Calculating Velocity Over Time

Velocity is the integral of acceleration. Acceleration is dependent on three things: the force generated by the engine, the force created by atmospheric drag (in the opposite direction of the velocity), and the mass of the rocket which decreases over time.

Mass over Time

$$m = m_i - t * \frac{T}{v_e}$$

Acceleration over Time

This is assuming that thrust is constant.

$$a = \frac{T}{m} = \frac{T}{m_i - t * \frac{T}{v_e}}$$

Velocity over Time

$$v = \int a = C - v_e * ln\left(m_i - t * \frac{T}{v_e}\right)$$

In this case the integration constant C is $v_e * ln\left(m_i\right)$ which (because $ln\left(\frac{x}{y}\right) = ln(x) - ln(y)$) makes this equation equivalent to the rocket equation.

$$v = v_e * ln\left(\frac{m_i}{m_i - t * \frac{T}{v_e}}\right)$$

Atmospheric Drag

$$F_d = \frac{C_d * A * d * v^2}{2}$$

Combined Acceleration Differential Equation

$$\frac{dv}{dt} = \frac{T - ½*C_d*A*d*v^2}{m_i - t*\frac{T}{v_e}}$$

Solve for v Given

T = Thrust at Sea Level

C_d = Coefficient of Drag

A = Cross-sectional Area

d = Atmospheric Density at Sea Level

m_i = Initial Craft Mass

v_e = Effective Exhaust Velocity (Isp * g)

$v(0)$ = Velocity at time 0 = 0

Re-organize as a Separable Differential Equation and Integrate

$$\int \frac{1}{T - ½*C_d*A*d*v^2} dv = \int \frac{1}{m_i - t*\frac{T}{v_e}} dt$$

The left hand side is integrable given:

$$\int \frac{1}{a - b*x^2} dx = \frac{tanh^{-1}\left(x\sqrt{\frac{b}{a}}\right)}{\sqrt{a*b}} + C$$

The right hand side of this equation is integrable given that:

$$\int \frac{1}{a - b*x} dx = -\frac{1}{b} ln(a - b * x) + C$$

Therefore:

$$\frac{\tanh^{-1}\left(v\sqrt{\frac{½*C_d*A*d}{T}}\right)}{\sqrt{½*C_d*A*d*T}} = -\frac{1}{\left(\frac{T}{v_e}\right)} ln\left(m_i - \frac{T}{v_e} * t\right) + C$$

Therefore:

$$v = \frac{\tanh\left(\left(C - \frac{v_e}{T} * ln\left(m_i - \frac{T}{v_e} * t\right)\right) * \sqrt{\frac{C_d*A*d*T}{2}}\right)}{\sqrt{\frac{C_d*A*d}{2*T}}}$$

And Solving for C at (0,0):

$$C = \frac{v_e * ln\left(m_i\right)}{T}$$

If the initial condition is not v(0) = 0 but v(0) = x where x is positive, then

Finding Displacement Over Time

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