

Linear Algebra

Lesson 27

Part I Vector Subspace

Part II Null Space

Part III Span

Part IV Basis

Part V Dimension

Part VI Hilbert Space: Infinite Dimensional Space

Part VII Fourier Series: Analog to Digital

Parts I-V are required as they review theorems and definitions we learned before for the final and analyze how they work on vector spaces in general. We have a different playlist for each of the parts. Parts VI and VII are important for math majors, physics majors, and engineers as they concern Hilbert Space and Fourier Series and the conversion of analog to digital sound.

Lesson 29

Part I Vector Subspaces

Part II Null Spaces

Part III Spans

Part IV Basis

Part V Dimension

Part VI Hilbert Space

Part VII Fourier Series

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V$
- Associativity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\forall t \in \mathbb{R} \quad \forall \vec{v} \in V \quad t\vec{v} \in V$
- Compatibility Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity Property: $\exists \mathbf{1} \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad \mathbf{1}\vec{v} = \vec{v}$
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
- Distribution over Scalar Addition Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$

Defn: A Linear Map $F: V \rightarrow W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(\vec{v} + \vec{u}) = F(\vec{v}) + F(\vec{u})$
- Preserves Scalar Mult: $\forall \vec{v} \in V \quad \forall t \in \mathbb{R} \quad F(t\vec{v}) = tF(\vec{v})$



Lesson 29 (not 29)

Part I Vector SubspacesPart II Null SpacesPart III SpansPart IV BasisPart V DimensionPart VI Hilbert SpacePart VII Fourier Series

Review of Concepts
we learned about
vectors in $\mathbb{R}^n$

and see now that
they are interesting
to apply to vector
spaces. (function
spaces)

← Analog ↔ Digital
Sound

Watch [Playlist 313F20-27-Part1](#)

Part I

Vector Subspaces

Defn V is a vector subspace of a vector space W if $V \subset W$ (it a subset of W) and has the same addition and scalar multiplication V is a vector space itself.

Thm: We need only check

- Closed under Vector Addition and
- Closed under Scalar Mult.

Proof We must check Add Identity $\in V$
We must check Inverses in V

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V$ *
- Associativity
Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$ ✓
- Commutativity
Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$ ✓
- Additive Identity
Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ ←
- Additive Inverses
Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$ ←

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V$ *
- Compatibility
Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$ ✓
- Scalar Identity
Property: $\exists \mathbf{1} \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad \mathbf{1}\vec{v} = \vec{v}$ ✓
- Distribution over Vector Addition
Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$ ✓
- Distribution over Scalar Addition
Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$ ✓

Defn: A Linear Map $F: V \rightarrow W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(\vec{v} + \vec{u}) = F(\vec{v}) + F(\vec{u})$
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Part I

Vector Subspaces

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Vector Subspace

Thm: We need only check

- Closed under Vector Addition and
- Closed under Scalar Mult.

Proof We must check Add Identity $\in V$
We must check Inverses in V

Some Useful Lemmas: are true on any vector space

Lemma: $\vec{0}$ is unique

There is only one vector $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0} \quad \forall \vec{v} \in V$.

Easy to see in $\mathbb{R}^2$ $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

in function spaces $0(x) = 0$ constant

Skip the proof in general.

Lemma: $0 \in \mathbb{R}$ and any $\vec{v} \in V$, $0\vec{v} = \vec{0}$.

Easy to see in $\mathbb{R}^2$ $0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0v_1 \\ 0v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$

in function spaces $(0f)(x) = 0 \cdot f(x) = 0$ const

Skip the proof in general.

Lemma: Inverses are Unique

Easy to see in $\mathbb{R}^2$ inv of $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ was $\begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$
in function spaces took $-f(x)$.

Lemma: $-1 \in \mathbb{R}$ and $\vec{v} \in V$ then

$-1\vec{v} = -\vec{v}$ the inverse of $\vec{v}$.

Easy to see in $\mathbb{R}^2$ and function spaces.

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V is a vector space itself.

Vector Subspace

Thm: If V nonempty subset of W which is

- Closed under Vector Addition
- and
- Closed under Scalar Mult

then V is a vector subspace of W .

Proof We must check Add Identity $\in V$

We must check inverses in V

Show Add Identity is in V

① Take $0 \in \mathbb{R}$ and any $\vec{v} \in V$ ① Lemma

$0\vec{v} = \vec{0}$

② $\vec{0} = 0\vec{v} \in V$ ② Given under Scalar Mult

Thus $\vec{0} \in V$

Show inverses are in V

[Hw!] Use our given properties and one of our lemmas to prove inverses are in V (2 lines) QED

Linear Algebra III

Some Useful Lemmas:
are true on any vector space

Lemma: $\vec{0}$ is unique

There is only one vector $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0} \forall \vec{v} \in V$.

Easy to see in $\mathbb{R}^2$ $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

in function spaces $0(x) = 0$ constant

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in function spaces took $-f(x)$.

Lemma: $-1 \in \mathbb{R}$ and $\vec{v} \in V$ then $-1\vec{v} = -\vec{v}$ the inverse of $\vec{v}$.

Easy to see in $\mathbb{R}^2$ and function spaces.

Writing again
more neatly:

Vector Subspace Thm:

If V is a nonempty subset of W which is

* Closed Under Addition

* Closed Under Scalar Mult.

then V is a vector subspace of W

Proof: Must check all properties of a vector space hold.

