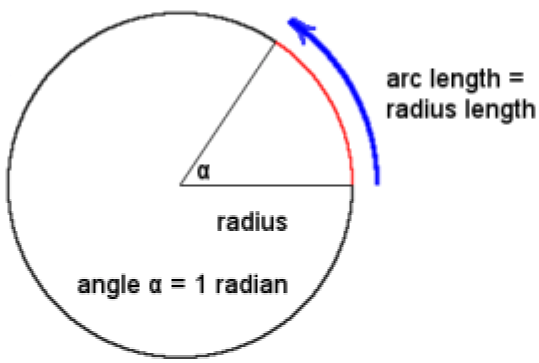
RAS Beirut public High School.
Class: Grade10-C.
Subject: Math.
Teacher: Mr. H. FAKIH.

Chapter 5: Trigonometric functions.

1-Definition of a radian

One radian is the angle made at the center of a circle by an arc whose length is equal to the radius of the circle (see the figure below).



2-Conversion Degrees ↔ Radians:

2-1 To convert from degrees to radians multiply by $\frac{\pi}{180}$.

2-2 To convert from radians to degrees multiply by $\frac{180}{\pi}$.

2-3Examples:

a) convert $15°$ *and* $315°$ *to radian.*

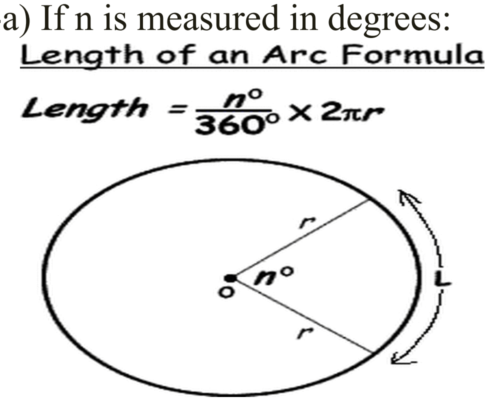
b) Convert $\frac{\pi}{5}$ *and* $\frac{3\pi}{8}$ to degrees.

3-Arc length vs arc measure:

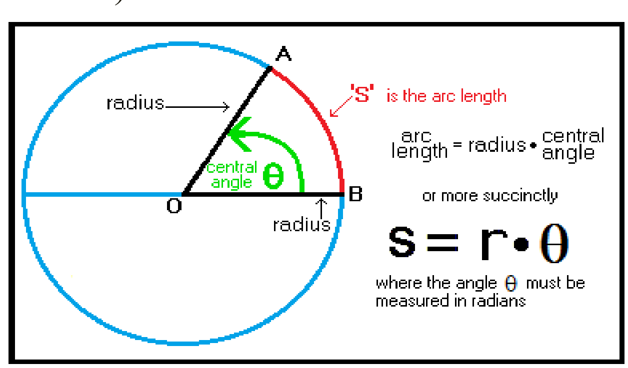
3-1)Arc measure: The measure (in degrees) of an arc is equal to the measure of the central angle that intercepts that arc.

3-2) Arc length: In addition to its measure, arcs have another property called arc length. And since it is a length, it may be given in centimeters, meters, inches, … or any unit of length. We distinguish two cases:

3-2-a) If n is measured in degrees:



3-2-b) If θ is measured in radians:



Note: When the arc length equals an entire circumference, we can use $s = r\times\theta$ to get $2\pi r = r\theta$ *and* $2\pi = \theta$ This implies that $2\pi = 360°$.

3-3)Example: How long is the arc subtended by an angle of $\frac{7\pi}{4}$ radians on a circle of radius 20 cm?

Answer: $S = r\times\theta = 20\times\frac{7\pi}{4} \approx 109.956\ cm$.

4-Trigonometric circle (unit circle):

The Unit Circle is a circle with its center at the origin (0,0) and a radius of one unit. Any angle has an initial ray and a terminal ray.

