

Proof:

① $F(\vec{v}) = F(\vec{w})$ then $F(\vec{v}) - F(\vec{w}) = 0$ — ① W is a vector space

② $F(\vec{v} - \vec{w}) = 0$ — ② F is a linear map lemma

③ $\vec{v} - \vec{w} \in Null(F)$ — ③ Defn $Null(F)$

④ $\vec{v} - \vec{w} = \vec{0}$ — ④ Given $Null(F) = \{\vec{0}\}$

⑤ $\vec{v} = \vec{w}$ — ⑤ adding $\vec{w}$ for V is a vector space

Thus F is one to one

QED

Defn F is one to one means $F(\vec{v}) = F(\vec{w}) \Rightarrow \vec{v} = \vec{w}$

Lemma If $F: V \to W$ is a linear map then $F(\vec{a} - \vec{b}) = F(\vec{a}) - F(\vec{b})$

Pf:

① $F(\vec{a} - \vec{b}) = F(\vec{a} + (-\vec{b}))$ — ① defn of $-$

② $= F(\vec{a}) + F(-\vec{b})$ — ② F pres add

③ $= F(\vec{a}) - F(\vec{b})$ — ③ F pres scalar mult

QED

$5p'(x) + 4p(x) = q(x)$ has at most one solution because Null space is trivial in HW6

4:48 PM Sat Dec 5

Linear Algebra III

$\boxed{\text{HW7}}$ Extra Credit

Prove:

Theorem: If $F: V \to W$ is a linear map between vector spaces V and W then $\text{Null}(F)$ is a vector subspace of V.

Hint: you only need to prove

- Closed under addition
- Closed under scalar mult

Linear Algebra III

$\boxed{\text{HW8}}$ Let $F: C^3([0,1]) \to C([0,1])$ be defined $F(h(x)) = h'''(x)$

Prove $\text{Null}(F) = P_2$

(this uses calculus)

Hint: $C^3([0,1])$ = functions whose third derivatives are continuous on $[0,1]$.

$C([0,1])$ = functions that are continuous on $[0,1]$

$P_2 = \{p_2x^2 + p_1x + p_0 \mid p_i \in \mathbb{R}\}$ polynomials of degree ≤ 2.

Facts from Calc:

If $f'(x) = 0$ function then $f(x) = k$

If $f'(x) = k$ then $f(x) = kx + C$

If $f''(x) = ax + b$ then $f(x) = \frac{a}{2}x^2 + bx + C$

Watch Playlist 313F20-27-Part3

Part III Spans

Defn: Given $\vec{v}_1 \dots \vec{v}_k \in V$

the span $\langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle =$

$$= \{ t_1\vec{v}_1 + \dots + t_k\vec{v}_k \mid t_i \in \mathbb{R} \}$$

$$= \left\{ \sum_{i=1}^{k} t_i \vec{v}_i \mid t_i \in \mathbb{R} \right\}$$

$\boxed{HW9}$ Is $\begin{pmatrix}1\\2\\3\end{pmatrix} \in \left\langle \begin{pmatrix}2\\0\\2\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$?

$\boxed{HW10}$ Is $f \in \langle g, h \rangle$ where

$f(x) = x^2 \quad g(x) = (x+2)^2 \quad h(x) = x+1$

is the vector space $C([0,1])$?

$\boxed{HW9}$ $\begin{pmatrix}1\\2\\3\end{pmatrix} = t_1\begin{pmatrix}2\\0\\2\end{pmatrix} + t_2\begin{pmatrix}1\\1\\1\end{pmatrix}$?

Solve this system?

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{array}\right) \xrightarrow{\rho_3 \to \rho_3 - \rho_1} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right)$$

row of zeroes ending in 2

so there is no solution

thus $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is not in the span.

If there was a solution then you say it is in the span.

Is $\begin{pmatrix}5\\1\\5\end{pmatrix} \in \left\langle \begin{pmatrix}2\\0\\2\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$?

$$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$ has a solution!

Yes $\begin{pmatrix}5\\1\\5\end{pmatrix}$ is in the span.

Part III Spans

Defn: Given $\vec{v}_1 \dots \vec{v}_k \in V$

the span $\langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle =$

$$= \{ t_1 \vec{v}_1 + \dots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \}$$

$$= \left\{ \sum_{i=1}^{k} t_i \vec{v}_i \mid t_i \in \mathbb{R} \right\}$$

$\boxed{\text{HW9}}$ Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \left\langle \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$?