* Closed Under vector addition (given)

* Closed Under scalar mult (given)

Check off 6 properties using that they are true for vectors in W and all vectors in V are in W //

→ Additive Id Property (proved using a lemma)

→ Additive Inverses Prop (HW also a lemma proven with a lemma)

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

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- Closed under Vector Addition: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V$ *
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- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$ ✓
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- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$ ←

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V$ *
- Compatibility Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$ ✓
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$ ✓
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$ ✓
- Distribution over Scalar Addition Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$ ✓

QED

Vector Subspace Thm:

If V is a nonempty subset of W which is

* Closed Under Addition

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then V is a vector subspace of W

Proof: Must check all properties of a vector space hold.

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Check off 6 properties using that they are true for vectors in W and all vectors in V are in W //

→ Additive Id Property (proved using a lemma)

→ Additive Inverses Prop (HW also proven with a lemma)
QED

HW2 $\left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid 2v_1 - 4v_2 = 0 \right\}$
is a vector subspace of $\mathbb{R}^3$

HW3 P_3 is a vector subspace $\subset (C[0,1])$

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Linear Algebra III
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SCREEN RECORDING
Screen Recording video saved to Photos

HW2 $\left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid 2v_1 - 4v_2 = 0 \right\} = V$
 is a vector subspace of $\mathbb{R}^3$
 • Closed under Vector Addition

① Given $\vec{v}, \vec{w} \in V$ ② defn of V
 $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ with $2v_1 - 4v_2 = 0$
 $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ with $2w_1 - 4w_2 = 0$

② $\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$ ② defn of + on V

③ $2(v_1 + w_1) - 4(v_2 + w_2) =$ ③ by algebra and step 1
 $= 2v_1 + 2w_1 - 4v_2 - 4w_2$
 $= \underbrace{2v_1 - 4v_2} + \underbrace{2w_1 - 4w_2}$
 $= 0 + 0 = 0$

④ $\vec{v} + \vec{w} \in V$ ④ by defn of V

Also check closed under scalar

HW3 P_3 is a vector subspace $C([0,1])$
 • $P_3 \subset C([0,1])$ because polynomials are continuous.

• Closed under addition

① Given $p, q \in P_3$ ① by defn of P_3
 $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$
 $q(x) = q_0 + q_1x + q_2x^2 + q_3x^3$

② $p(x) + q(x) = (p_0 + q_0) +$ ② by add in $\mathbb{R}$
 $+ (p_1 + q_1)x + (p_2 + q_2)x^2 + (p_3 + q_3)x^3$

③ $p(x) + q(x) \in P_3$ ③ defn of P_3

• Closed under scalar mult

You must do for HW3.

Watch [Playlist 313F20-27-Part2](#)

Part II Null Spaces

Defn: Given a linear map

$F: V \rightarrow W$ we have

$$\text{Null}(F) = \{ \vec{v} \in V \mid F(\vec{v}) = \vec{0} \}$$

Recall a Linear Map
preserves vector addition
and scalar multiplication.

HW4 Find $\text{Null}(F)$ where
 $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_3 \end{pmatrix}$

HW5 Find $\text{Null}(F)$ where
 $F: P_3 \rightarrow P_2 \quad F(p) = p'(x).$

Defn: A Vector Space V is a set
with addition and scalar multiplication
that has the following ten properties:

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- Closed under Vector Addition: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V$
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Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
- Commutativity
Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Additive Identity
Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$
- Additive Inverses
Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\forall t \in \mathbb{R} \quad \forall \vec{v} \in V \quad t\vec{v} \in V$
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Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity
Property: $\exists \mathbf{1} \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad \mathbf{1}\vec{v} = \vec{v}$
- Distribution over Vector Addition
Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
- Distribution over Scalar Addition
Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$

Defn: A Linear Map $F: V \rightarrow W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(\vec{v} + \vec{u}) = F(\vec{v}) + F(\vec{u})$
- Preserves Scalar Mult: $\forall \vec{v} \in V \quad \forall t \in \mathbb{R} \quad F(t\vec{v}) = tF(\vec{v})$

Part II Null Spaces

Defn: Given a linear map

$F: V \rightarrow W$ we have

$$\text{Null}(F) = \{ \vec{v} \in V \mid F(\vec{v}) = \vec{0} \}$$

Recall a Linear Map preserves vector addition and scalar multiplication.

HW4 Find $\text{Null}(F)$ where

$$F: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_3 \end{pmatrix}$$

HW5 Find $\text{Null}(F)$ where

$$F: P_3 \rightarrow P_2 \quad F(p) = p'(x).$$

HW6 Find $\text{Null}(F)$ where

$$F: P_2 \rightarrow P_2 \quad F(p) = 5p'(x) + 4p(x)$$

HW4 $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\{ \vec{v} \in \mathbb{R}^3 \mid F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} =$$

$$= \{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \mid \begin{pmatrix} v_1 - v_2 \\ v_1 + v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

solving the homogeneous system

$$\begin{cases} v_1 - v_2 = 0 \\ v_1 + v_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right)$$

$$R_2 \rightarrow R_2 - R_1 \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

$$R_1 \rightarrow R_1 + R_2 \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

$$\begin{aligned} x_1 + x_3 &= 0 & x_1 &= -x_3 \\ x_2 + x_3 &= 0 & x_2 &= -x_3 \\ x_3 &= x_3 & & \text{(free)} \end{aligned}$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \mid x_3 \in \mathbb{R} \right\} = \text{Null}(F).$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_1 + v_3 \end{pmatrix}$$

$$F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-1 \\ 0+0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