Angles are always measured counter-clockwise from the initial ray(here the positive x-axis) to the terminal ray. Angles measured counter-clockwise have *positive* values; angles measured clockwise have *negative* values

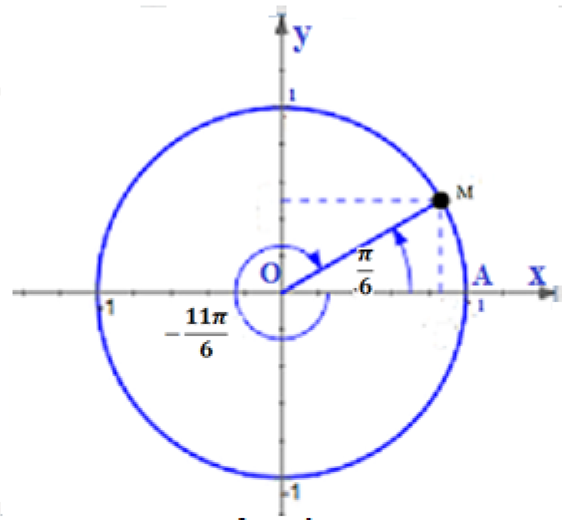
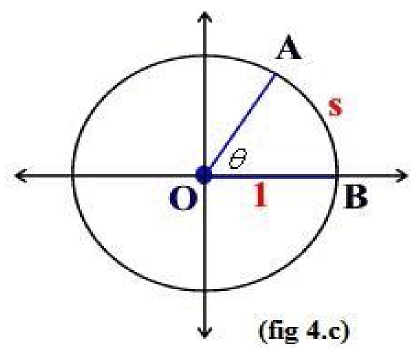


fig 4.b

In figure 4.b $\overset{\wedge}{AM} = \frac{\pi}{6}$ when rotating in the positive direction and $\overset{\wedge}{AM} = -\left(2\pi - \frac{\pi}{6}\right) = -\frac{11\pi}{6}$ when rotating in the negative direction

4-1) Note: When working in the unit circle, with radius 1, the length of the arc equals the radian measure of the angle. Since $S = r\times\theta = 1\times\theta = \theta$ so $S = \theta$.(see fig 4.c)



(fig 4.c)

Thus in figure 4.b we have $\overset{\wedge}{AM} = A\hat{O}M = \frac{\pi}{6}$.

And in figure 4.c $\overset{\wedge}{BA} = B\hat{O}A = \theta$.

4-2) Principal determination:

In figure 4.b we have:

$A\hat{O}M = \overset{\wedge}{AM} = \frac{\pi}{6}$ one can notice that

$\frac{\pi}{6} + 2\pi; \frac{\pi}{6} - 2\pi; \frac{\pi}{6} + 4\pi; \frac{\pi}{6} - 6\pi; \frac{\pi}{6} + 6\pi \ldots \frac{\pi}{6} + 2k\pi$ *where k is any integer*

all refer to the same point hence all are measures to the same arc that of initial point A and terminal point M, these values are obtained by performing a complete revolution(turn) several times either in the positive or negative orientation.

In general if θ is the measure in radians of the arc $\hat{AM}$ then $\theta + 2k\pi$ where k is any integer is another measure of the arc $\hat{AM}$ and we write:
$M(\theta) = M(\theta + 2\pi) = M(\theta + 4\pi) = M(\theta - 4\pi) = ... = M(\theta + 2k\pi)$ where k is any integer.
Among all these measures there is one who is called principal determination , and it belongs to the interval $]-\pi; +\pi]$.

4-3) Exercise : Find the principal determination of

$$\frac{13\pi}{6}; \frac{-7\pi}{4}; \frac{5\pi}{3}; \frac{32\pi}{7}; 225°; -927° \ and \ 3705°.$$

5-Trigonometric ratios:

5-1) Definitions:

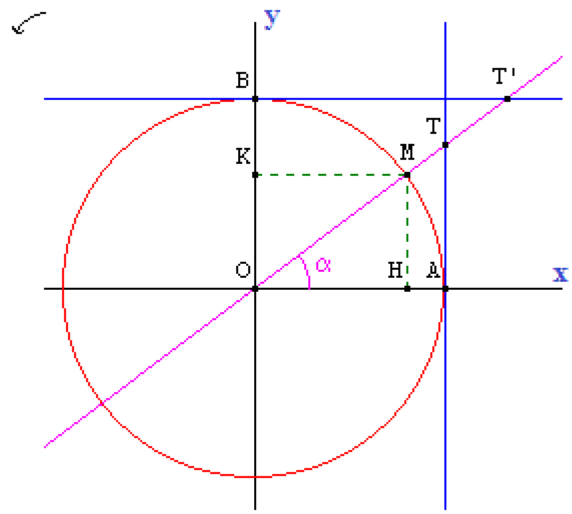
In the adjacent figure (C) is the trigonometric circle, M(x,y) a point on (C),and α the measure in radian of the central angle $\left(\vec{Ox}, \vec{OM}\right) = \hat{AM}$.
In the triangle OHM right angled at H we define:

