Some Tested Approaches to Topics in High School Mathematics

CME Project
Education Development Center, Inc.
55 Chapel St.
Newton, MA 02458

http://www.edc.org/cmeproject

With additional support from Texas Instruments

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The CME Project:
Some Distinguishing Features

The *CME Project*, developed by EDC's Center for Mathematics Education, is a coherent, four-year, NSF-funded high school program designed around how knowledge is organized and generated within mathematics: the themes of algebra, geometry, and analysis. Many standard curricula look at each of these areas as sets of results and techniques. Many integrated programs look at them as threads that run through varying contexts. The *CME Project* sees these branches of mathematics not only as compartments for certain kinds of results, but also as descriptors for *methods* and *approaches*—the habits of mind that determine how knowledge is organized and generated within mathematics itself. As such, they deserve to be centerpieces of a curriculum, not its byproducts.

Other important parts of the discipline—probability, statistics, combinatorics, number theory, measurement—are integrated into these themes.

The primary goal of the *CME Project* is to develop robust mathematical proficiency in students. To achieve this, the *CME Project* strikes a balance between the common wisdom and tradition in this country—that students need to focus on one piece of mathematics at a time—and what has been learned about the added value of seeing connections among mathematical topics and to fields outside mathematics. The program builds on lessons learned from high-performing countries: develop an idea thoroughly and then revisit it only to deepen it; organize ideas in a way that is faithful to how they are organized in mathematics; and reduce clutter and extraneous topics. It also employs the best American models that call for struggling with ideas and problems as preparation for instruction, moving from concrete problems to abstractions and general theories, and situating mathematics in engaging contexts (including mathematics itself). The *CME Project* is a comprehensive curriculum that meets the dual goals of mathematical rigor and accessibility for a broad range of students.

The CME Project provides teachers and schools with a third alternative to the choice between traditional texts driven by low-level skill development and more progressive texts that have unfamiliar organizations. The *CME Project* gives teachers the option of a problem-based, student-centered program, organized around the mathematical themes with which teachers and parents are familiar.

You can find a more detailed description of the design principles and philosophy for the *CME Project* later in this handout in the paper *Towards a Curriculum Design Based on Mathematical Thinking*.

The program also employs some unusual and effective approaches to mathematical topics—approaches that have been tested and refined, in some cases for several decades, by teachers and others affiliated with the program. The purpose of this note is to describe some of these approaches.

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Solving Simple Equations

In the Algebra 1 course, students begin to make the connection between finding solutions to equations and finding inverses for functions. Before any of the formalisms about solving linear equations, they use a method we call *backtracking* to solve equations like this

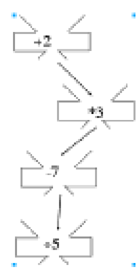
$$\frac{3(x+2)-7}{5} = 4$$

The course presents such equations with descriptions like

> "When I took a number, added 2, tripled the result, subtracted 7 from the answer, and divided the result by 5, I got 4. What number did I start with?"

So, the left-hand side becomes a description of an *algorithm*, a function defined by a sequence of arithmetic calculations. Students model this algorithm in many ways; one useful representation is as a *machine network*:

This machine image is useful as a starting point, especially if students build computational models of functions on their calculators. Later, in Algebra 2, we move from the machine metaphor to the more robust notion of function as pairing, so that students begin to see that a function is defined by its *behavior*.



A machine model for $\frac{3(x+2)-7}{5}$

Students practice running several inputs through the network, and then we ask them to "pull back" an output to get the corresponding input. To do this, they do the "inverse steps in reverse order," finding a solution of the equation. In fact, as an extension, we ask them to build a network that solves the equation $\frac{3(x+2)-7}{5} = k$ for any value k of the right-hand side:

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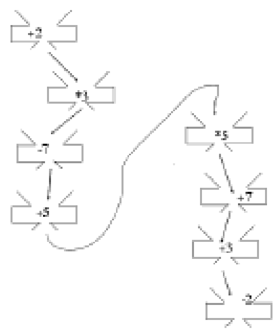

"Undoing" the algorithm

In another direction, one that connects to expression simplification and equality of functions, we ask students to find a simpler network that produces the same input-output pairs as the original network.