HW 6 Find $\text{Null}(F)$ where

$$F: P_2 \rightarrow P_2 \quad F(p) = 5p'(x) + 4p(x)$$

$$\text{Null}(F) = \{p_2x^2 + p_1x + p_0 \mid F(p) = 0\}$$

$$= \{p_2x^2 + p_1x + p_0 \mid 5(2p_2x + p_1) + 4(p_2x^2 + p_1x + p_0) = 0\}$$

$$= \{p_2x^2 + p_1x + p_0 \mid 4p_2x^2 + (10p_2 + 4p_1)x + (5p_1 + 4p_0) = 0\}$$

$$4p_2 = 0 \quad 10p_2 + 4p_1 = 0 \quad 5p_1 + 4p_0 = 0$$

Solve this linear system

$$\begin{pmatrix} 4 & 0 & 0 & | & 0 \\ 10 & 4 & 0 & | & 0 \\ 0 & 5 & 4 & | & 0 \end{pmatrix} \rightarrow \text{row reduction}$$

$$\begin{pmatrix} 4 & 0 & 0 & | & 0 \\ 10 & 4 & 0 & | & 0 \\ 0 & 5 & 4 & | & 0 \end{pmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{4}R_1} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 10 & 4 & 0 & | & 0 \\ 0 & 5 & 4 & | & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \rightarrow R_2 - 10R_1} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 4 & 0 & | & 0 \\ 0 & 5 & 4 & | & 0 \end{pmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{4}R_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 5 & 4 & | & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 5R_2} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 4 & | & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \rightarrow \frac{1}{4}R_3} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix} \quad \begin{matrix} p_2 = 0 \\ p_1 = 0 \\ p_0 = 0 \end{matrix}$$

$$\text{Null}(F) = \{0x^2 + 0x + 0\}$$

Null space is trivial
"Kernel is trivial"

If $F: V \rightarrow W$ has $\text{Null}(F) = 0$
then F is one to one.

Thm: If $F: V \rightarrow W$ is a linear map between vector spaces V and W and if $\text{Null}(F) = \{\vec{0}\}$ then F is one-to-one.

Proof:

- ① $F(\vec{v}) = F(\vec{w})$ ① W is a vector space
then $F(\vec{v}) - F(\vec{w}) = \vec{0}$
 - ② $F(\vec{v} - \vec{w}) = \vec{0}$ ② F is a linear map lemma
 - ③ $\vec{v} - \vec{w} \in \text{Null}(F)$ ③ Defn Null(F)
 - ④ $\vec{v} - \vec{w} = \vec{0}$ ④ Given $\text{Null}(F) = \{\vec{0}\}$
 - ⑤ $\vec{v} = \vec{w}$ ⑤ adding $\vec{w}$ for V is a vector space QED
- Thus F is one-to-one

Defn F is one-to-one means
 $F(\vec{v}) = F(\vec{w}) \Rightarrow \vec{v} = \vec{w}$

Lemma If $F: V \rightarrow W$ is a linear map then $F(\vec{a} - \vec{b}) = F(\vec{a}) - F(\vec{b})$

Pf:

- ① $F(\vec{a} - \vec{b}) = F(\vec{a} + (-\vec{b}))$ ① Defn
- ② $= F(\vec{a}) + F(-\vec{b})$ ② F pres add
- ③ $= F(\vec{a}) - F(\vec{b})$ ③ F pres scalar mult

$5p'(x) + 4p(x) = q(x)$ | QED
has at most one solution
because Null space is trivial in HW6

HW7 Extra Credit

Prove:

Theorem: If $F: V \rightarrow W$ is a linear map between vector spaces V and W then $\text{Null}(F)$ is a vector subspace of V .

Hint: you only need to prove

- Closed under addition
- Closed under scalar mult

HW8 Let $F: C^3([0,1]) \rightarrow C([0,1])$ be defined $F(h(x)) = h'''(x)$

Prove $\text{Null}(F) = P_2$
(this uses calculus)

Hint: $C^3([0,1])$ = functions whose third derivatives are continuous on $[0,1]$.

$C([0,1])$ = functions that are continuous on $[0,1]$

$$P_2 = \{p_2x^2 + p_1x + p_0 \mid p_i \in \mathbb{R}\}$$

polynomials of degree ≤ 2 .

Facts from Calc:

If $f'(x) = 0$ function then $f(x) = k$

If $f'(x) = k$ then $f(x) = kx + C$

If $f''(x) = ax + b$ then $f(x) = \frac{a}{2}x^2 + bx + C$

Watch [Playlist 313F20-27-Part3](#)

Part III Spans

Defn: Given $\vec{v}_1, \dots, \vec{v}_k \in V$
the span $\langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle =$
 $= \{ t_1 \vec{v}_1 + \dots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \}$
 $= \{ \sum_{i=1}^k t_i \vec{v}_i \mid t_i \in \mathbb{R} \}$

HW9 Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \langle \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle$?

HW10 Is $f \in \langle g, h \rangle$ where
 $f(x) = x^2$ $g(x) = (x+2)^2$ $h(x) = x+1$
is the vector space $C([0,1])$?

HW9 $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = t_1 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

Solve this system?

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 - R_1} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array} \right)$$

row of zeroes ending in 2
so there is no solution

thus $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is not in the span.

If there was a solution
then you say it is in
the span.

Is $\begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix} \in \langle \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle$?

$$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right) \text{ has a solution!}$$

Yes $\begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$ is in the span.