$\boxed{\text{HW10}}$ Is $f \in \langle g, h \rangle$ where

$f(x) = x^2 \quad g(x) = (x+2)^2 \quad h(x) = x+1$

is the vector space $C([0,1])$?

$\boxed{\text{HW10}}$ Can we find $t_1, t_2 \in \mathbb{R}$

s.t $f = t_1 g + t_2 h$

$$x^2 = t_1 (x+2)^2 + t_2 (x+1)$$

$$x^2 = t_1 (x^2 + 4x + 4) + t_2 x + t_2$$

$$x^2 = t_1 x^2 + 4t_1 x + 4t_1 + t_2 x + t_2$$

$$x^2 + 0x + 0 = t_1 x^2 + (4t_1 + t_2) x + (4t_1 + t_2)$$

Solve $1 = t_1 \quad 0 = 4t_1 + t_2 \quad 0 = 4t_1 + t_2$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

has a solution

Yes $f \in \langle g, h \rangle$.

5:20 PM Sat Dec 5 Linear Algebra III

Part III Spans

Defn: Given $\vec{v}_1 \dots \vec{v}_k \in V$ the span $\langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle =$

$$= \{ t_1\vec{v}_1 + \dots + t_k\vec{v}_k \mid t_i \in \mathbb{R} \}$$

$$= \{ \sum_{i=1}^{k} t_i \vec{v}_i \mid t_i \in \mathbb{R} \}$$

Thm: If $\vec{v}_1 \dots \vec{v}_k \in W$ and $V = \langle \vec{v}_1, \vec{v}_2, \dots \vec{v}_k \rangle$ then V is a vector subspace of W.

$\boxed{\text{HW11}}$ Prove this

Linear Algebra III

Defn: The Image of a map $F: V \to W$ is $F(V) = \{ F(\vec{v}) \mid \vec{v} \in V \}$

Thm: If $V = \langle \vec{v}_1, \dots, \vec{v}_k \rangle$ then $F(V) = \langle F(\vec{v}_1), \dots, F(\vec{v}_k) \rangle$

$\boxed{\text{HW12}}$ Prove this.

Defn: $F: V \to W$ is onto $\Leftrightarrow \forall \vec{w} \in W \; \exists \vec{v} \in V$ s.t $F(\vec{v}) = \vec{w}$

Thm: Onto $\Leftrightarrow F(V) = W$

Thm: Onto $\Leftrightarrow \exists \vec{v}_1 \dots \vec{v}_k \in V$ such that $W = \langle F(\vec{v}_1) \dots F(\vec{v}_k) \rangle$

$\boxed{\text{HW13}}$ Prove this

Part III Spans

Defn: Given $\vec{v}_1 \dots \vec{v}_k \in V$ the **span** $\langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle =$

$$= \{ t_1\vec{v}_1 + \dots + t_k\vec{v}_k \mid t_i \in \mathbb{R} \}$$

$$= \left\{ \sum_{i=1}^{k} t_i \vec{v}_i \;\middle|\; t_i \in \mathbb{R} \right\}$$

Thm: If $\vec{v}_1 \dots \vec{v}_k \in W$ and $V = \langle \vec{v}_1, \vec{v}_2, \dots \vec{v}_k \rangle$ then V is a vector subspace of W.

$\boxed{\text{HW11}}$ Prove this

Defn: The **Image** of a map $F: V \to W$ is $F(V) = \{ F(\vec{v}) \mid \vec{v} \in V \}$

Thm: If $V = \langle \vec{v}_1, \dots, \vec{v}_k \rangle$ then $F(V) = \langle F(\vec{v}_1), \dots, F(\vec{v}_k) \rangle$

$\boxed{\text{HW12}}$ Prove this.

Defn: $F: V \to W$ is **onto** $\iff \forall \vec{w} \in W \; \exists \vec{v} \in V$ s.t. $F(\vec{v}) = \vec{w}$

Thm: Onto $\iff F(V) = W$

Thm: Onto $\iff \exists \vec{v}_1 \dots \vec{v}_k \in V$ such that $W = \langle F(\vec{v}_1) \dots F(\vec{v}_k) \rangle$

$\boxed{\text{HW13}}$ Prove this

HW13 is a very short proof combining the two theorems above it with the definition of onto. You may skip it and I have also removed HW14.

Below we show the solutions to some of the more important homework HW11 and HW12 and HW15.

