Ian Stewart writes in his book “Letters to a young mathematician”,

“…A proof is a story. It is a story told by mathematicians to mathematicians, expressed in their common language. It has a beginning (the hypothesis) and an end (the conclusion), and it falls apart instantly if there are any logical gaps.”

I have not yet encountered a proof which I can describe as my favorite but I understood a proof of a theorem from L.V. Tarasov’s “Calculus” book: **how to prove that “a” is a limit of a sequence.**

And I am going to prove that the sequence $y_n = \frac{n}{n+1}$ has limit 1.

Definition of limit:

The number a is said to be the limit of the sequence if for any positive number is there is a real number N such that for all n > N the following inequality holds: $|y_n - a| < \varepsilon$

Tarasov write further,

“Don't hasten to remember. Try to comprehend this definition, to realize its structure and its inner logic. You will see that every word in this phrase carries a definite and necessary content and that no other definition of the limit of sequence could be more succinct (more delicate, even).”

And then he describes a beautiful and clear explanation of this definition.
So here I am going to retell the story I have read,

The sequence is,

$$y_n = \frac{n}{n+1}$$

We have to prove that its limit is 1.
Now from the definition let’s choose $\varepsilon > 0$
Let’s take, $\varepsilon = 0.1$
If a=1 is the limit of the sequence then, $|y_n - a| < \varepsilon$ should hold, so

$$|\frac{n}{n+1} - 1| < \varepsilon$$

$$|\frac{-1}{n+1}| < \varepsilon$$

$$\frac{1}{(n+1)} < \varepsilon$$

$$(n + 1) > \frac{1}{\varepsilon}$$

$$n > \frac{1}{\varepsilon} - 1$$

So we can write LHS using the definition as,

$$N = \frac{1}{\varepsilon} - 1$$

Now if $\varepsilon = 0.1$,

$$N = \frac{1}{0.1} - 1 = 9$$

And if $\varepsilon = 0.01$

$$N = \frac{1}{0.01} - 1 = 99$$

So it's proved that for every $\varepsilon > 0$ we can find N s.t. $\forall\, n > N,\ |y_n - a| < \varepsilon$

I can explain it clearly as follows,

For $\varepsilon = 0.1$ N=9
So n > 9
Let's take n=10 (Since n > N)

$$y_{10} = \frac{10}{10+1}$$

$$y_n = \frac{10}{11} = 0.909$$

$$|0.909 - 1| < 0.1$$

$$0.09 < 0.1$$

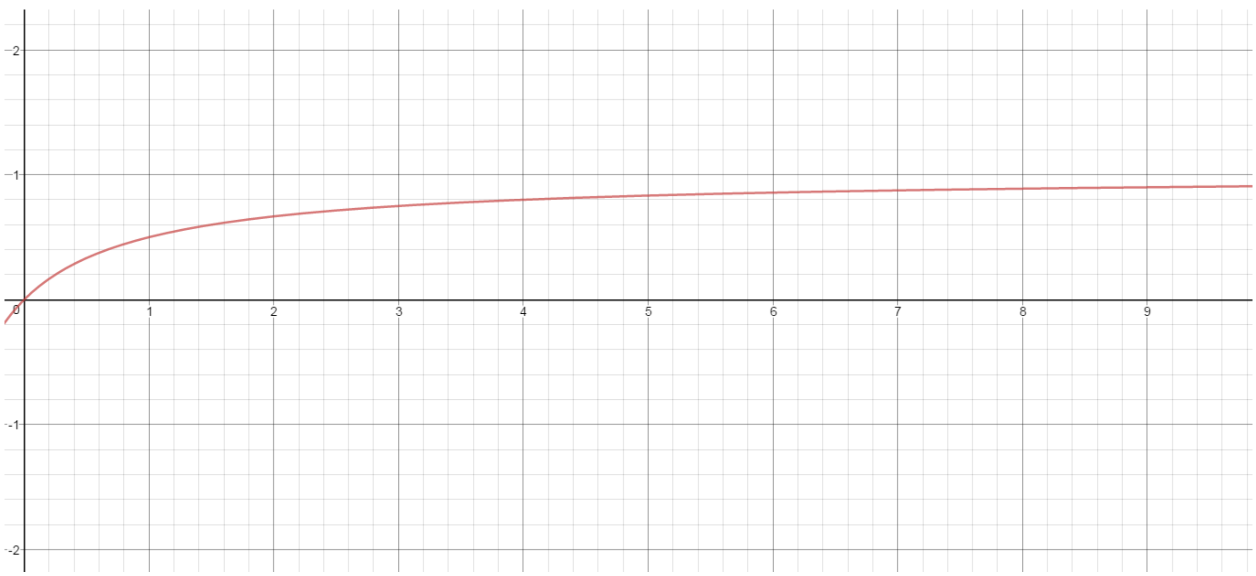
So it's proved that for every $\varepsilon > 0$ we can find N s.t. $\forall\, n > N,\ |y_n - a| < \varepsilon$
Or,