$$\sin sin\ \alpha = \frac{opposite\ side}{hypotenuse} = \frac{\overline{HM}}{OM} = \frac{y}{1} = y.$$

$$.\cos cos\ \alpha = \frac{adjacent\ side}{hypotenuse} = \frac{\overline{OH}}{OM} = \frac{x}{1} = x.$$

$$.\tan tan\ \alpha = \frac{opposite\ side}{adjacent\ side} = \frac{\overline{HM}}{\overline{OH}} = \frac{sin\alpha}{cos\alpha} = \frac{y}{x}.$$

$$.cot\alpha = \frac{adjacent\ side}{opposite\ side} = \frac{\overline{OH}}{\overline{HM}} = \frac{cos\alpha}{sin\alpha} = \frac{1}{\tan tan\ \alpha}$$



Trigonometric circle OM=1

We can prove easily that $tan\alpha = \overline{AT}\ and\ cot\alpha = \overline{BT'}$.

5-2 Pythagorean relation: If $\hat{AM}$ =x, where x is a real number that represents the measure in ***radian*** of the arc $\hat{AM}$ we have:

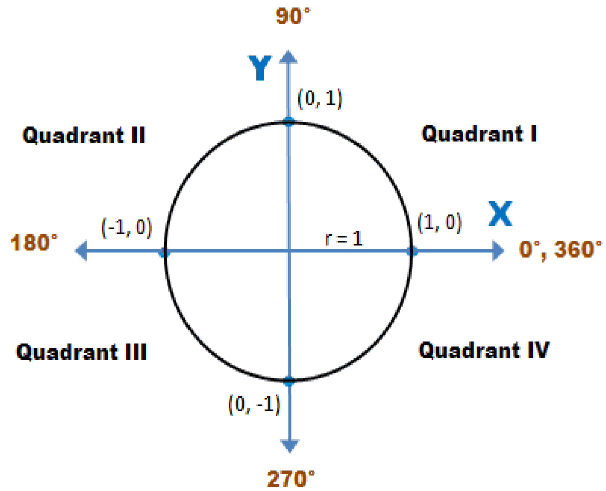
$$sin^2x + cos^2x = 1$$

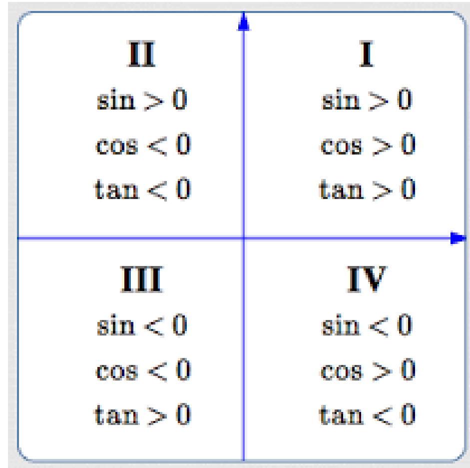
$$-1 \leq sinx \leq 1\ and\ -1 \leq cosx \leq 1.$$

5-3 Periodicity:

a) $\sin sin\ (x + 2k\pi) = sinx.$
b) $\cos cos\ (x + 2k\pi) = cosx.$
c) $\tan tan\ (x + k\pi) = \tan tan\ (x)$.
d) $\cot cot\ (x + k\pi) = \cot cot\ (x)$.

5-4 Sign of the trigonometric functions:





5-5

ind $cosx$ and $tanx$.

b) Given $cosx = \frac{-2}{3}$ and $\pi < x < \frac{3\pi}{2}$ Find $sinx$ and $tanx$.

c) Given $tanx = -2$ and $-\frac{\pi}{2} < x < 0$ Find $sinx$ and $cosx$.

