

Linear Algebra

Lesson 11

Eigenvectors and Eigenvalues

Linear Algebra MAT313 Fall 2022

Professor Sormani

Don't forget your team challenge!

Look for 313F22-Oct-Team in googledocs and check your teammates work and add a step of the proof!

Part I: Eigenvectors and Eigenvalues

Part II: Power Method for finding Eigenvalues

*You will cut and paste the **photos of your notes and completed classwork** in a googledoc entitled:*

MAT313F22-lesson11-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

*If you have a question, type **QUESTION** in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.*

This lesson has two parts:

Part I: Eigenvectors and Eigenvalues

Part II: Power Method

There are ten homework problems.

Part I: Eigenvectors and Eigenvalues

Begin by reading what an eigenvalue and an eigenvector is right here:

Given a square $n \times n$ matrix A

$\vec{v}$ is an eigenvector of A with eigenvalue λ

if $A\vec{v} = \lambda\vec{v}$

our
 $n \times n$
matrix

vector
in $\mathbb{R}^n$

real
number

$\vec{v}$ cannot be the $\vec{0}$ vector. λ can be zero.

An eigenvector can never be the zero vector, but an eigenvalue can be zero.

Classwork ①

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

Check that $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

is an eigenvector

and find its eigenvalue.

Solution:

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3(1) + 1(0) \\ 0(1) + 2(0) \end{pmatrix} = \begin{pmatrix} 3+0 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda(1) \\ \lambda(0) \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \leftarrow \text{so } \lambda = 3$$

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Classwork ②

$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ Check that $\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector and find its eigenvalue.

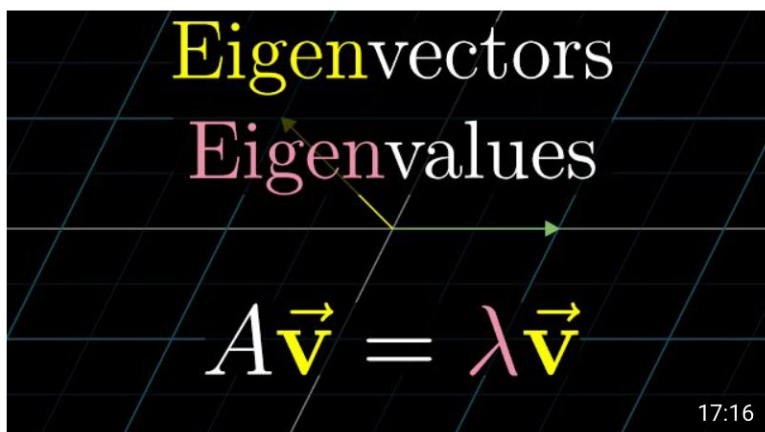
Solution:

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(-1) + 1(1) \\ 0(-1) + 2(1) \end{pmatrix} = \begin{pmatrix} -3 + 1 \\ 0 + 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda(-1) \\ \lambda(1) \end{pmatrix} = \begin{pmatrix} -\lambda \\ \lambda \end{pmatrix} \leftarrow \text{so } \lambda = 2$$

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Below we see this example in a three blue one brown video:

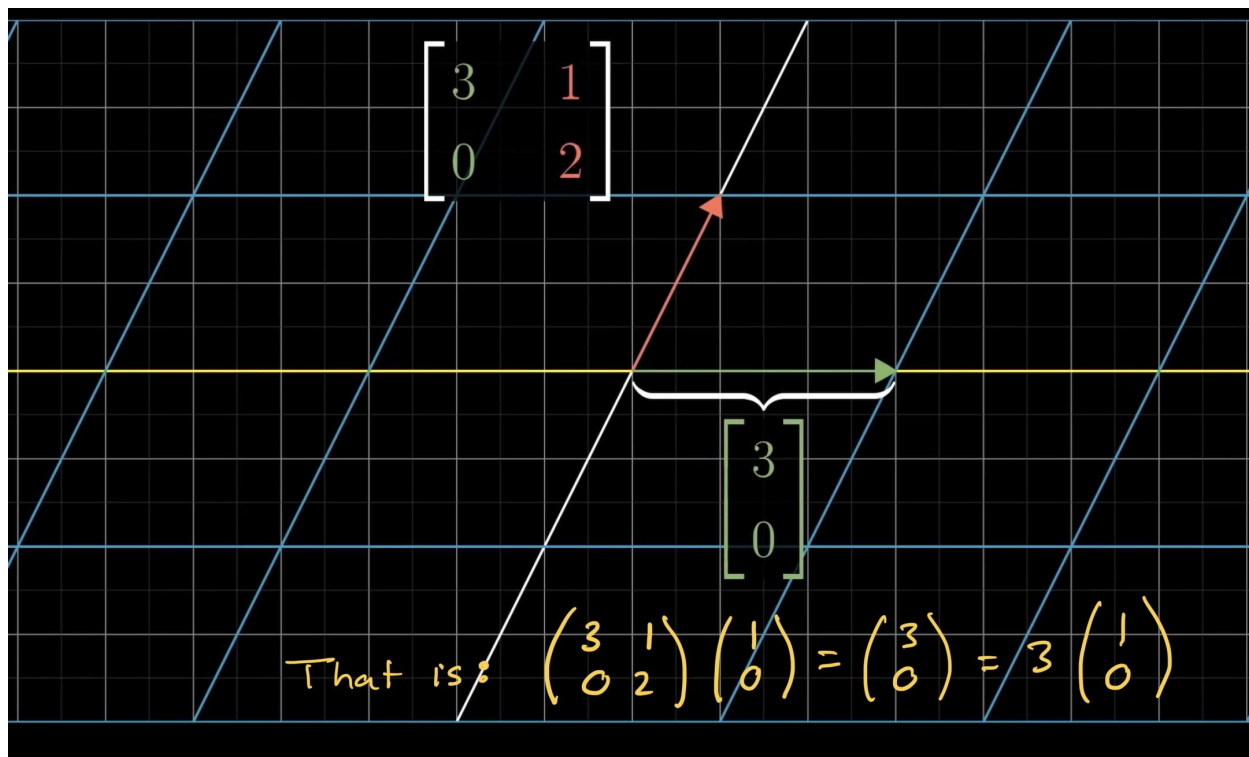
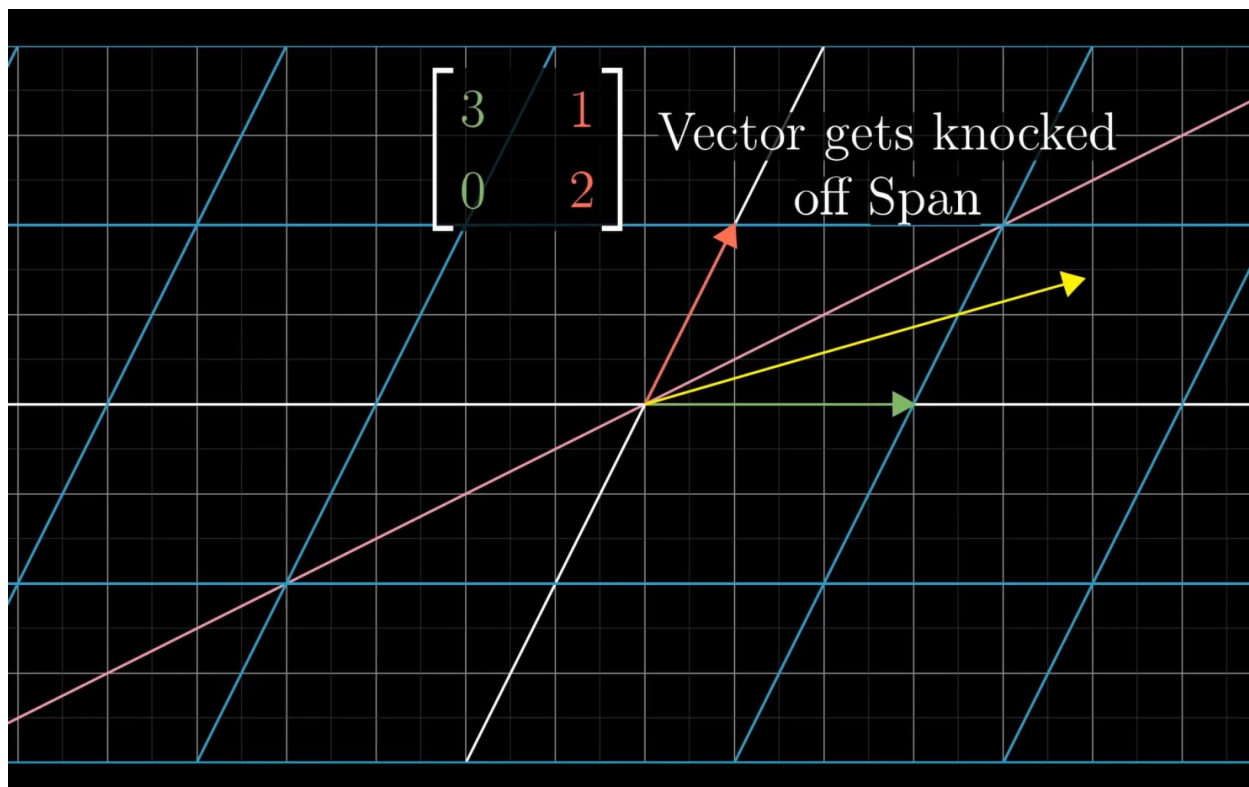