Part III Spans

Defn: Given $\vec{v}_1, \dots, \vec{v}_k \in V$
the span $\langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle =$
 $= \{ t_1 \vec{v}_1 + \dots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \}$
 $= \{ \sum_{i=1}^k t_i \vec{v}_i \mid t_i \in \mathbb{R} \}$

HW 9 Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \langle \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$?

HW 10 Is $f \in \langle g, h \rangle$ where
 $f(x) = \underline{x^2}$ $g(x) = \underline{(x+2)^2}$ $h(x) = \underline{x+1}$
is the vector space $C([0,1])$?

HW 10 Can we find $t_1, t_2 \in \mathbb{R}$
s.t. $f = t_1 g + t_2 h$

$$x^2 = t_1 (x+2)^2 + t_2 (x+1)$$

$$x^2 = t_1 (x^2 + 4x + 4) + t_2 x + t_2$$

$$x^2 = t_1 x^2 + 4t_1 x + 4t_1 + t_2 x + t_2$$

$$x^2 + 0x + 0 = t_1 x^2 + (4t_1 + t_2)x + (4t_1 + t_2)$$

$$\text{Solve } 1 = t_1 \quad 0 = 4t_1 + t_2 \quad 0 = 4t_1 + t_2$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

has a solution

Yes $f \in \langle g, h \rangle$.

Part III Spans

Defn: Given $\vec{v}_1, \dots, \vec{v}_k \in V$
the span $\langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle =$
 $= \{ t_1 \vec{v}_1 + \dots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \}$
 $= \{ \sum_{i=1}^k t_i \vec{v}_i \mid t_i \in \mathbb{R} \}$

Thm: If $\vec{v}_1, \dots, \vec{v}_k \in W$ and
 $V = \langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle$ then
 V is a vector subspace of W .

HW11 Prove this

Defn: The Image of
a map $F: V \rightarrow W$
is $F(V) = \{ F(\vec{v}) \mid \vec{v} \in V \}$

Thm: If $V = \langle \vec{v}_1, \dots, \vec{v}_k \rangle$
then $F(V) = \langle F(\vec{v}_1), \dots, F(\vec{v}_k) \rangle$

HW12 Prove this.

Defn: $F: V \rightarrow W$ is onto
 $\Leftrightarrow \forall \vec{w} \in W \exists \vec{v} \in V \text{ s.t. } F(\vec{v}) = \vec{w}$

Thm: $\text{Onto} \Leftrightarrow F(V) = W$

Thm: $\text{Onto} \Leftrightarrow \exists \vec{v}_1, \dots, \vec{v}_k \in V$
such that $W = \langle F(\vec{v}_1), \dots, F(\vec{v}_k) \rangle$

HW13 Prove this

Part III Spans

Defn: Given $\vec{v}_1, \dots, \vec{v}_k \in V$
the span $\langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle =$
 $= \{ t_1 \vec{v}_1 + \dots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \}$
 $= \{ \sum_{i=1}^k t_i \vec{v}_i \mid t_i \in \mathbb{R} \}$

Thm: If $\vec{v}_1, \dots, \vec{v}_k \in W$ and
 $V = \langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle$ then
 V is a vector subspace of W .

HW11 Prove this

Defn: The Image of
a map $F: V \rightarrow W$
is $F(V) = \{ F(\vec{v}) \mid \vec{v} \in V \}$

Thm: If $V = \langle \vec{v}_1, \dots, \vec{v}_k \rangle$
then $F(V) = \langle F(\vec{v}_1), \dots, F(\vec{v}_k) \rangle$

HW12 Prove this.

Defn: $F: V \rightarrow W$ is onto
 $\Leftrightarrow \forall \vec{w} \in W \exists \vec{v} \in V \text{ s.t. } F(\vec{v}) = \vec{w}$

Thm: Onto $\Leftrightarrow F(V) = W$

Thm: Onto $\Leftrightarrow \exists \vec{v}_1, \dots, \vec{v}_k \in V$
such that $W = \langle F(\vec{v}_1), \dots, F(\vec{v}_k) \rangle$

HW13 Prove this

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V$
- Associativity
Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
- Commutativity
Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Additive Identity
Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$
- Additive Inverses
Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V$
- Compatibility
Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity
Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$
- Distribution over Vector Addition
Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
- Distribution over Scalar Addition
Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$

Defn: A Linear Map $F: V \rightarrow W$

- Preserves Addition: $\forall \vec{v}, \vec{w} \in V \quad F(\vec{v} + \vec{w}) = F(\vec{v}) + F(\vec{w})$
- Preserves Scalar Mult: $\forall \vec{v} \in V \quad \forall t \in \mathbb{R} \quad F(t\vec{v}) = tF(\vec{v})$

HW 11

closed under addition (you do)

closed under scalar

Given $\vec{v} \in V = \langle \vec{v}_1, \dots, \vec{v}_k \rangle$

$t \in \mathbb{R}$

Show $t\vec{v} \in \langle \vec{v}_1, \dots, \vec{v}_k \rangle$

(1) $\vec{v} \in \langle \vec{v}_1, \dots, \vec{v}_k \rangle$ (1) by defn of span
 $\vec{v} = t_1\vec{v}_1 + \dots + t_k\vec{v}_k$

(2) $t\vec{v} = t(t_1\vec{v}_1 + \dots + t_k\vec{v}_k)$ (2) by defn scalar mult

(3) $= t(t_1\vec{v}_1) + \dots + t(t_k\vec{v}_k)$ (3) Distrib

(4) $= (tt_1)\vec{v}_1 + \dots + (tt_k)\vec{v}_k$ (4) Compat.

