Solutions

1. Point P is $\sqrt{3}$ units away from plane A. Let Q be a region of A such that every line through P that intersects A in Q intersects A at an angle between 30° and 60°. What is the largest possible area of Q?

   BMT 2019 INDIVIDUAL ROUND PROBLEM 5

Answer: 8π

Solution: Realize that the area is the difference of two circles with radius 1 and radius 3 respectively. The area is just $(3^2 - 1^2)\pi = \boxed{8\pi}$.

2. A standard 6-sided die is repeatedly rolled until every number from 1 to 6 inclusive appears on the top face at least once. Let m / n be the number of rolls, on average, that are necessary for this to occur. What is m + n?

   COWCONUTS 6th 2023 PROBLEM 20

Proposed by: David Zhang

Answer: $\boxed{157}$

Solution: The key idea is to consider the rolls as a 6-step process.

Our first step is straightforward: we roll the die once. If for example we roll a 3, we check 3 off our list of needed numbers. One number down, 5 more to go.

Now here's our second step: we keep rolling until we're able to check off another number from our list. Since we've only rolled 3 so far, this just means that we keep rolling until we get something other than a 3.

During our second step, we may need more than a single roll. Let E_2 be the expected number of rolls we need to get something other than a 3. If our first roll for this step is something other than a 3, great, we actually only needed one roll. Otherwise, we expect on average to need E_2 additional rolls, and in conjunction with the one roll we've already expended we arrive at $E_2 + 1$ rolls for this case. All this is to say that

$$E_2 = \frac{5}{6} \cdot 1 + \frac{1}{6} \cdot (E_2 + 1) \quad \rightarrow \quad E_2 = \frac{6}{5}.$$

Now we continue this process, at each step checking off one more number until we've finally rolled all 6 numbers at least once. For each step, we may run a calculation similar to the one used for the second step. Adding up the expected number of rolls used per step yields an expectation of

$$1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = \frac{147}{10}$$

total rolls, and thus an answer of $147 + 10 = 157$.

3.

Answer: 602

Solution: We will compute A and B with Fermat's little theorem. Note that $\varphi(10) = 4$. So, if $\gcd(x, 10) = 1$ then $x^{2020} \equiv 1 \mod 10$. So, this is equivalent to finding the last digits 1,3,7,9. So, there are $A = 803$.

Now if $\gcd(x, 10) \neq 1$, then we have $x^{2017} \equiv x \mod 10$. Hence, we need to figure out when $x^3 \equiv 6 \mod 10$ which occurs when $x \equiv 6 \mod 10$. So, $B = 201$. Hence $A - B = \boxed{602}$.