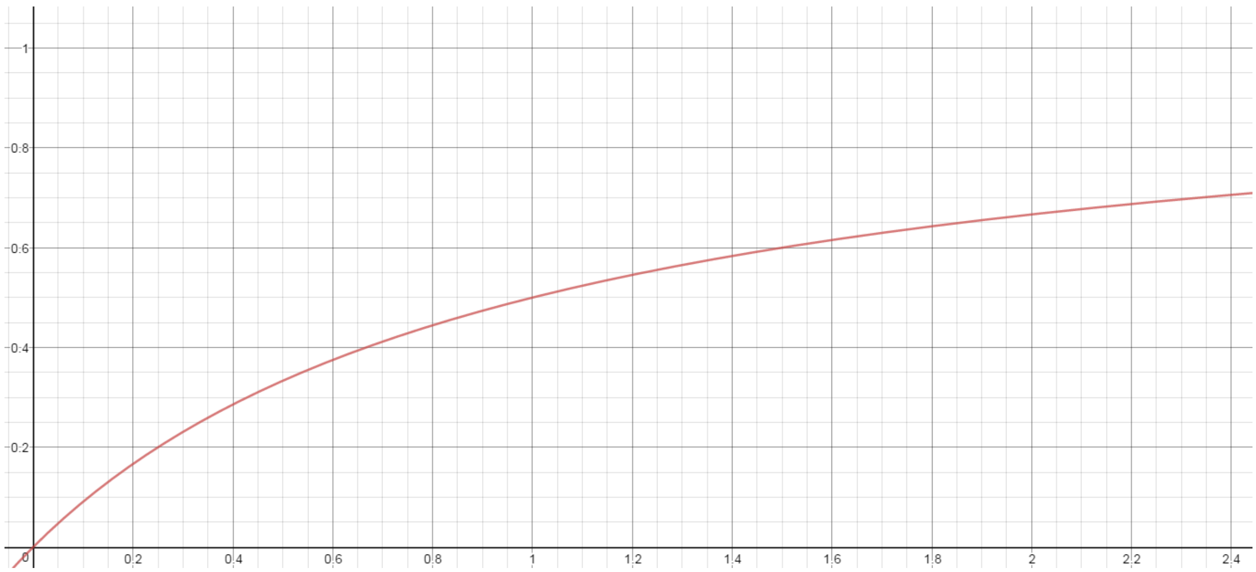
It's proved that for $\varepsilon = 0.1$ we found N=9 s.t. $\forall\, n > 9,\ |\frac{n}{n+1} - 1| < 0.1$

Generally in textbooks the following procedure is followed to prove limit of a sequence,

1. Evaluate $|y_n - a|$
2. Find N such that $\forall\, n > N,\ |y_n - a| < \varepsilon$
3. Check for arbitrary values of ε

On this website https://www.desmos.com/calculator/9o8vzqfblo you can see graph of the sequence and zoom in to see whether limit is actually 1. You will see that however far you look in the graph the line never exceeds 1. That's what means when we say that limit of the sequence is 1.





We can state the definition of limit in a different way,

If the limit of a sequence is "a" then for an arbitrary positive number $\varepsilon > 0$, in the interval $(a + \varepsilon)$ and $(a - \varepsilon)$ the inequality $|y_n - a| < \varepsilon, \forall\, n > N$ will always hold.

Let's take the interval $(1 + 0.1)$ and $(1 - 0.1)$, so in between 1.1 and 0.9 there should be some numbers of the sequence following inequality. Since $\varepsilon = 0.1$ and we have shown above that in this case $\forall\, n > 9$ the inequality $|y_n - a| < \varepsilon$ will hold. So it can be seen in the graph that after 9th term of the sequence all numbers fall inside the interval 1.1 and 0.9.

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