(5) $\in \langle \vec{v}_1, \dots, \vec{v}_k \rangle$ (5) $tt_i \in \mathbb{R}$ defn of span

You do closed under addition. carefully name the properties

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V$
- Associativity
Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
- Commutativity
Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Additive Identity
Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$
- Additive Inverses
Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V$
- Compatibility
Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity
Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$
- Distribution over Vector Addition
Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
- Distribution over Scalar Addition
Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$

Defn: A Linear Map $F: V \rightarrow W$

- Preserves Addition: $\forall \vec{v}, \vec{w} \in V \quad F(\vec{v} + \vec{w}) = F(\vec{v}) + F(\vec{w})$
- Preserves Scalar Mult: $\forall \vec{v} \in V \quad \forall t \in \mathbb{R} \quad F(t\vec{v}) = tF(\vec{v})$

HW12 $V = \langle v_1, \dots, v_k \rangle$

Show $F(V) = \langle F(v_1), \dots, F(v_k) \rangle$

Proof:

$$(1) F(V) = \{ F(\vec{v}) \mid \vec{v} \in V \}$$

(1) by defn of Image

$$(2) = \{ F(t_1 \vec{v}_1 + \dots + t_k \vec{v}_k) \mid t_i \in \mathbb{R} \}$$

(2) defn of span
 $V = \langle v_1, \dots, v_k \rangle$

$$(3) = \{ F(t_1 v_1) + \dots + F(t_k v_k) \mid t_i \in \mathbb{R} \}$$

(3) F preserves addition

$$(4) = \{ t_1 F(v_1) + \dots + t_k F(v_k) \mid t_i \in \mathbb{R} \}$$

(4) F preserves scalar

$$(5) = \langle F(v_1), \dots, F(v_k) \rangle$$

(5) by defn of span QED

HW15 Are $(x+1)^2$, $5x$, 8
a basis for P_2 ?

• Do they span P_2 ?

Given any $p \in P_2$ $p = p_2x^2 + p_1x + p_0$

Find $t_1, t_2, t_3 \in \mathbb{R}$ s.t.

$$p_2x^2 + p_1x + p_0 = t_1(x+1)^2 + t_2(5x) + t_3(8)$$

$$= t_1(x^2 + 2x + 1) + t_2(5x) + t_3(8)$$

$$= t_1x^2 + (t_1 \cdot 2 + t_2 \cdot 5)x + (t_1 + t_3 \cdot 8)$$

$$p_2 = t_1 \quad p_1 = t_1 \cdot 2 + t_2 \cdot 5 \quad p_0 = t_1 + t_3 \cdot 8$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & p_2 \\ 2 & 5 & 0 & p_1 \\ 1 & 0 & 8 & p_0 \end{array} \right) \text{ solve for } t_1, t_2, t_3$$

95 of 96 $\Rightarrow$ solve this system + show it has a solution.

Are they lin indep?

$$t_1(x+1)^2 + t_2(5x) + t_3(8) = 0_{\text{function}}$$

$\Rightarrow$ same work $p_i = 0$

$$\left\{ \begin{array}{l} 0 = t_1 \\ 0 = t_1 \cdot 2 + t_2 \cdot 5 \\ 0 = t_1 + t_3 \cdot 8 \end{array} \right. \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & 8 & 0 \end{array} \right) \text{ solve}$$

Is the only solution

$$t_1 = 0 \quad t_2 = 0 \quad t_3 = 0.$$

Finish this.

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Linear Algebra III

HW15 Are $(x+1)^2$, $5x$, 8
a basis for P_2 ?

Do they span P_2 ?

Given any $p \in P_2$ $p = p_2x^2 + p_1x + p_0$
Find $t_1, t_2, t_3 \in \mathbb{R}$ s.t.

$$p_2x^2 + p_1x + p_0 = t_1(x+1)^2 + t_2(5x) + t_3(8)$$

$$= t_1(x^2 + 2x + 1) + t_2(5x) + t_3(8)$$

$$= t_1x^2 + (t_1 \cdot 2 + t_2 \cdot 5)x + (t_1 + t_3 \cdot 8)$$

$$p_2 = t_1 \quad p_1 = t_1 \cdot 2 + t_2 \cdot 5 \quad p_0 = t_1 + t_3 \cdot 8$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & p_2 \\ 2 & 5 & 0 & p_1 \\ 1 & 0 & 8 & p_0 \end{array} \right)$$

Solve for t_1, t_2, t_3

Solve this system + show it has a solution.

Linear Algebra III

Are they lin indep?

$$t_1(x+1)^2 + t_2(5x) + t_3(8) = 0_{\text{function}}$$

same work $p_i = 0$

$$0 = t_1 \quad 0 = t_1 \cdot 2 + t_2 \cdot 5 \quad 0 = t_1 + t_3 \cdot 8$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & 8 & 0 \end{array} \right)$$

Solve

Is the only solution $t_1 = 0 \quad t_2 = 0 \quad t_3 = 0$.

Finish this.

Part IV Basis

Watch [Playlist 313F20-27-Part4](#)

Part IV Basis

Defn: $\vec{v}_1, \dots, \vec{v}_k$ are a basis for a vector space V if $V = \langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle$ and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are linearly independent.

Defn: $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent if

$$t_1 \vec{v}_1 + t_2 \vec{v}_2 + \dots + t_k \vec{v}_k = \vec{0}$$

$\Downarrow$

$$t_1 = t_2 = \dots = t_k = 0.$$

Standard Basis for $\mathbb{R}^2$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ easy check lin indep

$$\mathbb{R}^2 = \left\{ t_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid t_1, t_2 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \mid t_1, t_2 \in \mathbb{R} \right\}$$

for $\mathbb{R}^3$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

for P_2 $x^2 \ x \ 1$

$$P_2 = \{ t_2 x^2 + t_1 x + t_0 \mid t_i \in \mathbb{R} \} = \langle x^2, x, 1 \rangle$$

easy to check lin indep.

