3.2 – The Mean Value Theorem

I. FORMULAS

1. Rolle's Theorem

Conditions:

- $f(x)$ is continuous on $[a,\ b]$
- $f(x)$ is differentiable on $(a,\ b)$
- $f(a) = f(b)$

Then $\exists\ c \in (a,\ b) : f^{'}(c) = 0$

2. Mean Value Theorem (MVT)

Conditions:

- $f(x)$ is continuous on $[a,\ b]$
- $f(x)$ is differentiable on $(a,\ b)$

Then $\exists\ c \in (a,\ b) : f^{'}(c) = \frac{f(b)-f(a)}{b-a}$

3. Use these

- $f^{'}(x) = 0 \rightarrow$ for Rolle's Theorem
- $f^{'}(x) = \frac{f(b)-f(a)}{b-a} \rightarrow$ for alternative MVT form
- $f(b) - f(a) = f'(c)(b - a) \rightarrow$ alternative MVT form
- If $f^{'}(x) \geq m$, then $f(b) \geq f(a) + m(b - a)$

II. QUESTIONS

2. Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfies the conclusion of Rolle's Theorem.

$$f(x) = 5 - 6x + 3x^{2},\ \ [0, 2]$$

Since $f(x)$ is a polynomial, it is continuous and differentiable on R, so it is continuous on [0, 2] and differentiable on (0,2).

Also $f(0) = 5,\ \ f(2) = 5 - 12 + 12 = 5 \Rightarrow f(0) = f(2)$

Then there exists c such that: $f^{'}(c) = 0$

Set $f^{'}(c) = 0$

$$-6+6c=0 \Rightarrow c=\frac{6}{6} \Rightarrow c=1$$

3. Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfies the conclusion of Rolle's Theorem.

$$f(x)=\sqrt{x}-\frac{1}{9}x,\ [0,\ 81]$$

f, being the difference of a root function and a polynomial, is continuous and differentiable on $[0,\ \infty)$, so it so continuous on $[0,\ \infty)$ and differentiable on $(0,\ 81)$.

Also $f(0)=0$, $f(81)=9-9=0 \Rightarrow f(0)=f(81)$

Then $f'(c)=\frac{1}{2\sqrt{c}}-\frac{1}{9}$

Set $f(c)=0$:

$$\frac{1}{2\sqrt{c}}-\frac{1}{9}=0 \Rightarrow \frac{1}{2\sqrt{c}}=\frac{1}{9} \Rightarrow \sqrt{c}=\frac{9}{2} \Rightarrow c=\frac{81}{4}$$

4. Consider the following function

$$f(x)=1-x^{2/3}$$

Find $f(-1)$ and $f(1)$

$f(-1)=1-(-1)^{2/3}=1-1=0$

$f(1)=1-1=0$

Find all values c in $(-1,\ 1)$ such that $f'(c)=0$

$f'(c)=-\frac{2}{3}c^{-1/3}$

$f'(c)=0$ has no solution.

Based on this information, what conclusions can be made about Rolle's Theorem?

This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on (-1, 1)

5. Does the function satisfy the hypotheses of the Mean Value Theorem on the given interval?

$$f(x) = 4x^2 - 3x + 1, \ [0, 2]$$

Yes, f is continuous on $[0, 2]$ and differentiable on $(0, 2)$ since polynomials are continuous and differentiable on R.

If it satisfies the hypotheses, find all numbers c that satisfies the conclusion of the Mean Value Theorem.

$f(x)$ is continuous on $[0, 2]$ and differentiable on (0, 2) since polynomials are continuous and differentiable on R.

$$f'(x) = 8x - 3$$

$$f(2) = 16 - 6 + 1 = 11, \ f(0) = 1$$

$$\frac{f(2)-f(0)}{2-0} = \frac{10}{2} = 5$$

Set $f'(c) = 5$:

$8c - 3 = 5 \Rightarrow c = 1$

6. Let $f(x) = (x - 3)^{-2}$. Find all values of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$.

$f'(x) = -2(x - 3)^{-3}$

$f(4) = 1, \ f(1) = \frac{1}{4}, \frac{f(4)-f(1)}{4-1} = \frac{1-\frac{1}{4}}{3} = \frac{1}{4}$

Set $f'(c) = \frac{1}{4}$:

$-2(c - 3)^{-3} = \frac{1}{4} \Rightarrow (c - 3)^3 = -8 \Rightarrow c = 1$

But $c = 1$ not in (1,4) -> No c satisfies.

Based on this information, what conclusions can be made about the Mean Value Theorem?

This does not contradict the Mean Value Theorem since f is not continuous at $x = 3$.

7. If $f(4) = 9$ and $f'(x) \geq 3$ for $4 \leq x \leq 6$, how small can $f(6)$ possibly be?

By the mean value theorem, $f(6) - f(4) = f'(c)(6 - 4)$ for some $c \in (4, 6)$

But for every $c \in [4, 6]$, $f'(x) \geq 3$.

Putting $f'(c) \geq 3$ into the equation and substituting $f(4) = 9$:

$$f(6) = f(4) + f'(c)(2) \geq 9 + 3 * 2 = 15$$

$$f(6) \geq 15$$