5-6 Trigonometric functions of some special angles:

Degrees	0	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

5-7 Associated arcs:

$$\sin(-a) = -\sin a$$
$$\cos(-a) = \cos a$$
$$\tan(-a) = -\tan a$$

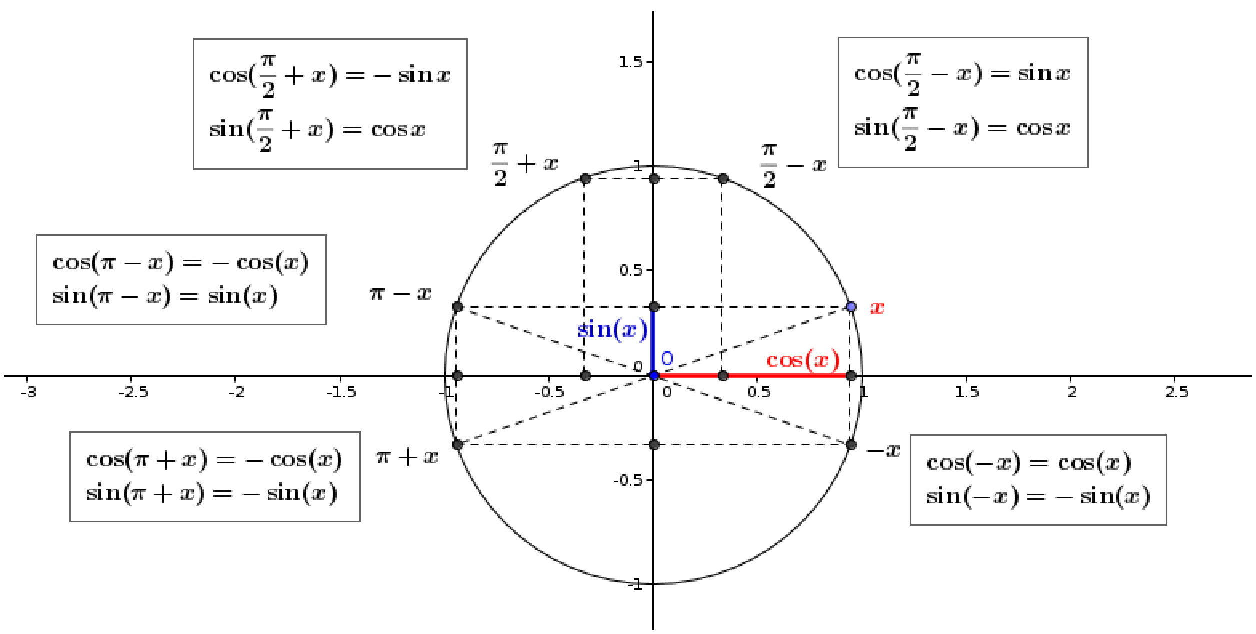
$$\sin(\pi - a) = \sin a$$
$$\cos(\pi - a) = -\cos a$$
$$\tan(\pi - a) = -\tan a$$

$$\sin(\pi + a) = -\sin a$$
$$\cos(\pi + a) = -\cos a$$
$$\tan(\pi + a) = \tan a$$

$$\sin\left(\frac{\pi}{2} - a\right) = \cos a$$

$$\sin\left(\frac{\pi}{2} + a\right) = \cos a$$

To memorize the above formulas you may use the diagram below:



5-8) Exercise: Calculate the following trigonometric ratios (Use the table of special angles and associated arcs):

a) $\cos cos\left(-\frac{\pi}{3}\right)$; $\sin sin\left(-\frac{\pi}{3}\right)$ *and* $\tan tan\left(-\frac{\pi}{3}\right)$.
b) $\cos cos\left(\frac{2\pi}{3}\right)$; $\sin sin\left(\frac{2\pi}{3}\right)$ *and* $\tan tan\left(\frac{2\pi}{3}\right)$.
c) $\cos cos\left(\frac{7\pi}{6}\right)$; $\sin sin\left(\frac{7\pi}{6}\right)$ *and* $\tan tan\left(\frac{7\pi}{6}\right)$.
d) $\cos cos\left(\frac{11\pi}{4}\right)$; $\sin sin\left(\frac{11\pi}{4}\right)$ *and* $\tan tan\left(\frac{11\pi}{4}\right)$.

5-9) Exercise: Given $cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$. Calculate:

$\sin sin\left(\frac{\pi}{5}\right)$; $cos\left(\frac{7\pi}{10}\right)$; $sin\left(-\frac{7\pi}{10}\right)$; $\cos cos\left(\frac{9\pi}{5}\right)$; $\tan tan\left(\frac{16\pi}{5}\right)$ $and\ tan\left(-\frac{6\pi}{5}\right)$

.