If our functions how
to check lin. indep?

$$\cos(x), \sin(x)$$

are these lin indep?

$$t_1 \cos(x) + t_2 \sin(x) = 0 \text{ function}$$

so this sum is 0 for any
x we plug in.

$$t_1 \cos(0) + t_2 \sin(0) = 0$$

$$t_1 \cdot 1 + t_2 \cdot 0 = 0$$

$$\boxed{t_1 = 0}$$

plug in $x = \frac{\pi}{2}$

$$t_1 \cos\left(\frac{\pi}{2}\right) + t_2 \sin\left(\frac{\pi}{2}\right) = 0$$

$$t_1 \cdot 0 + t_2 \cdot (1) = 0$$

$$\boxed{t_2 = 0}$$

Yes $\cos(x)$ and $\sin(x)$ are
lin indep.

HW16 Are

$\sin(x), \sin(2x), \sin(4x)$
linearly independent?

Try this
using the
same method
plugging in values
of x where
different functions
are 0.
Try $x = \frac{\pi}{2}$ $x = \frac{\pi}{4}$ and another x.

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Linear Algebra III

Thm If $\vec{v}_1, \dots, \vec{v}_k$ form a basis for V then every $\vec{v} \in V$ has exactly one set $t_1, \dots, t_k \in \mathbb{R}$ such that $\vec{v} = t_1 \vec{v}_1 + \dots + t_k \vec{v}_k$. (that is, there exists a unique solution)

Usually Difficult to find the t_i

65%

Linear Algebra III

Recall:

Orthonormal Basis

$\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$

s.t. $\vec{v}_i \cdot \vec{v}_i = 1$

$\vec{v}_i \cdot \vec{v}_j = 0$ if $i \neq j$

Easy to find t_i for any $\vec{v}$

$t_i = \vec{v} \cdot \vec{v}_i$

Check $\vec{v} = t_1 \vec{v}_1 + \dots + t_k \vec{v}_k$

HW21 check this works.

We will do HW17-20 in Part 5.

Part V Dimension

Watch [Playlist 313F20-27-Part5](#)

V Dimension

Thm: If $\vec{v}_1, \dots, \vec{v}_k$ are a basis for V and also $\vec{w}_1, \dots, \vec{w}_m$ are a basis for V then $m = k$

Defn: $\uparrow$ Dimension of V

HW17 Prove the theorem
Extra Credit: hint find our old proof in $\mathbb{R}^n$ and imitate.

HW18 Find the dimension of $P_3 = \{p_3x^3 + p_2x^2 + p_1x + p_0 \mid p_i \in \mathbb{R}\}$

Defn: V is infinite dimensional if it has no finite basis.

Thm: If there is a sequence $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \in V$ such that for every $k \in \mathbb{N}$ $v_1, v_2, \dots, v_k$ are lin. indep then V is infinite dim'.

HW19 Prove this theorem

HW20 Use this theorem to show $C([0,1])$ is infinite dim'.

HW19

Proof by Contradiction to show V has no finite basis

① Assume on the contrary it does have a finite basis $w_1, \dots, w_m$ ① Indirect Hypothesis Unjustified.

② $V = \langle w_1, \dots, w_m \rangle$ ② defn of basis. and $w_1, \dots, w_m$ are lin indep

③ Take $k = m+1$ ③ by given $v_1, v_2, \dots, v_{m+1}$ are info for linearly indep. our thm.

④ Each $v_i \in V = \langle w_1, \dots, w_m \rangle$
So $\exists t_{i1}, t_{i2}, \dots, t_{im} \in \mathbb{R}$ s.t.

$$v_i = t_{i1}w_1 + t_{i2}w_2 + \dots + t_{im}w_m$$

④ by defn of span

$$\textcircled{5} s_1 v_1 + s_2 v_2 + \dots + s_{m+1} v_{m+1} = 0$$

$$\Rightarrow s_1 = s_2 = \dots = s_{m+1} = 0$$

⑤ by defn of lin indep

$$\textcircled{6} s_1 (t_{11}w_1 + t_{12}w_2 + \dots + t_{1m}w_m)$$

$$+ s_2 (t_{21}w_1 + t_{22}w_2 + \dots + t_{2m}w_m)$$

$$+ \dots$$

$$+ s_m (t_{m1}w_1 + \dots + t_{mm}w_m)$$

$$+ s_{m+1} (t_{m+1,1}w_1 + \dots + t_{m+1,m}w_m) = 0$$

$$\Rightarrow s_1 = s_2 = \dots = s_{m+1} = 0$$

⑥ Sub step 4 into steps



$$\begin{aligned} & (s_1 t_{11} + s_2 t_{21} + \dots + s_m t_{m1} + s_{m+1} t_{m+1,1}) w_1 \\ & + (s_1 t_{12} + \dots + s_{m+1} t_{m+1,2}) w_2 \\ & + \dots + (s_1 t_{1m} + \dots + s_{m+1} t_{m+1,m}) w_m \\ & = 0 \\ \Rightarrow & s_1 = s_2 = \dots = 0. \end{aligned}$$

(8) Since w_1 to w_m span V and are lin indep.