Algebra Word Problems

The difficulties that high school students have with algebra word problems are legendary. The quintessential word problem ("Mary is 10 years older than her brother was 5 years ago . . . ") is the topic of cartoons and jokes. Teachers have devoted a great deal of effort to exposing the roots of the difficulties people have with word problems. Two very common perceptions are that students have difficulty with word problems because

- they have a general difficulty with reading
- they are often not familiar with the contexts described in the problems.

But an analysis by some middle and high school teachers in Woburn MA showed that there's got to be more to it. They observed that the following problem

> Mary drives from Boston to Washington, a trip of 500 miles. If she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back, how many hours does her trip take?

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> Mary drives from Boston to Washington, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. If the total trip takes $18\frac{1}{3}$ hours, how far is Boston from Washington?

is baffling to many of the same students a year later in algebra class. This analysis led to an effective method—that we call *Guess-Check-Generalize*—for solving these kinds of problems. Here's how it works for the second problem above:

Guess-Check-Generalize is different from the well-known Guess and Check strategy for finding solutions or approximate solutions to numerical problems.

The first step is to guess at an answer; suppose Boston is 500 miles from Washington. The purpose of the guess is *not* to stumble on a right answer; rather, it's to focus students on the steps they take to check the guess. So, if the guess is 500 miles, then Mary takes $\frac{500}{60} = 8\frac{2}{3}$ hours to drive down and $\frac{500}{50} = 10$ hours to get home. The total trip is $18\frac{2}{3}$ hours, so 500 is not the right answer, but that's OK. We ask students to be explicit about what they did to check the guess. If they are not sure, they take another guess, and another, and another, until they are able to articulate something like

In spite of our proclamations that the point is not to get the right answer by guessing, many students are at first reluctant to take a guess, fearing they'll be incorrect.

> "You take the guess, divide it by 60, then divide it by 50, add you answers and see if you get $18\frac{1}{3}$."

The generic "guess checker" is then

$$\frac{\text{guess}}{60} + \frac{\text{guess}}{50} \stackrel{?}{=} 18\frac{1}{3}$$

This gives them the equation that models the problem:

$$\frac{x}{60} + \frac{x}{50} = 18\frac{1}{3}$$

and from here, it's "pure" algebra.

This method was inspired in part by some educational theories about how people "encapsulate" isolated actions into coherent processes.

Extension

One of the main themes in the program is *extension*. Extension in the *CME Project* takes two forms: *algebraic* (extending operations via their defining properties) and *analytic* (extension by continuity).

Example: Arithmetic with Signed Numbers

Students have practiced arithmetic with non-negative integers since first grade. Our approach is to extend the "number facts"

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that many have memorized by extending patterns in the "ta- bles" in ways that preserve the properties of the operations. Here's a piece of the multiplication table:

				12	0	12	24	36	48	60	72	84	96	108	120	132	144
				11	0	11	22	33	44	55	66	77	88	99	110	121	132
				10	0	10	20	30	40	50	60	70	80	90	100	110	120
				9	0	9	18	27	36	45	54	63	72	81	90	99	108
				8	0	8	16	24	32	40	48	56	64	72	80	88	96
				7	0	7	14	21	28	35	42	49	56	63	70	77	84
				6	0	6	12	18	24	30	36	42	48	54	60	66	72
				5	0	5	10	15	20	25	30	35	40	45	50	55	60
				4	0	4	8	12	16	20	24	28	32	36	40	44	48
				3	0	3	6	9	12	15	18	21	24	27	30	33	36
				2	0	2	4	6	8	10	12	14	16	18	20	22	24
				1	0	1	2	3	4	5	6	7	8	9	10	11	12
				0	0	0	0	0	0	0	0	0	0	0	0	0	0
	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
				-1													
				-2													
				-3													
				-4													

The multiplication table, reoriented

Notice the reorientation to make the table look more like a coordinate system. This is on purpose. For example, by graphing the line with equation $x + y = 12$ on this table, you get a picture of all the products of integers that sum to 12. Which product is largest?