5-10) Exercise: Calculate the following:

a) $sin\left(\frac{13\pi}{6}\right) - cos\left(-\frac{11\pi}{3}\right) + \tan tan\left(\frac{4\pi}{3}\right)$.

b) $\cos cos\left(\frac{17\pi}{4}\right) - sin\left(\frac{23\pi}{3}\right) - tan\left(\frac{7\pi}{4}\right)$.

c) $cos^2\left(\frac{7\pi}{8}\right) + cos^2\left(\frac{5\pi}{8}\right) + cos^2\left(\frac{3\pi}{8}\right) + cos^2\left(\frac{\pi}{8}\right)$.

5-11) Exercise: Simplify:

a) E=sin(x+π)+sin(x+2 π)+sin(x- π)+sin(x-3 π).

b) F=sin($\frac{5\pi}{2}$+x)-cos(x- π)+sin(3 π+x)+sin(-x).

c) G=cos(5 π-x)-sin($\frac{3\pi}{2} + x$)+cos(-x-$\frac{7\pi}{2}$)+sin(x+3 π).

d) H=cos(-π-x)-sin($\frac{5\pi}{2}$ +x)+cos(-x$-\frac{\pi}{2}$)+sin(x-π).

e) I=tan(x+π)+tan(x-π)+tan(x$-\frac{\pi}{2}$)+tan($\frac{7\pi}{2}$+x).

f) J=$\frac{\sin sin\,(7\pi+x) + cos\left(\frac{3\pi}{2}-x\right)}{sin^2(\pi-x)+sin^2\left(\frac{5\pi}{2}-x\right)}$.

5-12)Exercise: Given cosx+sinx=$\frac{\sqrt{3}+1}{2}$ where x is in the interval]0;$\frac{\pi}{2}$[.

a) Show that $(cosx + sinx)^2 + (cosx - sinx)^2$=2.

b) Deduce the value of cosx-sinx.

c) Calculate cosx, sinx, and tanx.

5-13) Exercise: verify the following identities:

a) $\cos cos\left(\frac{\pi}{4} + x\right) = sin(\frac{3\pi}{4} + x)$.

b) $\sin sin\left(x + \frac{\pi}{12}\right) = \cos cos\left(\frac{19\pi}{12} + x\right)$.

c) $(sinx + cosx)^2 = 1 + 2sinxcosx$.

d) $tan^2x\left(1 - sin^2x\right) = sin^2x$.

e) $sinxcosx = \frac{1}{tanx+cotx}$.

f) $\frac{1+sinx}{cosx} = \frac{cosx}{1-sinx}$.

g) $sin^2x - sin^2y = \frac{1}{1+cot^2x} - \frac{1}{1+cot^2y}$

h) $cos^4x - sin^4x = 1 - 2sin^2x = 2cos^2x - 1$.

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Chapter 5: Trigonometric functions(continued)

5-10) Exercise(Solution): Calculate the following:

a) $sin\left(\frac{13\pi}{6}\right) - cos\left(-\frac{11\pi}{3}\right) + \tan tan\left(\frac{4\pi}{3}\right) = \sin sin\left(\frac{12\pi+\pi}{6}\right) - \cos cos\left(\frac{-12\pi+\pi}{3}\right) +$
$=\sqrt{3}$

b) $\cos cos\left(\frac{17\pi}{4}\right) - sin\left(\frac{23\pi}{3}\right) - tan\left(\frac{7\pi}{4}\right) = \cos cos\left(\frac{16\pi+\pi}{4}\right) - \sin sin\left(\frac{24\pi-\pi}{3}\right) - \tan$

c) $cos^2\left(\frac{7\pi}{8}\right) + cos^2\left(\frac{5\pi}{8}\right) + cos^2\left(\frac{3\pi}{8}\right) + cos^2\left(\frac{\pi}{8}\right) = cos^2\left(\frac{8\pi-\pi}{8}\right) + cos^2\left(\frac{4\pi+\pi}{8}\right) + cos^2($

5-11) Exercise(Solution): Simplify:

a) E=sin(x+π)+sin(x+2 π)+sin(x- π)+sin(x-3 π)= -sinx+sinx+sin[-(π-x)]+ sin(x-4π+π) =-sin(π-x)+sin(x+π)=-2sinx.