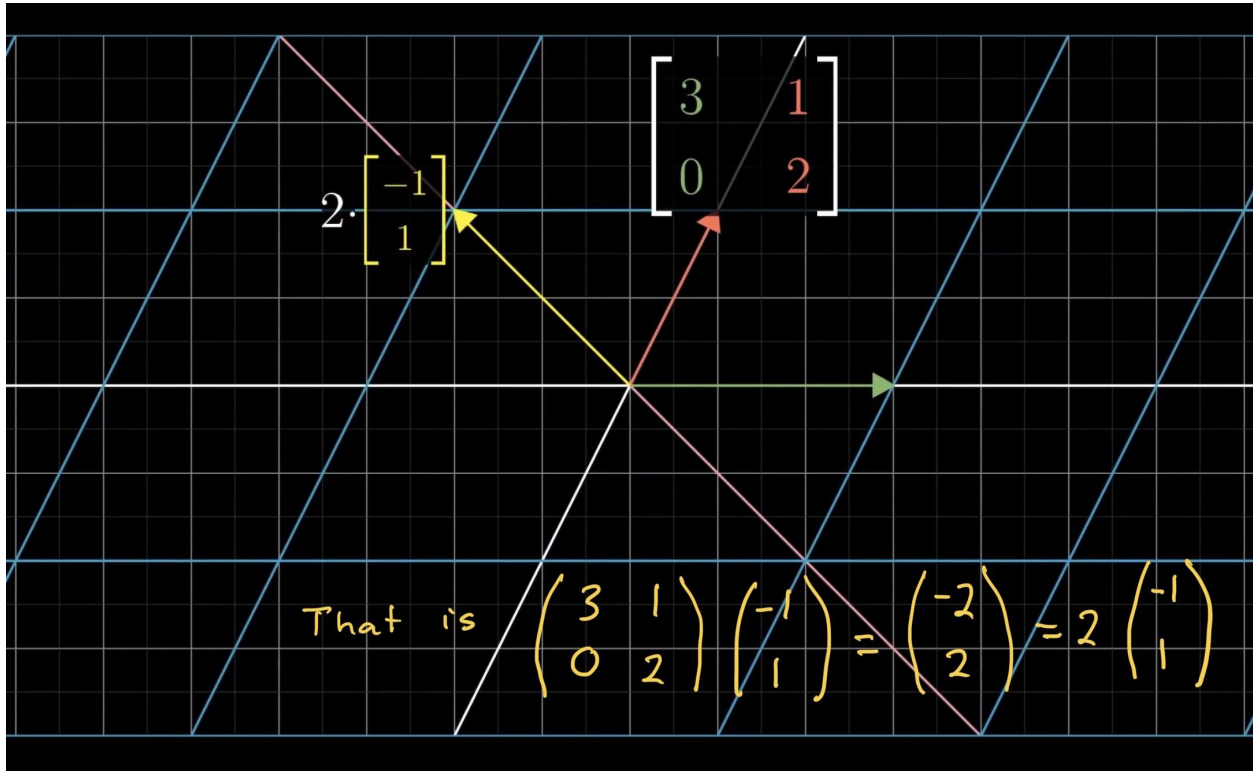
Eigenvectors and eigenvalues |

3.3M views · 6 years ago



3Blue1Brown





Please watch **only the first five minutes** of the [Three Blue One Brown video](#) on eigenvalues and eigenvectors where the above images are animated. **Do not watch the whole video** as it covers more advanced topics we will learn next month.

HW1 Show the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}$

has three eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and find the eigenvalues

HW2 Check your answer has $\lambda_1=5, \lambda_2=-2, \lambda_3=6$

Ask me for help if this did not work.

Do not ask a tutor for help with this. The tutor will show you extra complicated things you do not need yet. You will learn them later. Just practice multiplying the matrix times each eigenvector.

Solution to HW1 to check your answers

Try First!

HW1 Show the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}$

has three eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and find the eigenvalues

Check your answer has $\lambda_1=5, \lambda_2=-2, \lambda_3=6$

Ask me for help if this did not work.

Do not ask a tutor for help with this. The tutor will show you extra complicated things you do not need yet. You will learn them later. Just practice multiplying the matrix times each eigenvector.

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 + 4 \cdot 0 \\ 0 \cdot 1 + 4 \cdot 0 + 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \cdot 0 + 0 \cdot 1 + (-1) \cdot 0 \\ 5 \cdot 0 + 2 \cdot 1 + (-1) \cdot 4 \\ 5 \cdot 0 + 4 \cdot 1 + (-1) \cdot 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 - 4 \\ 4 - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 \\ 5 \cdot 0 + 2 \cdot 1 + 1 \cdot 4 \\ 5 \cdot 0 + 4 \cdot 1 + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 + 4 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

If your matrix multiplication is incorrect, go back to the previous lesson and review it.

Now try the following classwork:

Classwork ③

Consider a diagonal matrix $D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$

What is $D\hat{i}$? Is $\hat{i}$ an eigenvector?

What is $D\hat{j}$? Is $\hat{j}$ an eigenvector?

What is $D\hat{k}$? Is $\hat{k}$ an eigenvector?

Solution: Remember $\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

What is $D\hat{i}$? Is $\hat{i}$ an eigenvector?

$$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{11} \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + d_{22} \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 + d_{33} \cdot 0 \end{pmatrix} = \begin{pmatrix} d_{11} \\ 0 \\ 0 \end{pmatrix} = d_{11} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So yes $\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue $= d_{11}$

What is $D\hat{j}$? Is $\hat{j}$ an eigenvector?

You do the work ...

What is $D\hat{k}$? Is $\hat{k}$ an eigenvector?