So, how could one extend the table in ways that make the patterns in the rows and columns continue? There are several ways, but not surprisingly, the one that is most natural for many students is exactly the one that ensures that the extended arithmetic works the way it's supposed to: use the rows to continue the linear patterns to the left, and use the columns to extend down. We present students with some other ideas, as well (the rows increase again to the left of 0, for example) and we investigate why such choices "break" the rules of arithmetic. In other words, this is not an exercise in "extending the pattern;" rather, it is a search for an extension that *preserves rules for calculating*.

A detailed verification that the usual extension of the multiplications and addition tables preserves arithmetic properties like associativity and commutativity is too technical for most students at this stage. The development in the program is more informal, but it is faithful to the principles of this verification.

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χ

12

0 12 24 36 48 60 72 84 96 108 120 132 144

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0 11 22 33 44 55 66 77 88 99 110 121 132

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10

0 10 20 30 40 50 60 70 80 90 100 110 120

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0 9 18 27 36 45 54 63 72 81 90 99 108

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0 8 16 24 32 40 48 56 64 72 80 88 96

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0 7 14 21 28 35 42 49 56 63 70 77 84

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6

0 6 12 18 24 30 36 42 48 54 60 66 72

is largest?

5

0 5 10 15 20 25 30 35 40 45 50 55 60

4

0 4 8 12 16 20 24 28 32 36 40 44 48

3

0 3 6 9 12 15 18 21 24 27 30 33 36

2

0 2 4 6 8 10 12 14 16 18 20 22 24

1

0 1 2 3 4 5 6 7 8 9 10 11 12

χ

-4

-3 -2 -1 0

0 0 1 0 2 0 3 0 4 0 5 0 6 0 7 0 8 0 9 0 10 0 11 0 12 0 χ -1

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Example: Integer, Rational, and Real Exponents

Most students come from middle school with some experience with positive integer exponents; they know that 2^3 means $2 \times 2 \times 2$ and, more generally, if n is a positive integer,

$$2^n \text{ means } \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}}$$

In Algebra 1, we use this definition to develop rules for arithmetic with positive integer exponents:

1. $a^n \cdot a^m = a^{n+m}$
2. $(a^n)^m = a^{nm}$

If we insist that these definitions extend to all integers, we are forced to make some definitions:

This kind of extension is connected to what we call the "duck principle." To show that some expression is equal to $\sqrt{10}$, show that it is positive and that its square is 10. If it walks like a duck . . . .

- 3^0 would have to be 1 if we want rule 1 to extend:

  $$3^5 \cdot 3^0 = 3^{5+0} = 3^5$$

  but the only number that can be multiplied by 3^5 to get 3^5 is 1, so 3^0 would have to be 1.

- Similarly, 3^{-1} would have to be $\frac{1}{3}$ if we want rule 1 to extend to negative integers:

  $$3^{-1} \cdot 3^1 = 3^{-1+1} = 3^0 = 1$$

  but the only number that can be multiplied by 3^1 to get 1 is $\frac{1}{3}$, so 3^{-1} would have to be $\frac{1}{3}$.

In the same way, $3^{-2} \cdot 3^2 = 1$, so 3^{-2} would have to be $\frac{1}{3^2}$.

- If we want to extend the rules to fractional exponents, $3^{\frac{1}{2}}$ would have to be (by rule 2) a number whose square is 3:

  $$\left(3^{\frac{1}{2}}\right)^2 = 3^{\frac{1}{2} \cdot 2} = 3^1 = 3$$

  There are two choices here, $\sqrt{3}$ and $-\sqrt{3}$, and we make the choice that $3^{\frac{1}{2}} = \sqrt{3}$.

