Whiteboard: 1/60, f4.5, iso 640, EV+1.3, Auto Focus, H4n **INPUT** @ **95%** gain.
BBB 1/60, f6.7, iso 640, **WB Use Grey Card**, Manual Audio at level 11, focal length ½ way to ∞. H4n **INPUT** @ **95%** gain.
BBB Check Horizontal Level & Billy's Lights!! Middle line at bottom 1/3 of desk & remember Billy and Bo chair locations!
Indefinite Integral Introduction and 4 Kinematic (UAM) Equation Derivations

Mr.p: (30 seconds of listening for this group of videos)

Mr.p: [4k] Good morning. Up to this point we have only worked with definite integrals. Today we introduce the indefinite integral. … A definite integral has defined limits. … An indefinite integral does NOT have defined limits.

- Bobby: [♪ Flipping Physics ♪] Is that it?
- Billy: That seems…
- Bo: Too easy.

- Billy: Yeah.

Mr.p: Right. Well. That’s the basic difference, however, in practice it’s a bit more complicated.

BBB: Yea! Yeah. Yep.

Mr.p: Let’s start with the derivative definition of acceleration. Bobby, please rearrange the derivative definition of acceleration to form an antiderivative or an integral.

Bobby: Uh. … Acceleration equals the derivative of velocity with respect to time. … Multiply both sides by dt and we get dv equals acceleration times dt. … Take the integral of both sides and … I guess we do not add the limits because we are doing an indefinite integral and you said indefinite integrals do not have limits. … Taking the integral gives us … velocity equals acceleration times time. … And this integral assumes the acceleration is constant and does not change as a function of time.

$$a = \frac{dv}{dt} \Rightarrow \int dv = \int a\,dt \Rightarrow v = at$$

Mr.p: Thank you Bobby, however, I will point out that, because this is an indefinite integral, what you have said here is not quite correct. On the left-hand side, we do get velocity, however, notice that velocity changes as a function of time, so this is velocity as a function of time. … Also, whenever we take an indefinite integral, or an integral without limits, we need to add a constant of integration or capital C to the equation.

$$a = \frac{dv}{dt} \Rightarrow \int dv = \int a\,dt \Rightarrow v(t) = at + C$$

- Billy: What is a … constant of integration?
- Bo: Yeah, what is capital C?

Mr.p: Good question. In order to figure that out, let’s determine the value of the velocity function at the initial time or time equals zero seconds. Billy, please do that.

Billy: To determine the value of the velocity function at time equals zero seconds, we plug zero in for time. When we do that we get, the velocity at zero seconds equals … well, just the constant of integration, capital C. … Oh, that means the constant of integration equals the initial velocity of the function. $v(0) = a(0) + C \Rightarrow v_i = C$

Mr.p: In other words, the velocity as a function of time equals acceleration times time plus velocity initial. … And we can rearrange that slightly to get velocity final equals velocity initial plus acceleration times time. $\Rightarrow v(t) = at + v_i \Rightarrow v_f = v_i + at$

- Bo: That is one of our uniformly acceleration motion equations.

- Billy: Oh yeah! Velocity final equals velocity initial plus acceleration times time is a kinematic equation!
- Bo: Kinematic equation?
- Billy: The uniformly accelerated motion equations and kinematic equations are two different names for the same things.
- Bo: Oh.
- Bobby: … How did velocity as a function of time become velocity final?
- Bo: … Well, when you plug a time into the velocity as a function of time equation, you get a velocity at that time, which is exactly what velocity

final is.

- Bobby: Oh right. The initial time is zero, the time final is time t. The initial velocity is at time equals zero and the final velocity is at time equals t. That makes sense. Thanks.
- Bo: You're welcome. $t_i = 0; t_f = t; v(0) = v_i; v(t) = v_f$

v_i at $t_i = 0$ & v_f at $t_f = t$

- Billy: … But why did we not add a constant of integration to the left-hand side of the equation?

Mr.p: Bo, thanks for helping there. … Billy, we certainly could do that. Let's look at what happens when we add constants of integration to both sides. Constant 1 on the left side of the equation and Constant 2 to the right side of the equation. … We can subtract Constant 1 from the whole equation and we get Constant 2 minus Constant 1 on the right side.

$$\int dv = \int a\,dt \Rightarrow v(t) + C_1 = at + C_2 \Rightarrow v(t) = at + C_2 - C_1 = at + C$$

Billy: Oh, I get it. The quantity Constant 2 minus Constant 1 is still just a constant. So, it does not matter. Adding a single constant C on the right side is equivalent to adding constants to both sides. Thanks.

Mr.p: Thanks for the question. … Alright, let's get back to the equation we derived, velocity final equals velocity initial plus acceleration times time. Bo, please substitute the derivative definition of velocity in for final velocity and rearrange the equation to solve for another one of our uniformly accelerated motion equations.