You do the work ...

Theorem: A 3×3 diagonal matrix D

has $D\hat{i} = d_{11}\hat{i}$ $D\hat{j} = d_{22}\hat{j}$ and $D\hat{k} = d_{33}\hat{k}$

Definition: $A^2 \vec{v} = A \underbrace{A \vec{v}}$
 first find this vector
 then multiply by A again

Example

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 5 \\ 6 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 0 \cdot 5 + 1 \cdot 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 17 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 17 + 2 \cdot 6 \\ 0 \cdot 17 + 1 \cdot 6 \end{pmatrix} = \begin{pmatrix} 17 + 12 \\ 6 \end{pmatrix} = \begin{pmatrix} 29 \\ 6 \end{pmatrix} \end{aligned}$$

Definition: $A^k \vec{v} = \underbrace{A \times A \times \dots \times A}_{k \text{ times}} \vec{v}$

Example:

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^5 \begin{pmatrix} 5 \\ 6 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 17 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 29 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \cdot 29 + 2 \cdot 6 \\ 0 \cdot 29 + 1 \cdot 6 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 41 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \cdot 41 + 2 \cdot 6 \\ 0 \cdot 41 + 1 \cdot 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 53 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot 53 + 2 \cdot 6 \\ 0 \cdot 53 + 1 \cdot 6 \end{pmatrix} = \begin{pmatrix} 65 \\ 6 \end{pmatrix}$$

Note that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is a skew matrix

and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^5$ means skew five times

so the x term $5 \rightarrow 17 \rightarrow 29 \rightarrow 41 \rightarrow 53 \rightarrow 65$

but the y term did not change $6 \rightarrow \dots \rightarrow 6$

Classwork (5) $D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \quad \vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Find $D^2 \vec{v}$ and $D^3 \vec{v}$ and $D^4 \vec{v}$

What is $D^k \vec{v}$?

Solution:

$$D^2 \vec{v} = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} d_{11} a \\ d_{22} b \\ d_{33} c \end{pmatrix} = \begin{pmatrix} d_{11} d_{11} a \\ d_{22} d_{22} b \\ d_{33} d_{33} c \end{pmatrix} = \begin{pmatrix} d_{11}^2 a \\ d_{22}^2 b \\ d_{33}^2 c \end{pmatrix}$$

$$D^3 \vec{v} = D (D^2 \vec{v})$$

$$= \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} d_{11}^2 a \\ d_{22}^2 b \\ d_{33}^2 c \end{pmatrix} = \begin{pmatrix} d_{11}^3 a \\ d_{22}^3 b \\ d_{33}^3 c \end{pmatrix}$$

Similarly $D^4 \vec{v} = \begin{pmatrix} d_{11}^4 a \\ d_{22}^4 b \\ d_{33}^4 c \end{pmatrix}$

And $D^k \vec{v} = \begin{pmatrix} d_{11}^k a \\ d_{22}^k b \\ d_{33}^k c \end{pmatrix}$

This is a very special property
of diagonal matrices

We can do something similar if we know the eigenvectors of a matrix:
First recall this theorem from Lesson 9:

Thm: $A(k\vec{v}) = k(A\vec{v})$
 for all matrices $A \in M_{n \times m}$
 all real numbers $k \in \mathbb{R}$
 and all vectors $\vec{v} \in \mathbb{R}^m$

We can use it to prove a theorem about eigenvalues:

Theorem: If $A\vec{v} = \lambda\vec{v}$
 then $A^2\vec{v} = \lambda^2\vec{v}$

Proof: ① $A^2\vec{v} = A \cdot A\vec{v}$ ① by defn of $A^2\vec{v}$
 ② $= A \cdot \lambda\vec{v}$ ② by given
 ③ $= \lambda A\vec{v}$ ③ by $A(k\vec{v}) = kA\vec{v}$ from Lesson 9
 ④ $= \lambda \lambda\vec{v}$ ④ by given
 ⑤ $= \lambda^2\vec{v}$ ⑤ by defn of λ^2

QED

HW 3 Prove that

$$A\vec{v} = \lambda\vec{v} \implies A^3\vec{v} = \lambda^3\vec{v}$$

Theorem: $A\vec{v} = \lambda\vec{v} \implies A^k \vec{v} = \lambda^k \vec{v}$

So if we want to find $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^5 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

We can use classwork ①: $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\text{So } \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^5 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2^5 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2^5 \\ 2^5 \end{pmatrix} = \begin{pmatrix} -32 \\ 32 \end{pmatrix}$$

HW 4 Find $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^6 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ using this method.