So if $= 0$ then

$$\begin{aligned} s_1 t_{11} + s_2 t_{21} + \dots + s_{m+1} t_{m+1,1} &= 0 \\ \vdots \\ s_1 t_{1m} + \dots + s_{m+1} t_{m+1,m} &= 0 \end{aligned}$$

$$\begin{pmatrix} t_{11} & t_{21} & \dots & t_{m+1,1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1m} & t_{2m} & \dots & t_{m+1,m} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$



$m+1$ columns and m rows
So there is at least one free variable.

$$\left\{ \begin{pmatrix} s_1 \\ \vdots \\ s_{m+1} \end{pmatrix} = \text{something not just } 0 \right\}$$

Then we do **not** get

$$s_1 = s_2 = \dots = s_{m+1} = 0$$

Which means

$v_1, v_2, \dots, v_{m+1}$ are **not** linearly independent.

⊗ In thm they are lin. indep
So Indirect Hyp. is false
in step 1

So V does **not** have a finite basis QED

6:51 PM Sat Dec 5 Linear Algebra III

Thm If $\vec{v}_1, \dots, \vec{v}_k$ form a basis for V then every $\vec{v} \in V$ has exactly one set $t_1, \dots, t_k \in \mathbb{R}$ such that $\vec{v} = t_1 \vec{v}_1 + \dots + t_k \vec{v}_k$. (that is, there exists a unique solution)

Proven using properties of vector spaces.

66% Linear Algebra III

Proof:

① $\vec{v}_1, \dots, \vec{v}_k$ are a basis for V ① given

② $V = \langle \vec{v}_1, \dots, \vec{v}_k \rangle$ ② by defn of basis

③ $\vec{v} = t_1 \vec{v}_1 + \dots + t_k \vec{v}_k$ ③ by defn of span has a solution $t_1, \dots, t_k$

④ Is there only one solution? Check $\vec{v} = s_1 \vec{v}_1 + \dots + s_k \vec{v}_k$ ④ step 3

$t_1 \vec{v}_1 + \dots + t_k \vec{v}_k = s_1 \vec{v}_1 + \dots + s_k \vec{v}_k$

⑤ $t_1 \vec{v}_1 - s_1 \vec{v}_1 + \dots + t_k \vec{v}_k - s_k \vec{v}_k = \vec{0}$ ⑤ by add inverses

⑥ $(t_1 - s_1) \vec{v}_1 + \dots + (t_k - s_k) \vec{v}_k = \vec{0}$ ⑥ by dist prop

⑦ $t_1 - s_1 = t_2 - s_2 = \dots = t_k - s_k = 0$ ⑦ $\vec{v}_k$ are lin indep

⑧ $t_1 = s_1, t_2 = s_2, \dots, t_k = s_k$ ⑧ add s_i QED

Part VI Hilbert Space

Watch [Playlist 313F20-27-Part6](#)

Part VI

Hilbert Space

$$\ell_2 = \left\{ \underbrace{(v_1, v_2, v_3, \dots)}_{\substack{\text{infinite} \\ \text{sequence} \\ \text{of real numbers}}} \mid \sum_{i=1}^{\infty} v_i^2 < \infty \right\}$$

So $(1, 1, 1, 1, 1, \dots) \notin \ell_2$
 $1^2 + 1^2 + 1^2 + 1^2 + \dots = \infty$

But $(1, 1, 1, 1, 0, 0, 0, \dots) \in \ell_2$
 $1^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + \dots$
 $= 4 < \infty$

$$\begin{array}{c} | \quad | \quad | \quad | \\ 0 \quad \frac{1}{2} \quad 1 \end{array}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \leq 1$$

$$v_1 = \sqrt{\frac{1}{2}} \quad v_2 = \sqrt{\frac{1}{4}} \quad v_3 = \sqrt{\frac{1}{8}} \dots$$

this is in ℓ_2

Part VI

Hilbert Space

$$\ell_2 = \left\{ \underbrace{(v_1, v_2, v_3, \dots)}_{\substack{\text{infinite} \\ \text{sequence} \\ \text{of real numbers}}} \mid \sum_{i=1}^{\infty} v_i^2 < \infty \right\}$$

So $(1, 1, 1, 1, 1, \dots) \notin \ell_2$
 $1^2 + 1^2 + 1^2 + 1^2 + \dots = \infty$

But $(1, 1, 1, 1, 0, 0, 0, \dots) \in \ell_2$
 $1^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + \dots$
 $= 4 < \infty$

Hilbert Space is a Vector Space
 addition:

$$(v_1, v_2, \dots) + (w_1, w_2, \dots) \\ = (v_1 + w_1, v_2 + w_2, \dots)$$

- Check is in ℓ_2
 to see we are closed
 under addition

Show:
 $(v_1 + w_1)^2 + (v_2 + w_2)^2 + \dots$

is finite
 use: $v_1^2 + v_2^2 + \dots$ is finite
 $w_1^2 + w_2^2 + \dots$ is finite.