Similarly, $3^{\frac{5}{8}}$ is a number whose 8th power is 3^5, so we define it to be $\sqrt[8]{3^5}$. Later in the program, we show that this is the same as $\left(\sqrt[8]{3}\right)^5$. Even later in the program, we investigate the entire set of solutions to $x^8 = 3^5$ in the complex numbers.

- We can extend the meaning of exponents to all integers (in Algebra 1) and all rational numbers (in Algebra 2) in this way—by forcing rules 1 and 2 to extend. But what about irrational exponents? What "must" $3^{\sqrt{2}}$ mean? For this, we use extension by *continuity*. The graph of $y = 3^x$ for all rational x looks like this:

Example: Integer, Rational, and Real Exponents

Most students come from middle school with some experience with positive integer exponents; they know that 23 means 2 × 2 × 2 and, more generally, if n is a positive integer,

2n means 2 ⏟ × 2 ×···× ⏞⏞ 2 ⏟ n

times

In Algebra 1, we use this definition to develop rules for arith- metic with positive integer exponents:

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m

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• 30 would have to be 1 if we want rule 1 to extend:

show that some expression is equal to

35 · 30 = 35+0 = 35

but the only number that can be multiplied by 35 to get 35 is 1, so 30 would have to be 1.

• Similarly, 3−1 would have to be 1 3

√

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In the same way, 3−2 ·32 = 3−1 · 31 = 3−1+1 = 30 = 1

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= 3

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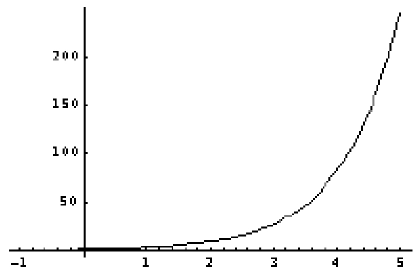

$y = 3^x$, full of holes

Even though it looks smooth, this graph is full of holes, one over each irrational number. If we fill in the holes we get the definition of 3^x for irrational x. Put another way, $3^{\sqrt{2}}$ is the number that is approached by the sequence

$$3^{1.4}, 3^{1.41}, 3^{1.414}, 3^{1.4142}, 3^{1.41421}, \ldots$$

where the sequence of rational exponents has $\sqrt{2}$ as a limit.

Again, this is all informal and intuitive, but the ideas can be made precise and *are* made precise in courses in post-calculus analysis.

Equations and Graphs

Many high school students do not understand the fact that one can test a point to see if it is on the graph of an equation by seeing if its coordinates satisfy the equation; equations are, for these students, a kind of code from which one can read off information that allows one to produce a graph.

Our approach to this phenomenon is to provide students with opportunities to connect equations and graphs without elaborate formalisms, using the idea that the equation of a graph is the *point-tester* for the graph: it tells you whether or not a point is on the graph by checking some numerical fact about its coordinates.

For example, we ask students to find the equation of the horizontal line that passes through $(5, 1)$. They typically have no trouble drawing the line, and, when we give them several points, they usually have no trouble explaining why each is on or off the line: to see if a point is on the line, check to see if its y-coordinate is 1. So, the point-tester is $y \stackrel{?}{=} 1$, and the equation is $y = 1$.

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= ?

1, and the equation is y = 1.

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IIIIIII

1 I 3 4 5

Another example: What's the equation of the line whose graph bisects quadrants 1 and 3? The check to see if a point is on the line is that its x- and y-coordinates are the same. The equation is thus $y = x$.

These are simple examples, but they reinforce the meaning of the correspondence between equations and their graphs. And the point-tester idea works well for more complex equations and their graphs—it's an idea that runs throughout the program.

Example: Equations of Lines and Slope

In Algebra 1, the point-tester idea helps students find equations for lines. The method involves a somewhat unorthodox approach to slope.

Our teaching experience tells us that "the slope of a line" approach places some undue cognitive demands on students—students are asked to think about a number (slope) that is an invariant of an infinite geometric object (the line). This is difficult for a couple reasons:

- The invariant is not part of the geometric object itself—it is a numerical quantity derived from the geometry of the line.
- And slope is derived via a calculation that seems at first glance to depend on a *choice* of two points on the line.