Another Theorem from Lesson 9:

313F22-Lesson9

⊕

$$\underline{\text{Thm:}} \quad A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} \\ \forall \text{ matrices } A \in M_{n \times m} \text{ and } \vec{v}, \vec{w} \in \mathbb{R}^m$$

We can use it to prove a theorem about eigenvectors:

Theorem If $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

$$\text{Then } A(c_1 \vec{v}_1 + c_2 \vec{v}_2) = \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2$$

Proof:

$$\textcircled{1} A(c_1 \vec{v}_1 + c_2 \vec{v}_2) = A c_1 \vec{v}_1 + A c_2 \vec{v}_2 \quad \textcircled{1} \text{ by } A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$$

$$\textcircled{2} = c_1 A \vec{v}_1 + c_2 A \vec{v}_2 \quad \textcircled{2} \text{ by } A k \vec{v} = k A \vec{v}$$

$$\textcircled{3} = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 \quad \textcircled{3} \text{ by given}$$

$$\textcircled{4} = \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2 \quad \textcircled{4} \text{ by } ab = ba \text{ for reals}$$

QED

HW5 Prove that:

$$\text{If } A\vec{v}_1 = \lambda_1 \vec{v}_1 \text{ and } A\vec{v}_2 = \lambda_2 \vec{v}_2$$

$$\text{Then } A^2(c_1 \vec{v}_1 + c_2 \vec{v}_2) = \lambda_1^2 c_1 \vec{v}_1 + \lambda_2^2 c_2 \vec{v}_2$$

Theorem If $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

$$\text{Then } A^k(c_1 \vec{v}_1 + c_2 \vec{v}_2) = \lambda_1^k c_1 \vec{v}_1 + \lambda_2^k c_2 \vec{v}_2$$

Classwork: Use this theorem to

$$\text{find } \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{10} \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Warning $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is not an eigenvector!

First we recall the eigenvalues we found in Classwork ①

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To use the thm we need c_1 and c_2 : $\begin{pmatrix} 5 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

This is a system:

$$\begin{cases} -1c_1 + 0c_2 = 5 \\ 1c_1 + 1c_2 = 6 \end{cases}$$

$$\left[\begin{array}{cc|c} -1 & 0 & 5 \\ 1 & 1 & 6 \end{array} \right] \xrightarrow{P_1 \rightarrow -P_1} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 1 & 1 & 6 \end{array} \right]$$

$$\xrightarrow{P_2 \rightarrow P_2 - P_1} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 11 \end{array} \right] \quad \begin{cases} c_1 = -5 \\ c_2 = 11 \end{cases}$$

$$\text{Check: } -5 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 11 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} + \begin{pmatrix} 0 \\ 11 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

So now

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{10} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{10} \left(-5 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 11 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$2^{10} = ?$$

$$2^1 = 2$$

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

$$2^6 = 64$$

$$2^7 = 128$$

$$2^8 = 256$$

$$2^9 = 512$$

$$2^{10} = 1024$$

$$= 2^{10}(-5) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 2^{10} \cdot 11 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 1024(-5) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 1024(11) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5120 \\ -5120 \end{pmatrix} + \begin{pmatrix} 0 \\ 11264 \end{pmatrix}$$

$$= \begin{pmatrix} 5120 \\ 6144 \end{pmatrix}$$

This may seem like a lot of work but is easier than doing it the long way.

It is almost as easy as a diagonal matrix.

Check your answer has $\lambda_1=5$, $\lambda_2=-2$, $\lambda_3=6$

Ask me for help if this did not work.

HW6 Use your work from **HW1**

where you showed

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

to find

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^3 \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$$

Hint: find c_1 and c_2 s.t.

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$$

HW6 Use your work from **HW1**
where you showed
 $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
to find $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^3 \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$ Hint: find c_1 and c_2 s.t.
 $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$

Solution $\begin{cases} c_1 + c_2 = 8 \\ 0 \cdot c_1 + c_2 = 6 \\ 0 \cdot c_1 + c_2 = 6 \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = 6 \end{cases}$

$$\begin{aligned} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^3 \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix} &= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^3 \left(2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \\ &= 8 \cdot 5^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 6 \cdot 6^3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= 8 \cdot 125 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 6 \cdot 216 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= 1000 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1296 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1000 \\ 1296 \\ 1296 \end{pmatrix} \end{aligned}$$

After you
do HW6
check your
answer
here

$$\begin{array}{r} 3 \\ 216 \\ \times 6 \\ \hline 1296 \end{array}$$

Theorem Suppose $A \in \mathbb{R}^{n \times n}$

has $A \vec{v}_i = \lambda_i \vec{v}_i$ for $i=1$ to n

then $A^k \left(\sum_{i=1}^n c_i \vec{v}_i \right) = \sum_{i=1}^n c_i \lambda_i^k \vec{v}_i$

and if you can solve the system

$$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{w}$$

then $A^k \vec{w} = c_1 \lambda_1^k \vec{v}_1 + \dots + c_n \lambda_n^k \vec{v}_n$

Theorem Suppose $A \in \mathbb{R}^{n \times n}$

has $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i=1$ to n

then $A^k \left(\sum_{i=1}^n c_i \vec{v}_i \right) = \sum_{i=1}^n c_i \lambda_i^k \vec{v}_i$

and if you can solve the system

$$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{w}$$

then $A^k \vec{w} = c_1 \lambda_1^k \vec{v}_1 + \dots + c_n \lambda_n^k \vec{v}_n$

This is the same as saying the matrix whose columns are $\vec{v}_1, \dots, \vec{v}_n$ is a nonsingular matrix

