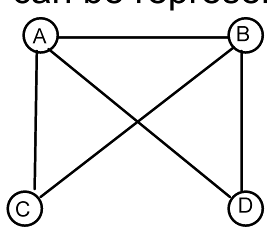
Graph Theory

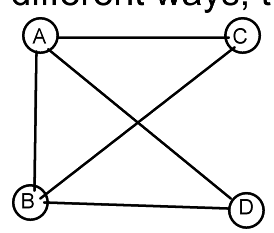
This topic is one of the most applicable to real-life applications because all networks (computer, transportation, communication, organizational, etc.) can be represented with a graph. For example, a school building has rooms connected by hallways, an airline map has cities connected by routes, and a rumor network has friends connected by conversations. Graphs can be used to model such situations.

Defining a Graph

A *graph* is a collection of vertices and edges. An *edge* is a connection between two *vertices* (or *nodes*). One can draw a graph by marking points for the vertices and drawing segments connecting them for the edges, but it must be borne in mind that the graph is defined independently of the representation.

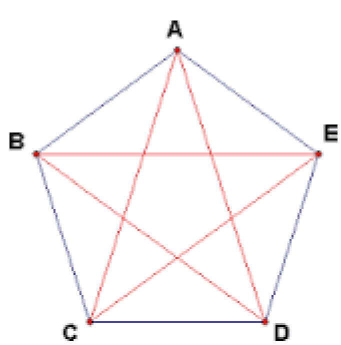
The precise way to represent a graph is to say that it consists of the set of vertices {*A, B, C, D*} and the set of edges between these vertices {*AB, AC, BC, AD, DB*}. This graph can be represented in many different ways, two of which are as follows:

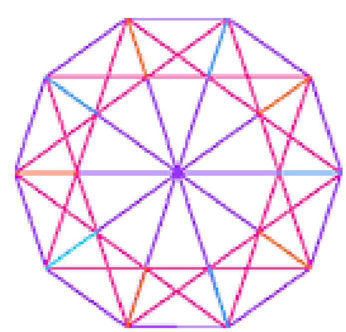




These are called *undirected* graphs because there are no arrows on the lines. An edge DB is the same as edge BD. Only *undirected* graphs will be considered in this topic for the Elementary Division.

One special kind of *undirected* graph is a *complete* graph which means that there is one edge from every vertex to each of the other vertices. These graphs are the same as drawing a polygon of any number of sides and drawing all of its diagonals. In every *complete* graph, the following relationship exists *E(dges) = V(ertices) + D(iagonals)*. In a pentagon as shown below, there are 5 vertices and 5 diagonals for a total of 10 edges. In a decagon, there are 10 vertices, 35 diagonals, and 45 edges.





Different Kinds of Paths

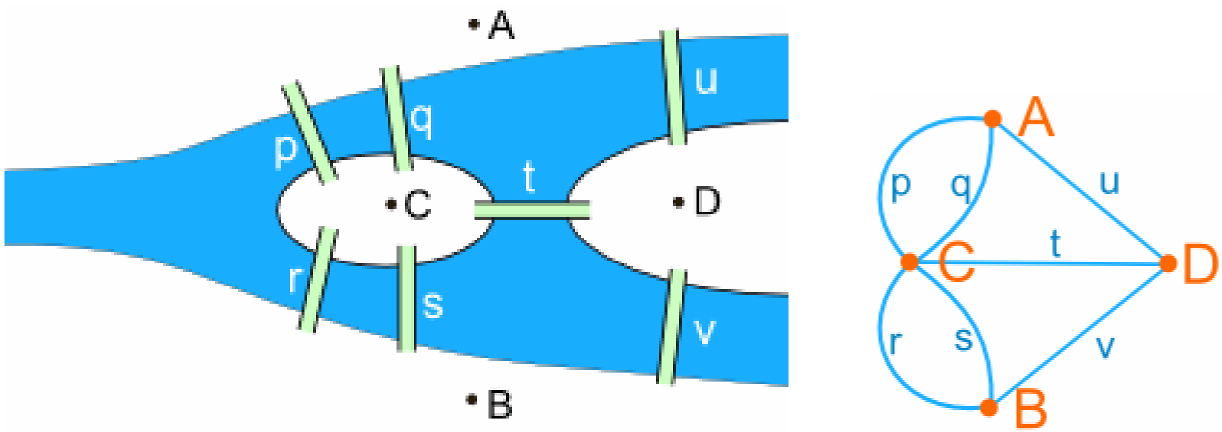
A *path* between two vertices in a graph is a list of vertices, in which successive vertices are connected by edges in the graph. For example, BD is a path of length 1 while BAD is a path of length 2 from vertex B to vertex D. A *simple path* is a path with no vertex repeated. For example, BACBD is a path of length 4, but it is not a simple path. BD and BAD are both simple paths.