- Bo: Sure. … Velocity equals the derivative of position with respect to time. … Multiply both sides by dt and take the integral or both sides. … Do we do a definite or indefinite integral?
- **Mr.p: (Let's do an indefinite integral again.)**
- Bo: Okay. Indefinite integral, so no limits. … The integral of dx is just x … but it is position as a

function of time. … And that equals … well, we are taking the integral with respect to time, so … acceleration times time squared divided by 2 plus … uh … is it velocity initial squared over 2?

- Bobby: … No. Velocity initial is a constant. So, the integral of velocity initial with respect to time is velocity initial times time.
- Bo: Oh right. … And, plus the constant of integration C. … And we need to solve for C. So, we plug in zero for the initial time. … And we get that the constant equals the initial position of the function. … Ah, yeah. We get position final equals position initial plus velocity initial times time plus

one half acceleration times time squared. Which is one of our uniformly accelerated motion equations. Nice.

$$v_f = v_i + at \Rightarrow \frac{dx}{dt} = at + v_i \Rightarrow \int dx = \int \left(at + v_i\right) dt \Rightarrow x\left(t\right) = \frac{at^2}{2} + v_i t + C$$

$$\Rightarrow x\left(0\right) = \frac{a\left(0\right)^2}{2} + v_i\left(0\right) + C \Rightarrow x_i = C \Rightarrow x\left(t\right) = \frac{at^2}{2} + v_i t + x_i \Rightarrow x_f = x_i + v_i t + \frac{1}{2}at^2$$

Mr.p: Correct Bo. And notice the constant of integration, C, is the initial value of the function for both of the equations we just solved for.

Billy: The constant of integration is the initial value of the function.

Mr.p: Right, that is true when the acceleration is constant, like it is here. … Now, let’s go back to the derivative definition of acceleration and approach it a bit differently. Hopefully you can see that the derivative of velocity as a function of time is equivalent to the derivative of velocity with respect to position times the derivative of position with respect to time.

- Bobby: … I don’t see it.
- Bo: The dx’s cancel out.
- Bobby: Woah, you can do that?
- Bo: Yep. It’s the chain rule.
- Bobby: The chain rule? $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

- Bo: Yeah. The derivative of y with respect to x equals the derivative of y with respect to u times the derivative of u with respect to x.
- Bobby: Cool.
- Bo: Yep.

Mr.p: And the derivative of position with respect to time equals velocity. So, now, Billy, please rearrange this and use a definite integral to solve for another kinematic equation.

- Billy: Absolutely. That means we can multiply both sides by dx and get the definite integral of acceleration with respect to position from position initial to position final equals the definite integral of velocity with respect to velocity from velocity initial to velocity final. … The integral of acceleration with respect to position equals … well, acceleration is a constant so that integral is just acceleration times position and add the limits. … Substituting in the limits gives us acceleration times position final minus acceleration times position initial or acceleration times change in position. … The integral of velocity with respect to velocity equals velocity squared over 2 and add the limits. … Substituting in the limits gives us velocity final squared over 2 minus velocity initial squared over 2. … And … Oh, I see it! We can multiply the whole equation by 2 and reverse the sides, … and then add velocity initial squared to both sides and we get velocity final squared equals velocity initial squared plus 2 times acceleration times displacement. And that is a third uniformly accelerated motion or kinematic equation!

- Bo: … Why did we use a definite integral and not an indefinite integral this time?

$$a=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=\frac{dv}{dx}v\Rightarrow\int_{x_i}^{x_f}a\,dx=\int_{v_i}^{v_f}v\,dv\Rightarrow ax\Big]_{x_i}^{x_f}=ax_f-ax_i=a\Delta x=\frac{v^2}{2}\Big]_{v_i}^{v_f}=\frac{v_f^{\,2}}{2}-\frac{v_i^{\,2}}{2}$$

$$\Rightarrow v_f^{\,2}-v_i^{\,2}=2a\Delta x\Rightarrow v_f^{\,2}=v_i^{\,2}+2a\Delta x$$