Note The 3×3 matrix in **HW1**

is especially nice because you

can always solve the system

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^k \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

We know this because

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \omega_1 \\ 0 & 1 & 1 & \omega_2 \\ 0 & -1 & 1 & \omega_3 \end{array} \right) \xrightarrow{p_3 \rightarrow p_3 + p_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \omega_1 \\ 0 & 1 & 1 & \omega_2 \\ 0 & 0 & 2 & \omega_2 + \omega_3 \end{array} \right)$$

$$\xrightarrow{p_3 \rightarrow \frac{1}{2}p_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \omega_1 \\ 0 & 1 & 1 & \omega_2 \\ 0 & 0 & 1 & (\omega_2 + \omega_3)/2 \end{array} \right)$$

$$\xrightarrow{p_2 \rightarrow p_2 - p_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \omega_1 \\ 0 & 1 & 0 & \omega_2 - (\omega_2 + \omega_3)/2 \\ 0 & 0 & 1 & (\omega_2 + \omega_3)/2 \end{array} \right)$$

always has the solution

$$c_1 = \omega_1$$

$$c_2 = \omega_2 - (\omega_2 + \omega_3)/2$$

$$c_3 = (\omega_2 + \omega_3)/2$$

So

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^k \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^k \left(c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$= c_1 (5)^k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 (-2)^k \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 (6)^k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

can always be found for any $\vec{\omega}$.

Warning

Not all matrices have real eigenvalues and eigenvectors!

Classwork:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ rotation by } 90^\circ$$

Let us show A has no eigenvectors!

$$\text{Try to solve } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{We get } \begin{pmatrix} 0x - 1y \\ 1x + 0y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

$$\text{So } \begin{matrix} -y = \lambda x \\ x = \lambda y \end{matrix} \Rightarrow \begin{matrix} y = -\lambda x \\ x = \lambda y \end{matrix}$$

$$\text{Then } x = \lambda y = \lambda(-\lambda x)$$

$$\text{So } x = -\lambda^2 x$$

$$\text{So } -\lambda^2 = 1 \Rightarrow \lambda^2 = -1$$

There is no real number λ that works!

Complex numbers can be used

$$\lambda^2 = -1 \Rightarrow \lambda = \pm i$$

For $\lambda = i$ we get $x = iy$

so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ might be an eigenvector

Check it:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot i + (-1) \cdot 1 \\ 1 \cdot i + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix}$$

$$i \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i \cdot i \\ i \cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} \leftarrow \text{match!}$$

$$\text{so } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} i \\ 1 \end{pmatrix}$$

HW7

Show the eigenvector for $\lambda = -i$ is $\begin{pmatrix} i \\ -1 \end{pmatrix}$

If you think about the three blue one brown video you watched, you will remember that eigenvectors describe directions that are not changed under the linear transformation. If the linear transformation is a rotation by 90 degrees, then all directions are changed. This is why the eigenvalues and eigenvectors are not real.

If you wish, watch **the first five minutes** of the [Three Blue One Brown video](#) a second time. There are many many applications of eigenvectors and eigenvalues and it is good to have an understanding how they describe linear transformations. We will learn the material covered in the rest of the video later.

Extra Credit: Prove that if v is an eigenvector for a matrix A , and you rescale v by a real number r , then rv is also an eigenvector of A with the same eigenvalue.

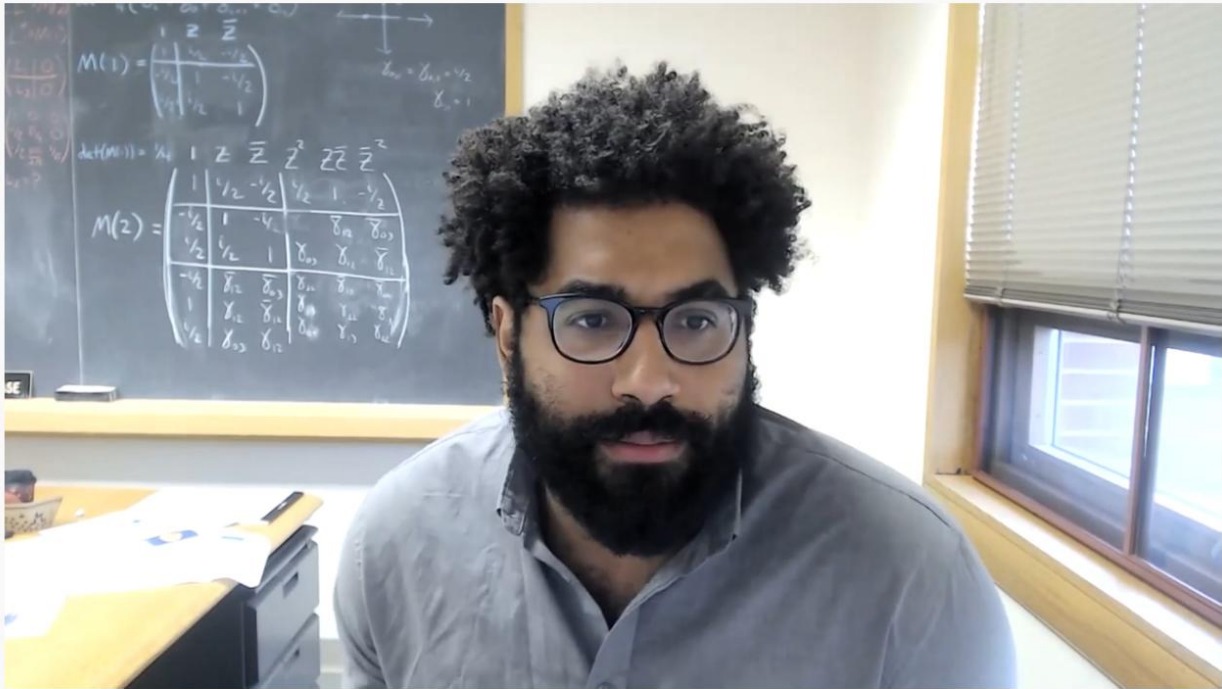
Part II: Methods to find Eigenvalues and Eigenvectors