A graph is *connected* if there is a path from every vertex to every other vertex in the graph. Intuitively, if the vertices were physical objects and the edges were strings connecting them, a connected graph would stay in one piece if picked up by any vertex. The above graph is connected, but if edges BC and AC were missing, then there would be 2 unconnected graphs, one with just vertex C and the other with vertices {A, B, D} with the remaining 3 edges.

A *cycle* is a path, which is simple except that the first and last vertex are the same (a path from a point back to itself). For example, the path *ABDA* is a cycle in our example. Vertices must be listed in the order that they are traveled to make the path, but any of the vertices may be listed first. Thus, *ABDA* and *BDAB* are different ways to identify the same cycle. For clarity, we list the start/end vertex twice, once at the start of the cycle and once at the end. In an *undirected* graph, cycles are always 3 or more vertices. In other words, a single edge of the graph cannot be a cycle such as *ABA*. In addition, there are always an even number of cycles in an undirected graph. In the first graph shown above, ABDA, ADBA, ABCA, ACBA, ACBDA, and ADBCA are all cycles so there are 6 of them. One of the most difficult problems for computers to solve is called the *Traveling Salesman Problem* which tries to find the best route for a salesperson to take if he/she wants to reach every stop (or vertex) once and only once while starting and ending at the same place. This isn't hard to do by hand with a limited number of vertices, but no algorithm has been developed to find a general solution to this problem without using some form of artificial intelligence.

Traversable Graphs

Traversability of a graph refers to whether or not you can use every edge once and only once without lifting your pencil. The most famous traversability example is the **Seven Bridges of Königsberg**, a historically notable problem in mathematics. Wikipedia states, "The city of Königsberg in Prussia was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. The problem was to devise a walk through the city that would cross each bridge once and only once, with the provisos that: the islands could only be reached by the bridges and every bridge once accessed must be crossed to its other end. The starting and ending points of the walk need not be the same." A graph can be used to model the city as follows:



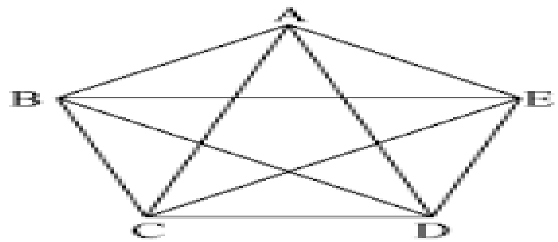
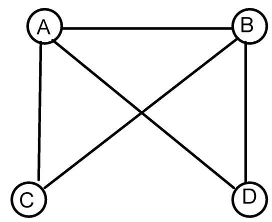
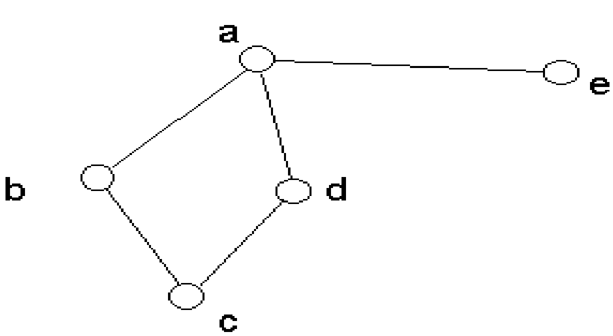
In 1735 Leonard Euler solved the problem by identifying each vertex as having a degree which represented the number of edges that came into that vertex. Therefore, vertex A has degree 3, vertex B has degree 3, vertex C has degree 5, and vertex D has degree 3. Euler discovered that the number of vertices of odd degree must be either 0 or 2 for a graph to be traversable.