Linear Algebra III

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{w} \in V \quad \vec{v}+\vec{w} \in V}$
- Associativity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v}+\vec{w})+\vec{u} = \vec{v}+(\vec{u}+\vec{w})$
- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v}+\vec{w} = \vec{w}+\vec{v}$
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0}+\vec{v} = \vec{v} = \vec{v}+\vec{0}$
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v}+(-\vec{v}) = (-\vec{v})+\vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$.
- Compatibility Property: $\forall s,t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v}+\vec{w}) = t\vec{v}+t\vec{w}$
- Distribution over Scalar Addition Property: $\forall s,t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v}+t\vec{v}$

Defn: A Linear Map $F: V \to W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+u) = F(v)+F(u)$
- Preserves Scalar Mult: $\forall v \in V \quad \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

Linear Algebra III

HW 11

closed under addition (you do)

Closed under scalar

Given $\vec{v} \in V = \langle \vec{v}_1 \dots \vec{v}_k \rangle$

$t \in \mathbb{R}$

Show $t\vec{v} \in \langle \vec{v}_1 \dots \vec{v}_k \rangle$

(1) $\vec{v} \in \langle \vec{v}_1 \dots \vec{v}_k \rangle$ — (1) by defn of span

$\vec{v} = t_1\vec{v}_1 + \dots + t_k\vec{v}_k$

(2) $t\vec{v} = t(t_1\vec{v}_1 + \dots + t_k\vec{v}_k)$ — (2) by defn scalar mult

(3) $= t(t_1\vec{v}_1) + \dots + t(t_k\vec{v}_k)$ — (3) Distri

(4) $= (tt_1)\vec{v}_1 + \dots + (tt_k)\vec{v}_k$ — (4) Compat.

(5) $\in \langle v_1 \dots v_k \rangle$ — (5) $tt_i \in \mathbb{R}$ defn of span.

You do closed under addition.

carefully name the properties

5:32 PM Sat Dec 5

Linear Algebra III

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{\omega} \in V \quad \vec{v} + \vec{\omega} \in V}$
- Associativity Property: $\forall \vec{v}, \vec{\omega}, \vec{u} \in V \quad (\vec{v}+\vec{\omega})+\vec{u} = \vec{v} + (\vec{u}+\vec{\omega})$
- Commutativity Property: $\forall \vec{v}, \vec{\omega} \in V \quad \vec{v}+\vec{\omega} = \vec{\omega}+\vec{v}$
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0}+\vec{v} = \vec{v} = \vec{v}+\vec{0}$
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t $\vec{v}+(-\vec{v}) = (-\vec{v})+\vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$.
- Compatibility Property: $\forall s,t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{\omega} \in V \quad t(\vec{v}+\vec{\omega}) = t\vec{v}+t\vec{\omega}$
- Distribution over Scalar Addition Property: $\forall s,t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v}+t\vec{v}$

Defn: A Linear Map $F: V \to W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+u) = F(v)+F(u)$
- Preserves Scalar Mult: $\forall v \in V \quad \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

5:32 PM 18%

Linear Algebra III

$\boxed{HW12}$ $\boxed{V = \langle v_1, \dots, v_k \rangle}$

Show $F(V) = \langle F(v_1) \dots F(v_k) \rangle$

Proof:

① $F(V) = \{F(\vec{v}) \mid \vec{v} \in V\}$

① by defn of Image

② $= \{F(t_1\vec{v}_1 + \dots + t_k\vec{v}_k) \mid t_i \in \mathbb{R}\}$

② defn of span $V = \langle v_1 \dots v_k \rangle$

③ $= \{F(t_1 v_1) + \dots + F(t_k \vec{v}_k) \mid t_i \in \mathbb{R}\}$

③ F pres addition

④ $= \{t_1 F(v_1) + \dots + t_k F(v_k) \mid t_i \in \mathbb{R}\}$

④ F pres scalar

⑤ $= \langle F(v_1), \dots, F(v_k) \rangle$

⑤ by defn of span

QED

5:45 PM Sat Dec 5

Linear Algebra III

$\boxed{\text{HW15}}$ Are $(x+1)^2$, $5x$, 8 a basis for P_2?

$\boxed{\bullet \text{ Do they span } P_2?}$

Given any $p \in P_2$ $p = p_2x^2 + p_1x + p_0$

Find $t_1, t_2, t_3 \in \mathbb{R}$ s.t.

$$p_2x^2 + p_1x + p_0 = t_1(x+1)^2 + t_2(5x) + t_3 8$$
$$= t_1(x^2+2x+1) + t_2 5x + t_3 8$$
$$= t_1x^2 + (t_1 2 + t_2 5)x + (t_1 + t_3 8)$$

$p_2 = t_1 \quad p_1 = t_1 2 + t_2 5 \quad p_0 = t_1 + t_3 8$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & p_2 \\ 2 & 5 & 0 & p_1 \\ 1 & 0 & 8 & p_0 \end{array}\right)$$

Solve to t_1, t_2, t_3

Solve this system + show it has a solution.