b) F=sin($\frac{5\pi}{2}$+x)-cos(x- π)+sin(3 π+x)+sin(-x)= sin($\frac{4\pi+\pi}{2}$+x)-cos[-(π-x)]+ sin(2π+π+x)+sin(-x)=sin(2π+ $\frac{\pi}{2}$+x)- cos(π-x)+sin(π+x)-sinx= sin($\frac{\pi}{2}$ +x)+cosx-sinx-sinx.=2cosx-2sinx.

c) G=cos(5 π-x)-sin($\frac{3\pi}{2} + x$)+cos(-x-$\frac{7\pi}{2}$)+sin(x+3 π)

=cos(4π+π-x)-sin($\frac{2\pi+\pi}{2} + x$)+cos(-x$-\frac{8\pi-\pi}{2}$)+sin(x+2π+π)

=cos(π-x)-sin[π+($\frac{\pi}{2} + x$)]+cos(-x-4π+$\frac{\pi}{2}$)+sin(x+π)

=-cosx+sin($\frac{\pi}{2}+x$)+cos($\frac{\pi}{2}-x$)-sinx

=-cosx+cosx+sinx-sinx.=0.

d) H=cos(-π-x)-sin($\frac{5\pi}{2}$ +x)+cos(-x$-\frac{\pi}{2}$)+sin(x-π)

= cos[-(π+x)]-sin($\frac{4\pi+\pi}{2}$ +x)+cos[-(x+ $\frac{\pi}{2}$)]+sin[-(π-x)].

= cos(π+x)-sin(2π $+\frac{\pi}{2}$ +x)+cos(x+ $\frac{\pi}{2}$)-sin(π-x).

=-cosx-sin($\frac{\pi}{2}$ +x)-sinx-sinx=-cosx-cosx-2sinx=-2cosx-2sinx

e) I=tan(x+π)+tan(x-π)+tan(x$-\frac{\pi}{2}$)+tan($\frac{7\pi}{2}$+x).

= tan(x)+tan[-(π-x)]+tan[-($\frac{\pi}{2}-x$)]+tan(3π $+\frac{\pi}{2}$+x).

tan(x)-tan(π-x)-tan($\frac{\pi}{2}$ −x)+tan($\frac{\pi}{2}$+x).

tanx+tanx-cotx-cotx=2tanx-2cotx.

f) J=$\frac{\sin sin\ (7\pi+x)\ +cos\left(\frac{3\pi}{2}-x\right)}{sin^2(\pi-x)+sin^2\left(\frac{5\pi}{2}-x\right)}=\frac{\sin sin\ (6\pi+\pi+x)\ +cos\left(\frac{4\pi-\pi}{2}-x\right)}{sin^2(x)+sin^2\left(\frac{4\pi+\pi}{2}-x\right)}=$

$\frac{\sin sin\ (\pi+x)\ +cos\left(2\pi-\frac{\pi}{2}-x\right)}{sin^2(x)+sin^2\left(2\pi+\frac{\pi}{2}-x\right)}=\frac{(x)\ +cos[-\left(\frac{\pi}{2}+x\right)]}{sin^2(x)+sin^2\left(\frac{\pi}{2}-x\right)}=\frac{(x)\ +cos\left(\frac{\pi}{2}+x\right)}{sin^2(x)+cos^2(x)}=$

.$\frac{(x)\ (x)}{1}=-\ 2sin(x).$

5-13) Exercise(Solution): verify the following identities:

a) $\cos cos\left(\frac{\pi}{4}+x\right)\ =sin(\frac{3\pi}{4}+x).$

Solution: $\cos cos\left(\frac{\pi}{4}+x\right)=sin[\frac{\pi}{2}+(\frac{\pi}{4}+x)]=sin(\frac{3\pi}{4}+x).$

Another method: $sin(\frac{3\pi}{4}+x)=$

$\sin sin\left(\frac{2\pi+\pi}{4}+x\right)\ =\sin sin\ [\frac{\pi}{2}+\left(\frac{\pi}{4}+4\right)]\ =\cos cos\left(\frac{\pi}{4}+x\right).$

b) $\sin sin\left(x+\frac{\pi}{12}\right)\ =\cos cos\left(\frac{19\pi}{12}+x\right).$