Indeed, the slope of a line is an example of the *derivative* that students will study in calculus.

Our development starts with the more concrete idea of "slope between 2 points," a number that can be calculated directly from coordinates.

We use the notation $m(A, B)$ for the slope between A and B.

Our approach to equations for lines synthesizes this perspective on slope with the point-tester idea. We make an explicit assumption (that will be proved in the Geometry course):

Assumption

Three points A, B, and C lie on the same line if and only if

$$m(A, B) = m(B, C)$$

The proof requires results about similar triangles.

Suppose you are given two points, say $A = (3, -1)$ and $B = (5, 3)$. What is the equation of the line that contains A and B?. Students develop the habit of checking several points to see if they are on the line, *keeping track of their steps*. At first, we

So, the *Guess-Check-Generalize* theme plays a major role in this method.

Another example: What's the equation of the line whose graph bisects quadrants 1 and 3? The check to see if a point is on the line is that its x- and y-coordinates are the same. The equation is

thus y = x.

These are simple examples, but they reinforce the meaning of the correspondence between equations and their graphs. And the point-tester idea works well for more complex equations and their graphs—it's an idea that runs throughout the program.

Example: Equations of Lines and Slope

In Algebra 1, the point-tester idea helps students find equations for lines. The method involves a somewhat unorthodox approach to slope.

Our teaching experience tells us that "the slope of a line" approach places some undue cognitive demands on students— students are asked to think about a number (slope) that is an invariant of an infinite geometric object (the line). This is difficult for a couple reasons:

• The invariant is not part of the geometric object itself—it

Indeed, the slope of a is a

numerical quantity derived from the geometry of the line.

line is an example of the derivative that students will study in calculus.

• And slope is derived via a calculation that seems at first glance to depend on a choice of two points on the line.

Our development starts with the more concrete idea of "slope

We use the notation

between 2 points," a number that can be calculated directly from coordinates.

m(A, B) for the slope between A and B.

Our approach to equations for lines synthesizes this perspec- tive on slope with the point-tester idea. We make an explicit assumption (that will be proved in the Geometry course):

Assumption Three points A, B, and C lie on the same line if and only if

The proof requires results about similar triangles. m(A, B) = m(B,C)

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give them some points to check—say $X = (7,6)$, $P = (1,-5)$, and $Q = (9.5, 10.5)$. In each case, they find the slope between the point to be tested and, say, B. Then they check whether it is equal to $m(A, B)$ (that is, 2). The generic check is that the slope from (x, y) to $(5, 3)$ should equal 2, so the point-tester is

$$\frac{y-3}{x-5} = 2$$

Some care has to be taken with the fact that x can't be 5 on the left-hand side of this equation.

This equation is then simplified and transformed into a linear equation in x and y. The course proceeds to develop fluency in sketching lines from their equations and finding equations for given lines, but only after this foundation is solid.

Our Uses of Technology

Students in the *CME Project* Algebra 2 course use technology in many of the same ways that technology is used in other programs: to test out conjectures, to reduce computational drudgery, to graph equations and functions, to perform statistical analyses on data, and to provide examples of theorems and results. And we also use the calculator as a context—figuring out the what's "behind" the built-in functions. For example, one lesson helps students understand the mathematics that underlies the functions on a calculator that compute standard deviation, variance, mean, and best fit lines.

We make another use of technology that is less standard: *students use technology to build computational models of mathematical objects.*

Example: Modeling Functions

One of the most important examples of this model-building activity in the last two courses is that students build computational models of mathematical *functions*.

Current high-end mathematical calculators (the TI-89 family, for example) and most computer mathematics systems contain a capability that will eventually be available on most machines—something we call a *functional language*. What this means is that one can create user-defined functions—we call them *models*—in a language that is quite close to ordinary mathematical notation, and then they can use the functions as if they were built-ins. For example, to build a model of the function f de-

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