Linear Algebra III

Are they lin indep?

$$t_1(x+1)^2 + t_2(5x) + t_3(8) = \underset{\text{function}}{0}$$

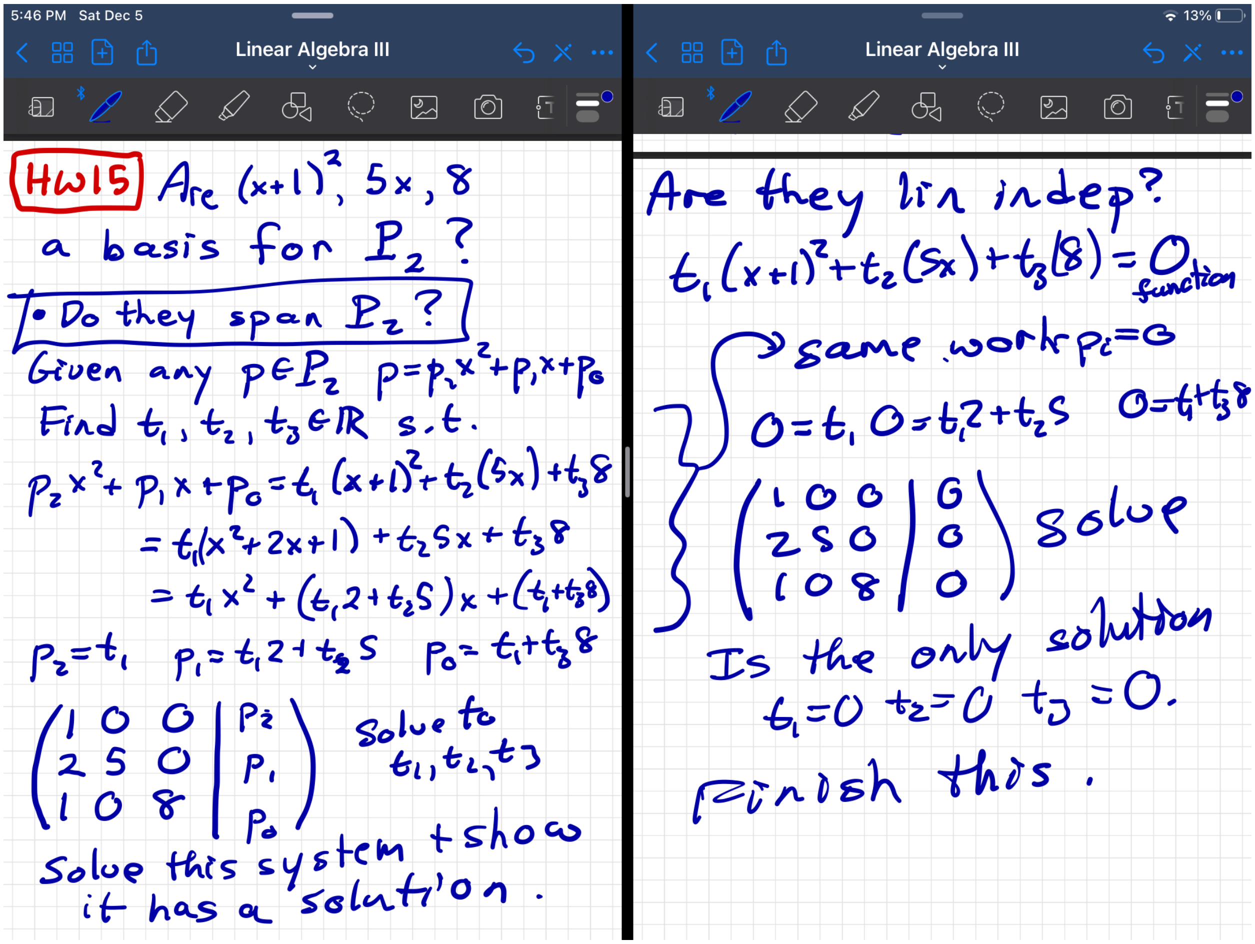
same work $p_i = 0$

$0 = t_1 \quad 0 = t_1 2 + t_2 5 \quad 0 = t_1 + t_3 8$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & 8 & 0 \end{array}\right) \text{ solve}$$

Is the only solution $t_1 = 0$ $t_2 = 0$ $t_3 = 0$.

Finish this.