8:31 PM Sat Dec 5

Linear Algebra III

Hilbert Space is a Vector Space

ADDITION

$$(v_1, v_2, \dots) + (w_1, w_2, \dots)$$

$$= (v_1 + w_1, v_2 + w_2, \dots)$$

must check this is in ℓ_2

- Closed under Addition

show:

$$(v_1 + w_1)^2 + (v_2 + w_2)^2 + \dots$$

is finite

use:

$$v_1^2 + v_2^2 + \dots \text{ is finite}$$

$$w_1^2 + w_2^2 + \dots \text{ is finite}$$

} difficult

- Additive Identity
(0, 0, 0, ...) (easy)
- Additive Inverse
(-v_1, -v_2, -v_3, ...) $\in \ell_2$ (easy)
- Associative (easy)
- Commutative (easy)

Linear Algebra III

SCALAR MULTIPLICATION

$$t(v_1, v_2, v_3, \dots) = (tv_1, tv_2, tv_3, \dots)$$

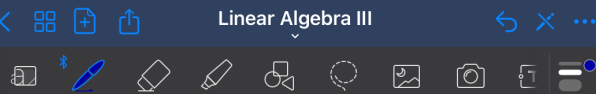
$t \in \mathbb{R}$ $v_i \in \ell_2$ $show \ tv_i \in \ell_2$

- Closed under scalar mult
(difficult)
- Compatibility
(easy) $(st)v = s(tv)$
- Scalar Identity $1v = v$
(easy)
- Distribution $t(v+w) = tv + tw$
(easy)
- Distribution $(s+t)v = sv + tv$
(easy)

Extra Credit
for completing
the proof.

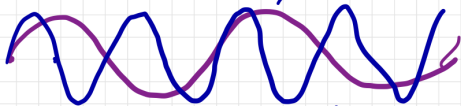
Part VII Fourier Series Analog to Digital

Watch [Playlist 313F20-27-Part7](#) which has one video made by me and then a few others with professional sound and graphics.



VII Fournier Series.

Consider $\{\sin(j\pi x) \mid j=0,1,2,\dots\}$



$\sin(\pi x)$ $\sin(2\pi x)$
higher and higher
frequencies

Consider linear combinations
of these sine waves

$f \in \langle \sin(\pi x), \sin(2\pi x), \sin(3\pi x) \rangle$

then f is periodic



Solve for $t_1, t_2, t_3 \in \mathbb{R}$
s.t.

$$f = t_1 \sin(\pi x) + t_2 \sin(2\pi x) + t_3 \sin(3\pi x)$$

Trick $\int_0^{2\pi} \sin(m\pi x) \sin(n\pi x) dx = 0$
if $m \neq n$

$$\boxed{K} = \int_0^{2\pi} \sin(m\pi x) \sin(n\pi x) dx$$

↑
find this

We can call them L_2 orthogonal

$$\frac{1}{K} \int_0^{2\pi} f(x)g(x) dx = \begin{cases} 0 & f \neq g \\ 1 & f = g \end{cases}$$

"dot product" for function



Consider

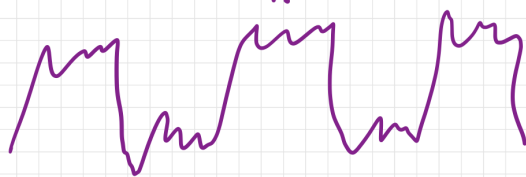
$\sin(k\pi x)$ to be
a note played

$t \sin(k\pi x)$ frequency

amplitude - loud

Play a few notes at
once

$$t_1 \sin(\pi x) + t_2 \sin(2\pi x) + \dots + t_k \sin(2\pi k x)$$



Start with a "nice"
function $f \in L_2$

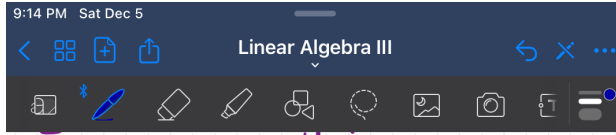
$$\int_0^{2\pi} f^2(x) dx \quad \text{Lebesgue finite.}$$

$$t_j = \frac{1}{k} \int_0^{2\pi} f(x) \sin(j\pi x) dx$$

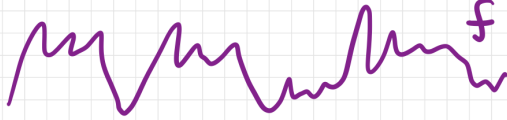
$$f(x) = t_1 \sin(\pi x) + t_2 \sin(2\pi x) + \dots + t_n \sin(n\pi x)$$

Fourier Coefficients of f

$$\text{Fourier series } f(x) = \sum_{j=1}^{\infty} t_j \sin(j\pi x)$$



Start with a sound
Sound wave

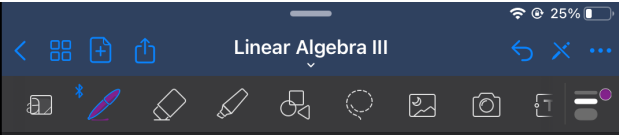


Compute the Fourier
coefficients

$\{t_1, t_2, t_3, t_4, \dots, t_N\}$

$f \mapsto (t_1, \dots, t_N)$
analog sound digital sound

Stop N when the frequency
is high enough a human
cannot hear.



$f \xrightarrow{F} (t_1, t_2, t_3, \dots)$

$f \in L_2 = \{f \mid \int_0^{2\pi} f^2(x) dx < \infty\}$

$(t_1, t_2, \dots) \in l_2 = \{(t_i, \dots) \mid \sum_{i=1}^{\infty} t_i^2 < \infty\}$

Hilbert Spaces

F is a linear
isomorphism
one to one
and
onto

Consider $\mathbb{C}$ valued functions

$f(s) = a(s) + ib(s)$

$t_i \in \mathbb{C}$ Fourier series.



Direct links to the professional videos:

Sounds <https://youtu.be/3IAMpH4xF9Q>

saw wave <https://youtu.be/YUBe-ro89I4>

3blue1brown <https://youtu.be/r6sGWTCMz2k>