Solution:

$\sin sin\left(x+\frac{\pi}{12}\right)\ =-\cos cos\left[\frac{\pi}{2}+\left(x+\frac{\pi}{12}\right)\right]\ =-\cos cos\left(x+\frac{6\pi}{12}+\frac{\pi}{12}\right)\ =-\cos c$

.

c) $(sinx+cosx)^2=1+2sinxcosx.$

Solution:

$(sinx+cosx)^2=sin^2x+2sinxcosx+cos^2x=1+2sinxcosx.$

d) $tan^2x\left(1-sin^2x\right)=sin^2x.$

Solution: $tan^2x\left(1 - sin^2x\right) = \frac{sin^2x}{cos^2x} \times cos^2x = sin^2x.$

Recall that: $sin^2x + cos^2x = 1 \Leftrightarrow cos^2x = 1 - sin^2x \Leftrightarrow sin^2x = 1 - cos^2x$

e) $sinxcosx = \frac{1}{tanx+cotx}$.

Solution: $\frac{1}{tanx+cotx} = \frac{1}{\frac{sinx}{cosx}+\frac{cosx}{sinx}} = \frac{1}{\frac{sin^2x}{cosx\times sinx}+\frac{cos^2x}{sinx\times cosx}} = \frac{1}{\frac{1}{sinxcosx}} = sinxcosx.$

f) $\frac{1+sinx}{cosx} = \frac{cosx}{1-sinx}$.

Solution: $\frac{1+sinx}{cosx} = \frac{cosx}{1-sinx} \Leftrightarrow (1 + sinx)(1 - sinx) = cos^2x$ (cross multiplying)

.$\Leftrightarrow 1 - sin^2x = cos^2x \Leftrightarrow cos^2x = cos^2x$ (verified)

g) $sin^2x - sin^2y = \frac{1}{1+cot^2x} - \frac{1}{1+cot^2y}$

Solution: $\frac{1}{1+cot^2x} - \frac{1}{1+cot^2y} = \frac{1}{1+\left(\frac{cosx}{sinx}\right)^2} - \frac{1}{1+\left(\frac{cosy}{siny}\right)^2} = \frac{1}{\frac{sin^2x}{sin^2x}+\frac{cos^2x}{sin^2x}} - \frac{1}{\frac{sin^2y}{sin^2y}+\frac{cos^2y}{sin^2y}} =$

$= \frac{1}{\frac{1}{sin^2x}} - \frac{1}{\frac{1}{sin^2y}}$.

h) $cos^4x - sin^4x = 1 - 2sin^2x = 2cos^2x - 1.$

Solution: $cos^4x - sin^4x = \left(cos^2x - sin^2x\right)\left(cos^2x + sin^2x\right) =$
$cos^2x - sin^2x = \left(1 - sin^2x\right) - sin^2x = 1 - 2sin^2x = cos^2x - (1 - cos^2x) =$
$.2cos^2x - 1.$

5-16) Exercise: Prove the following identities:

a) $\cos cos\,x + sinxtanx = \frac{1}{cosx}$

b) $sinx + cosxcotx = \frac{1}{sinx}$

c) $\frac{sin^2x}{cos^4x} = tan^2x + tan^4x.$

d) $\frac{1+cosx}{sinx} + \frac{sinx}{1+cosx} = \frac{2}{sinx}$.

e) $sin^2xcosx + cos^3x = cosx.$

f) $\frac{1-2sin^2x}{cosxsinx} = \frac{1-tan^2x}{tanx}$.

5-17) Exercise:

a) Prove the following identities:

$sin^2x = \frac{tan^2x}{1+tan^2x}$, $cos^2x = \frac{1}{1+tan^2x}$ and $sinxcosx = \frac{tanx}{1+tan^2x}$.

b) Deduce that $\frac{1-2sin^2x}{cosxsinx} = \frac{1-tan^2x}{tanx}$.

c) Write the function $F(x)=\frac{sin^3x+2sinxcos^2x}{sinx+cosx}$ in terms of $tanx$.

5-18) Exercise: Let α be an arc such that $0 < \alpha < \frac{\pi}{2}$.

Show that $\sqrt{\frac{1}{cos^2\alpha} + \frac{1}{sin^2\alpha}} = \frac{1+tan^2\alpha}{tan\alpha}$.