$\boxed{\text{HW15}}$ Are $(x+1)^2, 5x, 8$ a basis for P_2?

• Do they span P_2?

Given any $p \in P_2$ $\quad p = p_2x^2+p_1x+p_0$

Find $t_1, t_2, t_3 \in \mathbb{R}$ s.t.

$$p_2x^2+p_1x+p_0 = t_1(x+1)^2+t_2(5x)+t_3 8$$
$$= t_1(x^2+2x+1)+t_2 5x+t_3 8$$
$$= t_1x^2+(t_1 2+t_2 5)x+(t_1+t_3 8)$$

$p_2=t_1 \quad p_1=t_1 2+t_2 5 \quad p_0=t_1+t_3 8$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & p_2 \\ 2 & 5 & 0 & p_1 \\ 1 & 0 & 8 & p_0 \end{array}\right)$$ Solve for t_1, t_2, t_3

Solve this system + show it has a solution.

Are they lin indep?

$t_1(x+1)^2+t_2(5x)+t_3(8) = 0_{\text{function}}$

same work $p_i=0$

$0=t_1 \quad 0=t_1 2+t_2 5 \quad 0=t_1+t_3 8$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & 8 & 0 \end{array}\right)$$ solve

Is the only solution $t_1=0$ $t_2=0$ $t_3=0$.

Finish this.

Part IV Basis

Watch [Playlist 313F20-27-Part4]

Part IV Basis

Defn: $\vec{v}_1 \dots \vec{v}_k$ are a basis for a vector space V if $V = \langle \vec{v}_1, \vec{v}_2, \dots \vec{v}_k \rangle$ and $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ are linearly independent.

Defn: $\vec{v}_1 \dots \vec{v}_k$ are linearly independent if

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k = \vec{0}$$
$$\Downarrow$$
$$t_1 = t_2 = \dots = t_k = 0.$$

Standard Basis

for $\mathbb{R}^2$ $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$ easy check lin indep

$$\mathbb{R}^2 = \left\{ t_1\begin{pmatrix}1\\0\end{pmatrix} + t_2\begin{pmatrix}0\\1\end{pmatrix} \,\middle|\, t_1, t_2 \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix}t_1\\t_2\end{pmatrix} \,\middle|\, t_1, t_2 \in \mathbb{R} \right\}$$

for $\mathbb{R}^3$ $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\boxed{\text{for } P_2 \quad x^2 \quad x \quad 1}$

$$P_2 = \{ t_2x^2 + t_1x + t_0 1 \mid t_i \in \mathbb{R} \}$$
$$= \langle x^2, x, 1 \rangle$$

easy to check lin indep.

5:55 PM Sat Dec 5

Linear Algebra III

If our functions how to check lin. indep?

$\boxed{\cos(x), \sin(x)}$

are thes lin indep?

$t_1\cos(x) + t_2\sin(x) = 0$ function

so this sum is 0 for any x we plug in.

plug in $x=0$

$t_1\cos(0) + t_2\sin(0) = 0$

$t_1 1 + t_2 \cdot 0 = 0$

$\boxed{t_1 = 0}$

plug in $x = \frac{\pi}{2}$

$t_1\cos\left(\frac{\pi}{2}\right) + t_2\sin\left(\frac{\pi}{2}\right) = 0$

$t_1 \cdot 0 + t_2(1) = 0$

$\boxed{t_2 = 0}$

Yes $\cos(x)$ and $\sin(x)$ are lin indep.

Linear Algebra III

HW16 Are

$\sin(x), \sin(2x), \sin(4x)$

linearly independent?

Try this using the same method plugging in values of x where different functions are 0.

Try $x = \frac{\pi}{2}$ $x = \frac{\pi}{4}$ and another x.

6:53 PM Sat Dec 5

Linear Algebra III

Thm If $\vec{v}_1 \dots \vec{v}_k$ form a basis for V then every $\vec{v} \in V$ has exactly one set $t_1, \dots t_k \in \mathbb{R}$ such that $\vec{v} = t_1\vec{v}_1 + \dots + t_k\vec{v}_k$.

(that is, there exists a unique solution)

Usually Difficult to find the t_i

Linear Algebra III

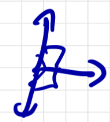
65%

Recall: Orthonormal Basis

$\vec{v}_1 \dots \vec{v}_k \in \mathbb{R}^n$

s.t $\vec{v}_i \cdot \vec{v}_i = 1$

$\vec{v}_i \cdot \vec{v}_j = 0$ if $i \neq j$



Easy to find t_i for any $\vec{v}$

$\boxed{t_i = \vec{v} \cdot \vec{v}_i}$

Check $\vec{v} = t_1\vec{v}_1 + \dots + t_k\vec{v}_k$

HW21 Check this works.