There are various techniques for finding the eigenvalues and eigenvectors of matrices. For large matrices, algorithms have been developed and coded into computers. See for example the [wikipedia article about eigenvalue algorithms](#) which you may consult if one comes up in a future course or in the workplace.

We will learn [Power Iteration](#) today.

Another method is to use the [Characteristic Polynomial](#) which we will learn next month.

A third is the [Jacobi Eigenvalue Algorithm](#). We will have a guest speaker, Dr. Urschel, present this method:



Jacobi's Eigenvalue Algorithm. An undergrad talk for Lehman College.

469 views • Sep 23, 2021



16



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John Urschel

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His talk is a little too advanced for the class to follow yet, so we will wait until later in the semester to include it in the course.

So let's learn power iteration now from Dr. Panagos:



Eigenvalue Power Method Example #1

Linear Algebra Example Problems

Dr. Adam Panagos

<http://www.adampanagos.org>

[Linear Algebra Example Problems](#)

The Eigenvalue Power Method Example #1 - Linear Algebra Example Problems

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Watch Dr. Panagos on youtube: <https://youtu.be/yBiQh1vsCLU>

Example #1 $A = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix}$ $\lambda_1 = 16$
 $\lambda_2 = -2$

① $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

② $k=0$
 $p = Ax_0 = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$

$n_0 = 9$

$x_1 = \begin{bmatrix} 0.77 \\ 1 \end{bmatrix}$

$k=1$
 $p = Ax_1 = \begin{bmatrix} 14.39 \\ 13.93 \end{bmatrix}$

$n_1 = 14.39$

$x_2 = \begin{bmatrix} 1.00 \\ 0.97 \end{bmatrix}$

$k=2$
 $p = Ax_2 = \begin{bmatrix} 15.73 \\ 15.79 \end{bmatrix}$

$n_2 = 15.79$

$x_3 = \begin{bmatrix} 0.996 \\ 1 \end{bmatrix}$

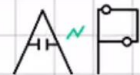
$k=3$

$p = Ax_3 = \begin{bmatrix} 15.972 \\ 15.964 \end{bmatrix}$

$n_3 = 15.972$

$x_4 = \begin{bmatrix} 1 \\ 0.999 \end{bmatrix}$

③ $n \rightarrow 16 = \lambda_1$



In fact we can adapt the power method iteratively to find all the eigenvalues of a good matrix, but we will learn this later.

When does the eigenvalue
power method work?

It works if the matrix $A \in \mathbb{R}^{n \times n}$
has $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i=1$ to n
such that the system

$$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{w}$$

can always be solved

The power method is using

$$A^k \vec{w} = c_1 \lambda_1^k \vec{v}_1 + \dots + c_n \lambda_n^k \vec{v}_n$$

If $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$

Then $c_1 \lambda_1^k \vec{v}_1$ gets much larger
than the other terms.

The division in the algorithm
keeps it away from $\rightarrow \infty$.

We won't prove this but
you can think about it.

HW8 Use the power method
to find the largest eigenvalue
of $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ starting with $\vec{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Example: $A = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix}$ $\lambda_1 = 16$
 $\lambda_2 = -2$

① $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

② $k=0$
 $p = Ax_0 = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$
 $n_0 = 9$
 $\tilde{x}_1 = \begin{bmatrix} 0.77 \\ 1 \end{bmatrix}$

$k=1$
 $p = Ax_1 = \begin{bmatrix} 14.39 \\ 15.93 \end{bmatrix}$
 $n_1 = 15.93$
 $\tilde{x}_2 = \begin{bmatrix} 0.90 \\ 0.97 \end{bmatrix}$

$k=2$
 $p = Ax_2 = \begin{bmatrix} 15.79 \\ 15.79 \end{bmatrix}$
 $n_2 = 15.79$
 $\tilde{x}_3 = \begin{bmatrix} 0.996 \\ 1 \end{bmatrix}$