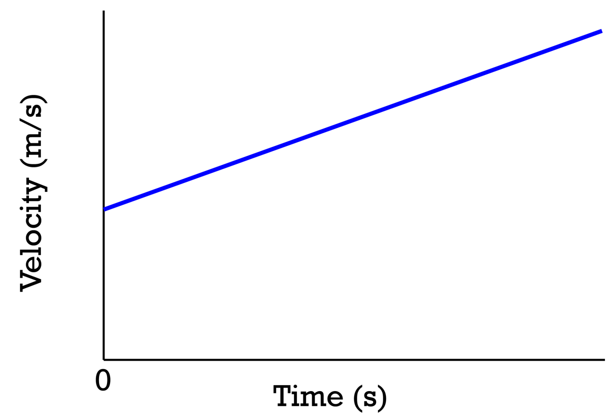
Mr.p: Thank you Billy. … Bo, the reason we used a definite integral this time and not an indefinite integral is because, in this case, the indefinite integral solution is a bit more cumbersome. I have included that solution in my lecture notes on my website if you would like to see it. … Now, there is one more uniformly accelerated motion equation we can solve for. We can use two of the equations we just derived to derive a fourth UAM equation. … Starting with velocity final equals velocity initial plus acceleration times time, we can rearrange that equation to solve for acceleration. … Then we can substitute that equation for acceleration into the equation position final equals position initial times velocity initial times time plus one half acceleration times time squared and subtract velocity initial from both sides. … Time squared over time reduces to just time. … And we can factor out time. … Velocity initial minus velocity initial over 2 equals positive velocity initial over 2. … Factoring out one half gives us the fourth, and sadly, often ignored, kinematic equation, displacement equals one half times the quantity velocity initial plus velocity final all times time. {hear}

$$v_f = v_i + at \Rightarrow a = \frac{v_f - v_i}{t} \ \& \ x_f = x_i + v_i t + \frac{1}{2}at^2 \Rightarrow x_f - x_i = v_i t + \frac{1}{2}\left(\frac{v_f - v_i}{t}\right)t^2$$

$$\Rightarrow \Delta x = v_i t + \frac{1}{2}\left(v_f - v_i\right)t = t\left(v_i + \frac{v_f}{2} - \frac{v_i}{2}\right) = t\left(\frac{v_i}{2} + \frac{v_f}{2}\right) \Rightarrow \Delta x = \frac{1}{2}\left(v_i + v_f\right)t$$

Bobby: (uh, mr.p?) I thought we were going to use integrals to solve for all four UAM equations.

Mr.p: Sure. How about y'all do that? Use the fact that uniformly accelerated motion means the acceleration is constant, … displacement equals the definite integral of velocity with respect to time from time initial to time final, and this example graph of velocity as a function of time for uniformly accelerated motion.



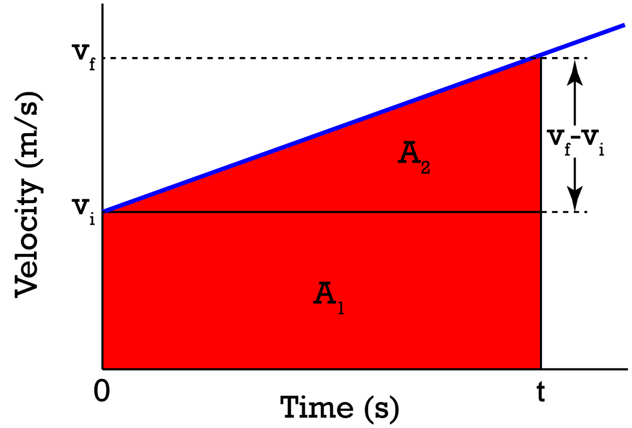
- Bobby: Uh …
- Bo: I agree.

- Billy: … … We know an integral is the area "under" a function, right?
- Bobby and Bo: So?
- Billy: … Well, we could pick a time t and determine the area between the velocity function and the time axis.
- Bo: … Okay. That would be the red area in the graph.
- Bobby: … We could split that into two different areas, area 1 and area 2. And the total area would be the sum of those two areas.
- Billy: Area 2 is a triangle and has an area equal to one half base times height.

- Bobby: Using those variables, area 1 is a rectangle of area equal to base times height. … And the base of both areas has a length equal to just t.
- Billy: The height of area 1 is velocity initial. … Oh, and the height of area 2 is velocity final minus velocity initial!
- Bo: … And that is the same equation we had halfway through the last solution, so this also gives us the same uniformly accelerated motion equation. Nice.
- Billy: Yeah.
- Bobby: … … But what if the velocity function is

not a straight line?

- Billy: Yeah?
- Bo: … These equations are for uniformly accelerated motion.
- Billy and Bobby: Oh!
- Billy: and a constant acceleration is a constant slope on a velocity as a function of time graph. So, the velocity function for uniformly accelerated motion is always a straight line.



- Bo: Yep.

$$\Delta x = \int_{t_i}^{t_f} v\,dt = \text{Area "under" Function} = A_1 + A_2 = b_1 h_1 + \frac{1}{2} b_2 h_2 = v_i t + \frac{1}{2}(t)(v_f - v_i)$$

Mr.p: Very nice everybody. … Realize we just used the definitions of velocity and acceleration, … and calculus to derive four uniformly accelerated motion equations, which are also called kinematic equations. … I think that is awesome. {hear} Thank you very much for learning with me today, I enjoyed learning with you.

BBB: (Absolutely! Yeah. Yep.)