We will do HW17-20 in Part 5.

Part V Dimension

Watch [Playlist 313F20-27-Part5]

V Dimension

Thm: If $\vec{v}_1 \ldots \vec{v}_k$ are a basis for V and also $\vec{w}_1, \ldots \vec{w}_m$ are a basis for V then $m = k$

Defn: ↑ Dimension of V

$\boxed{HW17}$ Prove the theorem
Extra Credit: hint find our old proof in $\mathbb{R}^n$ and imitate.

$\boxed{HW18}$ Find the dimension of $P_3 = \{p_3x^3 + p_2x^2 + p_1x + p_0 \mid p_i \in \mathbb{R}\}$

Defn: V is infinite dimensional if it has no finite basis.

Thm: If there is a sequence $\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots \in V$ such that for every $k \in \mathbb{N}$ $v_1, v_2 \ldots v_k$ are lin. indep then V is infinite dim'l.

$\boxed{HW19}$ Prove this theorem

$\boxed{HW20}$ Use this theorem to show $C([0,1])$ is infinite dim'l.

6:40 PM Sat Dec 5

Linear Algebra III

$\boxed{HW19}$

Proof by Contradiction to show V has no finite basis

① Assume on the contrary it does have a finite basis $w_1, \dots, w_m$

① Indirect Hypothesis Unjustified.

② $V = \langle w_1, \dots, w_m \rangle$ and $w_1 \dots w_m$ are lin indep

② defn of basis.

③ Take $k = m+1$
$v_1\ v_2 \dots v_{m+1}$ are linearly indep.

③ by given info for our thm.

④ Each $v_i \in V = \langle w_1, \dots, w_m \rangle$
So $\exists\, t_{i1}\ t_{i2} \dots t_{im} \in \mathbb{R}$ s.t.

$$\boxed{v_i = t_{i1} w_1 + t_{i2} w_2 + \dots + t_{im} w_m}$$

④ by defn of span

⑤ $s_1 v_1 + s_2 v_2 + \dots + s_{m+1} v_{m+1} = 0$

$\Rightarrow s_1 = s_2 = \dots = s_{m+1} = 0$

⑤ by defn of lin indep

⑥ $s_1 (t_{11} w_1 + t_{12} w_2 + \dots + t_{1m} w_m)$
$+ s_2 (t_{21} w_1 + t_{22} w_2 + \dots + t_{2m} w_m)$
$+ \dots$
$+ s_m (t_{m1} w_1 + \dots + t_{mm} w_m)$
$+ s_{m+1} (t_{m+1\,1} w_1 + \dots + t_{m+1\,m} w_m) = 0$

$\Rightarrow s_1 = s_2 = \dots = s_{m+1} = 0$

⑥ Sub step 4 into step 5

6:40 PM Sat Dec 5

Linear Algebra III

⑦ $(s_1 t_{11} + s_2 t_{21} + \ldots + s_m t_{m1} + s_{m+1} t_{m+1\,1})\, w_1$

$+ (s_1 t_{12} + \ldots + s_{m+1} t_{m+1\,2})\, w_2$

$+ \ldots + (s_1 t_{1m} + \ldots + s_{m+1} t_{m+1,m})\, w_m$

$= 0$

$\Rightarrow \boxed{s_1 = s_2 = \ldots = 0.}$

⑧ Since w_1 to w_m span V and are lin indep.

So if $= 0$ then

$$s_1 t_{11} + s_2 t_{21} + \ldots + s_{m+1} t_{m+1,1} = 0$$
$$\ldots$$
$$s_1 t_{1m} + \ldots + s_{m+1} t_{m+1,m} = 0$$

$$\begin{pmatrix} t_{11} & t_{21} & \cdots & t_{m+1,1} \\ & \cdots & & \\ t_{1m} & t_{2m} & \cdots & t_{m+1\,m} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$m+1$ columns and m rows

So there is at least one free variable.

$$\left\{ \begin{pmatrix} s_1 \\ \vdots \\ s_{m+1} \end{pmatrix} = \text{something not just } 0 \right\}$$

Then we do **not** get

$$\boxed{s_1 = s_2 = \ldots = s_{m+1} = 0}$$

which means $v_1, v_2 \ldots v_{m+1}$ are **not** linearly independent.

