Computing particle energy and center-of-mass energy from “effective energy” data

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We would like to know the particle energy and center-of-mass energy obtained by various particle accelerators. However, data showing progress in particle accelerators typically uses a calculated quantity called “effective energy”. This allows comparisons between older accelerators (like electron accelerators with stationary targets) and newer accelerators (like electron-positron or proton-antiproton colliders). The effective energy is the energy that is required for a moving particle colliding with a stationary proton, such that the energy in the center of mass frame is equivalent to the actual energy in the accelerator. So, for accelerators with a stationary target, the effective energy is the same as the particle energy. For a collider, we need to perform a lorentz transform from the frame in which one particle is stationary into the reference frame in which the center of mass is stationary, in order to obtain center-of-mass energy and particle energy.

I make the following assumptions:

1. For non-colliders, the energy given is the particle kinetic energy[1]
2. For colliders, the energy given is the particle kinetic energy such that the center of mass energy for that particle on a stationary proton is the same as the center of mass energy of the collisions that actually occur in the accelerator (the “equivalent energy”)

So our task is to use these values to find this particle energy.

Proton-proton and electron-electron Collisions

We’ll start with the simpler case, which is for colliders, because this is just a collision between particles of equal mass. We have a particle of mass $M$and speed u incident on another particle of mass M and velocity zero. We want to perform a lorentz transform on the velocities such that the total momentum of the system is zero:

$$\gamma' M u' - \gamma_v M v = 0 \Rightarrow \gamma' u' = \gamma_v v$$

[1] We can see that the charts are not using total energy (including rest energy) because they start at energies substantially lower than the rest mass of the proton

With the lorentz factor $\gamma = (1 - u^2/c^2)^{-1/2}$[2], and corresponding formulas relating γ_v to v and $\gamma'$to u'. Here I have assumed that the particles are moving in opposite directions, and written this equation in terms of speed, rather than velocity. We can infer from this that $u' = v$, which we could have intuited anyway from the symmetry of the system. We don't need to do a full Lorentz transform; we can just use the relativistic velocity addition formula[3]:

$$u' = \frac{u-v}{1-uv/c^2} = v$$

After some algebra, we use this to write v/cin terms of u/c:

$$v/c = (1 - \sqrt{1 - u^2/c^2})c/u$$

That square root term is just $1/\gamma$, and we can invert the equation for gamma to write u/c in terms of gamma:

$$u/c = \sqrt{1 - 1/\gamma^2}$$

Combining these and inserting them into the equation for γ' we find:

$$\gamma' = \left[1 - \frac{(1-1/\gamma)^2}{1-1/\gamma^2}\right]^{-1/2}$$

We can use this to find the center-of-mass frame kinetic energy, using the usual relativistic energy equation:

$$T = E - Mc^2 = (\gamma - 1)Mc^2 \text{(kinetic energy)}$$

For small values of gamma, the kinetic energy is the usual classical expression:

$$T = \frac{1}{2}Mu^2$$

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[2] This is the usual lorentz factor for special relativity https://en.wikipedia.org/wiki/Lorentz_factor
[3] https://en.wikipedia.org/wiki/Velocity-addition_formula#Special_relativity

This approximation is reasonably accurate for our purposes (<5% error) up to $\gamma \approx 1.03$.

For very large values of gamma, we might expect problems with precision[4], since $1 - 1/\gamma^2$ is so close to unity when gamma is very large. In this case, we can neglect $1/\gamma^2$ terms in, and we find a much simpler expression[5]:

$$\gamma' \approx \sqrt{\gamma/2}$$

Since the particles have the same mass, this straightforwardly gives us the total energy and the particle energy:

$T' = (\gamma' - 1)mc^2$ (particle kinetic energy)

$T_{total} = (\gamma' - 1)mc^2 + (\gamma' - 1)mc^2 = 2T'$ (center-of-mass kinetic energy)

Electron-Proton Collisions

For unequal masses, we have the zero-momentum equation:

$$\gamma' M u' - \gamma_v m v = 0$$

With the proton to electron mass ratio $\alpha \equiv M/m \approx 1836$. This, unfortunately, does not immediately reduce to equal velocities. To do anything useful with this, we need to choose appropriate energy regimes and make the appropriate approximations. I have tabulated these in the following table:

γ	u/c	v/c**(proton)**	v/c**(collider)**	γ_v
<1.05	<.3	<<1	<1.5	≈1
1.1-3	Use exact	<<1	No approx.	≈1
3-10	≈1	<<1	No approx.	≈1

[4] Google sheets seems to handle precision very well, so this turns out to be less of a problem than I would have thought

[5] I am uncertain, but I think the authors of many accelerator energy plots may just use this equation for all colliders.

10-200	≈1	<<1	$\gamma' \approx \sqrt{\gamma/2}$	≈1
200-1000	≈1	Use exact	$\gamma' \approx \sqrt{\gamma/2}$	≈1
1000-4000	≈1	Use exact	$\gamma' \approx \sqrt{\gamma/2}$	Use exact
>4000	≈1	≈1	$\gamma' \approx \sqrt{\gamma/2}$	Use exact

Additionally, I use the substitutions $b = 1 - u/c$, $b' = 1 - u'/c$, and $b_v = 1 - v/c$, because this allows us to neglect terms of order b^2 or higher in many cases (but not all!) when the corresponding velocity is approximately 1. In particular, we can often use $\gamma \approx 1/\sqrt{2b}$.

The velocity addition formula in terms of b's is

$$b' = 1 - \frac{b_v - b}{b_v + b - b_v b}$$

For the lowest energies, we observe that the system is already essentially in center-of-mass frame, due to the high mass of the proton. Nearly all of the kinetic energy of the electron is lost in the collision, and we do not need to perform a transform (these are the accelerators that the authors are trying to allow us to make comparisons to by using equivalent energy in the first place!). As it turns out, this takes us up to approximately $\gamma = 200$ before the error reaches 5%, so half the table is largely unneeded. You can see the values for kinetic energy remaining after the collision in the data tables.

For energies that are higher than this, but still relatively low, we use the approximations listed in the table for values of gamma between 1000 and 4000[6], and find from momentum conservation, after some algebra, that:

$$b' = \frac{2b}{\alpha\sqrt{2b} - 1}$$

[6] Keep in mind we can always use *fewer* approximations.

Inserting this into the velocity addition formula, and doing some algebra, we find:

$$2\alpha^2(2 - b_v)(1 - b_v)^2 = 1 + b/b_v$$

We want b_v in terms of b. This is annoying to solve by hand, so I just plugged it into Wolfram Alpha with the appropriate values for the handful of accelerators at appropriate energies.

Once we have b and b_v we can use the velocity addition formula to find b', and then we can use this to find and sum the energies of the lorentz-transformed proton and electron:

$T_e = \left(1 - 1/\sqrt{1 - (1 - b')^2}\right)m_e c^2$ and $T_p = \left(1 - 1/\sqrt{1 - (1 - b_v)^2}\right)m_p c^2$

The particle energy is T_e and the center-of-mass energy is $T_e + T_p$.

Above this energy, the accelerators are all lepton-lepton colliders[7], so we can use the equal-mass expressions from the previous section.

[7] I expect they're all electron-positron colliders, but I am uncertain