$k=3$
 $p = Ax_3 = \begin{bmatrix} 15.972 \\ 15.944 \end{bmatrix}$
 $n_3 = 15.972$
 $\tilde{x}_4 = \begin{bmatrix} 0.999 \\ 1 \end{bmatrix}$

③ $n \rightarrow 16 = \lambda_1$

Hint for HW8

follow this method

Solution for HW8
 is below to
 check your work

HW8 Use the power method
 to find the largest eigenvalue
 of $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ starting with $x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

① $x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

② $k=0$

$Ax_0 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 + 0 \cdot 2 \\ 0 \cdot 1 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
 n_0 is the largest number here
 so $n_0 = 4$
 $\tilde{x}_1 = \frac{1}{n_0} Ax_0 = \frac{1}{4} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$

$k=1$
 $Ax_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(3/4) + 0 \cdot 1 \\ 0(3/4) + 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 9/4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.25 \\ 2 \end{pmatrix}$
 n_1 is the largest number here
 so $n_1 = 2.25$
 $\tilde{x}_2 = \frac{1}{n_1} Ax_1 = \frac{1}{2.25} \begin{pmatrix} 2.25 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 8/9 \end{pmatrix}$

$k=2$
 $Ax_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 8/9 \end{pmatrix} = \begin{pmatrix} 3 \\ 16/9 \end{pmatrix}$
 $n_2 = 3$
 $\tilde{x}_3 = \frac{1}{n_2} Ax_2 = \frac{1}{3} \begin{pmatrix} 3 \\ 16/9 \end{pmatrix} = \begin{pmatrix} 1 \\ 16/27 \end{pmatrix}$

$k=3$
 $Ax_3 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 16/27 \end{pmatrix} = \begin{pmatrix} 3 \\ 32/27 \end{pmatrix}$
 $n_3 = 3$
 $\tilde{x}_4 = \frac{1}{n_3} Ax_3 = \begin{pmatrix} 1 \\ 32/81 \end{pmatrix}$
 same as n_2 so 3 is the eigenvalue

$k=4$
 $Ax_4 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 32/81 \end{pmatrix} = \begin{pmatrix} 3 \\ 64/81 \end{pmatrix}$
 $n_4 = 3$
 $\tilde{x}_5 = \frac{1}{n_4} Ax_4 = \frac{1}{3} \begin{pmatrix} 3 \\ 64/81 \end{pmatrix} = \begin{pmatrix} 1 \\ 64/243 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 21/81 \end{pmatrix}$

I approximated here

$k=5$
 $Ax_5 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 21/81 \end{pmatrix} = \begin{pmatrix} 3 \\ 42/81 \end{pmatrix}$
 $n_5 = 3$
 $\tilde{x}_6 = \frac{1}{n_5} Ax_5 = \begin{pmatrix} 1 \\ 14/81 \end{pmatrix}$

Notice $\tilde{x}_k = \begin{pmatrix} 1 \\ \text{something small} \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $k \rightarrow \infty$

check $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \checkmark$

Why does HW8 work?

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \left(1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 1 \cdot 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$n_1 = 4$

$$\text{So } \vec{x}_1 = \frac{1}{4} \left(1 \cdot 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{4} \left(1 \cdot 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= \frac{1}{4} \left(1 \cdot 3^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot 2^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 9/4 \\ 2 \end{pmatrix}$$

$n_2 = \frac{9}{4}$

$$\text{So } \vec{x}_2 = \left(\frac{4}{9} \right) \left(\frac{1}{4} \right) \left(1 \cdot 3^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot 2^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}_2 = \frac{1}{9} \left(1 \cdot 3^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot 2^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 3 \\ 2^4/9 \end{pmatrix}$$

$n_3 = 3$

$$\text{So } \vec{x}_3 = \frac{1}{3} \frac{1}{9} \left(1 \cdot 3^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot 2^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

...

$$\text{So } \vec{x}_k = \frac{1}{3^k} \left(1 \cdot 3^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot 2^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2 \cdot 2^k}{3^k} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as $k \rightarrow \infty$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

first eigenvector dominates

HW 9 Use the power method to find the largest eigenvalue of $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ starting with $\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

HW 10 What goes wrong if you try to use the power method to find the largest eigenvalue of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ starting with any vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

HW 10 What goes wrong if you try to use the power method to find the largest eigenvalue of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ starting with any vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution to HW 10

Try First!

$$\begin{aligned}
 \vec{x}_0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & k=1 & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & n_1 &= 1 & \vec{x}_1 &= \frac{1}{1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 & & k=2 & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} & n_2 &= |-1| = 1 & \vec{x}_2 &= \frac{1}{1} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
 & & k=3 & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & n_3 &= |1| = 1 & \vec{x}_3 &= \frac{1}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
 \end{aligned}$$

It seems may be the eigenvalue is 1?

$$\begin{aligned}
 k=4 & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow \text{WAIT!} \\
 & n_4 = 1 & \vec{x}_4 &= \frac{1}{1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
 \end{aligned}$$

We are back to $\vec{x}_0$!!!

No eigenvector!

Once you have completed this lesson, email asking me to check your work and you may practice the [Sample Exam 2](#).