⊗ In thm they are lin. indep

So Indirect Hyp. is false in step 1

So V does **not** have a finite basis **QED**

Thm If $\vec{v}_1 \ldots \vec{v}_k$ form a basis for V then every $\vec{v} \in V$ has exactly one set $t_1, \ldots t_k \in \mathbb{R}$ such that $\vec{v} = t_1\vec{v}_1 + \ldots + t_k\vec{v}_k$.

(that is, there exists a unique solution)

Proven using properties of vector spaces.

Proof:

(1) $\vec{v}_1 \ldots \vec{v}_k$ are a basis for V	(1) given
(2) $V = \langle \vec{v}_1 \ldots \vec{v}_k \rangle$	(2) by defn of basis
(3) $\vec{v} = t_1\vec{v}_1 + \ldots + t_k\vec{v}_k$ has a solution $t_1 \ldots t_k$	(3) by defn of span
(4) Is there only one solution? Check $\vec{v} = s_1\vec{v}_1 + \ldots + s_k\vec{v}_k$ $t_1\vec{v}_1 + \ldots + t_k\vec{v}_k = s_1\vec{v}_1 + \ldots + s_k\vec{v}_k$	(4) step 3
(5) $t_1v_1 - s_1v_1 + \ldots + t_kv_k - s_kv_k = \vec{0}$	(5) by add inverses
(6) $(t_1 - s_1)v_1 + \ldots + (t_k - s_k)v_k = \vec{0}$	(6) by dist prop
(7) $t_1 - s_1 = t_2 - s_2 = \ldots = t_k - s_k = 0$	(7) v_k are lin indep
(8) $t_1 = s_1 \quad t_2 = s_2 \ldots t_k = s_k$	(9) add s_i

QED

Part VI Hilbert Space

Watch [Playlist 313F20-27-Part6]

Part VI

Hilbert Space

$$\ell_2 = \{ (\underbrace{v_1, v_2, v_3, \ldots}_{\text{infinite sequence of real numbers}}) \mid \sum_{i=1}^{\infty} v_i^2 < \infty \}$$

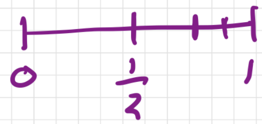
So $(1,1,1,1,1\ldots) \notin \ell_2$

$1^2+1^2+1^2+1^2+\ldots = \infty$

But $(1,1,1,1,0,0,0\ldots) \in \ell_2$

$1^2+1^2+1^2+1^2+0^2+0^2+\ldots$

$= 4 < \infty$



$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots \leq 1$

$v_1 = \sqrt{\frac{1}{2}}$ $v_2 = \sqrt{\frac{1}{4}}$ $v_3 = \sqrt{\frac{1}{8}}$ $\ldots$

this is in ℓ_2

8:22 PM Sat Dec 5

Linear Algebra III

Part VI

Hilbert Space

$$\ell_2 = \{ \underbrace{(v_1, v_2, v_3, \dots)}_{\text{infinite sequence of real numbers}} \mid \sum_{i=1}^{\infty} v_i^2 < \infty \}$$

So $(1,1,1,1,1\dots) \notin \ell_2$

$1^2+1^2+1^2+1^2+\dots = \infty$

But $(1,1,1,1,0,0,0\dots) \in \ell_2$

$1^2+1^2+1^2+1^2+0^2+0^2+\dots$

$= 4 < \infty$

Linear Algebra III

Hilbert Space is a Vector Space

addition:

$(v_1, v_2, \dots) + (w_1, w_2, \dots)$

$= \underbrace{(v_1+w_1, v_2+w_2, \dots)}$

• Check is in ℓ_2 to see we are closed under addition

Show: $(v_1+w_1)^2 + (v_2+w_2)^2 + \dots$ is finite

Use: $v_1^2+v_2^2+\dots$ ~~is~~ finite

$w_1^2+w_2^2+\dots$ is finite.

8:31 PM Sat Dec 5

Linear Algebra III

Hilbert Space is a Vector Space

ADDITION

$(v_1, v_2, \ldots) + (w_1, w_2, \ldots)$

$= (\underbrace{v_1+w_1, v_2+w_2, \ldots}_{\text{must check this is in } \ell_2})$

- Closed under Addition

Show: $(v_1+w_1)^2 + (v_2+w_2)^2 + \ldots$ is finite

Use: $v_1^2 + v_2^2 + \ldots$ is finite

$w_1^2 + w_2^2 + \ldots$ is finite

difficult

- Additive Identity $(0, 0, 0, \ldots)$ (easy)
- Additive Inverse $(-v_1, -v_2, -v_3 \ldots) \in \ell_2$ (easy)
- Associative (easy)
- Commutative (easy)