A path that visits each edge once is called an *Euler path*. Therefore, the graph representing the bridges of **Königsberg** is not traversable since there is no Euler path. If there are 0 odd vertices, the Euler path can start with any vertex, but if there are 2 odd vertices, one must be the starting point and one must be the ending point.

References

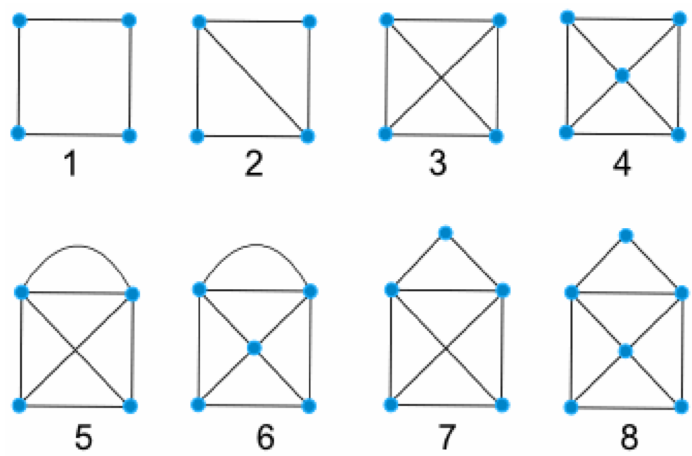
Math for eight-year-olds: graph theory for kids!
Geogebra - Graph Theory for Kids
Kiddle - Graph Theory Facts
Activity: The Seven Bridges of Königsberg

Sample Problems

Draw a complete graph with 5 vertices by using V = {A, B, C, D, E}.	
Find the number of different cycles contained in the graph with vertices {A,B,C,D} and edges {AB, BC, AC, AD, DB}.	The graph is as follows:  By inspection, the cycles are ABDA, ADBA, ACBA, ABCA, ADBCA, and ACBDA. Thus, there are 6 cycles in the graph.
In the above graph, list all simple paths of length 3 starting from vertex C.	The paths are CADB, CABD, CBAD, and CBDA. You cannot go to vertex D first.
Given the following graph, write the precise definition by listing the set of vertices and the set of edges. 	The set of vertices is {a,b,c,d,e} and the set of edges is {ab,ad,bc,cd,ae}. Edges can be written in either order since it is an undirected graph.

Using the above graph, identify all of the simple paths from vertex **b** to vertex **d**.	All of the simple paths from vertex '**b**' to vertex '**d**' are 'bad' and 'bcd'.
Using the above graph, how many paths of length 2 are there?	There are 12 paths of length 2 including 'abc', 'adc', 'bae', 'bad', 'bcd', 'cba', 'cda', 'dae', 'dcb', 'dab', 'eab', and 'ead'.
Using the above graph, if the graph is traversable, identify its possible starting and ending vertices. If it is not, write NO.	This graph is traversable starting with vertex '**a**' and ending with vertex '**e**' or starting with '**e**' and ending with '**a**'.
Using the above graph, identify all cycles.	The only cycles are 'abcda' and 'adcba'. Letters must be in that order but it can start and end with any of the letters. In undirected graphs, there are no cycles between just 2 vertices.
How many edges are there in a complete graph with 6 vertices?	A complete graph with 6 vertices is a hexagon with all of its diagonals. There are 9 diagonals and 6 sides so that makes 15 edges. This is also the number of handshakes that 6 people must make for everyone to shake each other person's hand once and only once.

Which of the following graphs is NOT traversable?



By inspection, the following table shows the number of vertices and the degree of each. Vertices are listed from the top of each graph to the bottom and left to right.

	V	A	B	C	D	E	F	Y/N
1	4	2	2	2	2			Y
2	4	3	2	2	3			Y
3	4	3	3	3	3			N
4	5	3	3	4	3	3		N
5	4	4	4	3	3			Y
6	5	4	4	4	3	3		Y
7	5	2	4	4	3	3		Y
8	6	2	4	4	4	3	3	Y

For example, an Euler path in graph 1 is ABCDA and an Euler path for graph 2 is ABDACD. An Euler path in graph 5 must start with either C or D and end with the other one and an Euler path for graph 8 must start and end with vertices E and F which could be EDBACBEFDCF or FDCFEDBACBE. The answer is 3 & 4.