SCALAR MULTIPLICATION

$\underbrace{t}_{\text{in } \mathbb{R}}\,\underbrace{(v_1, v_2, v_3 \ldots)}_{\text{in } \ell_2} = \underbrace{(tv_1, tv_2, tv_3 \ldots)}_{\text{show in } \ell_2}$

- Closed under scalar mult (difficult)
- Compatibility (easy) $(st)v = s(tv)$
- Scalar Identity $1v = v$ (easy)
- Distribution $t(v+w) = tv + tw$ (easy)
- Distribution $(s+t)v = sv + tv$ (easy)

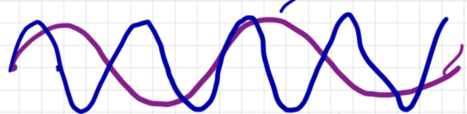
Extra Credit for completing the proof.

Part VII Fourier Series Analog to Digital

Watch Playlist 313F20-27-Part7 which has one video made by me and then a few others with professional sound and graphics.

VII Fourier Series

Consider $\{\underbrace{\sin(j\pi x)}\ |\ j=0,1,2\ldots\}$



$\sin(\pi x)$ $\sin(2\pi x)$

higher and higher frequencies

Consider linear combinations of these sine waves

$f \in \langle \sin(\pi x), \sin(2\pi x), \sin(3\pi x) \rangle$

then f is periodic

Linear Algebra III

Solve for $t_1, t_2, t_3 \in \mathbb{R}$ s.t.

$$f = t_1 \sin(\pi x) + t_2 \sin(2\pi x) + t_3 \sin(3\pi x)$$

Trick $\displaystyle\int_0^{2\pi} \sin(m\pi x)\sin(n\pi x) = 0$ if $m \neq n$

$$\boxed{k} = \int_0^{2\pi} \sin(m\pi x)\sin(m\pi x)\,dx$$

Find this

we can call them L_2 orthonormal

$$\frac{1}{k}\int_0^{2\pi} f(x)g(x)\,dx = \begin{cases} 0 & f \neq g \\ 1 & f = g \end{cases}$$

"dot product" for function

Consider

$\sin(k\pi x)$ to be a note played

$t \sin(k\pi x)$ frequency

amplitude — loud

Play a few notes at once

$t_1 \sin(\pi x) + t_2 \sin(2\pi x) + \dots + t_k \sin(2\pi k x)$

Linear Algebra III

Start with a "nice" function $f \in L_2$

$\int_0^{2\pi} f^2(x)\,dx$ finite. Lebesgue

$$t_j = \frac{1}{k}\int_0^{2\pi} f(x)\sin(j\pi x)\,dx$$

$$f(x) = t_1 \sin(\pi x) + t_2 \sin(2\pi x) + \dots + t_n \sin(\pi n x)$$

Fourier Coefficients of f

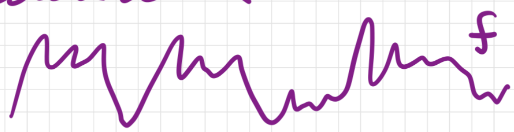
Fourier series $f(x) = \sum_{j=1}^{\infty} t_j \sin(j\pi x)$

9:14 PM Sat Dec 5

Linear Algebra III

Start with a sound

Sound wave



Compute the Fourier coefficients

$\{t_1, t_2, t_3, t_4 \ldots t_N\}$

$f \longmapsto \underbrace{(t_1 \ldots t_N)}_{\text{digital sound}}$

analog sound

Stop N when the frequency is high enough a human cannot hear.

Linear Algebra III

$f \xrightarrow{F} (t_1, t_2, t_3 \ldots\ldots)$

$f \in L_2 = \left\{ f \;\middle|\; \int_0^{2\pi} f^2(x)\,dx < \infty \right\}$

$(t_1, t_2 \ldots) \in \ell_2 = \left\{ (t_1 \ldots) \;\middle|\; \sum_{i=1}^{\infty} t_i^2 < \infty \right\}$

Hilbert Spaces

F is a linear isomorphism

one to one and onto

Consider $\mathbb{C}$ valued functions

$f(s) = a(s) + i b(s)$

$t_i \in \mathbb{C}$ $\quad$ $\mathbb{C}$ Fourier series.

Direct links to the professional videos:

Sounds https://youtu.be/3IAMpH4xF9Q

saw wave https://youtu.be/YUBe-ro89I4

3blue1brown https://youtu.be/r6sGWTCMz2k

There is a review for the final linked to from the course webpage.