Risk Modeling and Stochastic Programming

Where does it occur?

$$
\begin{aligned}
\text{Maximize} \quad & CX \\
\text{Subject to} \quad & AX \le b \\
& X \ge 0
\end{aligned}
$$

Uncertain objective function returns - C

- variability in prices
- variability in production quantities
- variability in costs
- variability in market sales

Resource usages - A

- variability in raw input quality
- variability in working conditions
- variability in intermediate product yields
- variability in product requirements

Resource endowments - b

- variability in demand firm faces
- variability in resources available
- variability in working conditions

Risk Modeling and Stochastic Programming

Including Risk

When incorporating risk there are three big issues

1. What is the nature of risk?
   a. What parameters of the model are uncertain?
   b. How do we describe their distribution?

2. When are risk outcomes revealed?
   Does the producer receive information about uncertain events and will make adaptive decisions?

3. How do we model behavioral reaction to risk?
   Is expected profit maximization not the proper objective but rather some degree of aversion to the variation caused by risk?

Risk Modeling and Stochastic Programming
Why Model Risk

Why not just solve for all values of risky parameters

Curses of dimensionality and certainty

Dimensionality: Number of possible plans

(5 possible values for 3 parameters $3^5 = 243$)

Certainty: Each plan would be certain of data so we would have 243 different things we could do

—What would we do?

General Risk Modeling Aim

Generate Robust in the face of the Uncertainty

Not necessarily a best performer in any setting, but a good performer across many or most or the most likely spectrum

Risk Modeling and Stochastic Programming

Forms of assumed reaction to risk

Non Recourse or non adaptive decision making

- Decisions made now consequence felt later
- No decisions made between now and when consequences felt

Example—Buy stock now make no decision for one year

Recourse or adaptive decision making

Later time during model additional decisions made.

In this later decision period

- Decision maker knows what happened between first decision and now.
- Decision makers cannot revise prior actions but can adjust current decisions ie current decisions can be employed to make adjustments in the face of realized events -- phenomena called irreversibility and recourse

Example – Buy stock now make review decisions quarterly possibly selling and buying other stocks

Including Non Adaptive Risk

Decision Maker reaction to risk

Expected Value Maximization

$$\begin{aligned} \max \quad & \bar{c}X \\ s.t. \quad & \bar{A}X \leq \bar{b} \\ & X \geq o \end{aligned}$$

Conservative - Fat or thin coefficients

$$\begin{aligned} \max \quad & \tilde{c}X \\ s.t. \quad & \tilde{A}X \leq \tilde{b} \\ & X \geq o \end{aligned}$$

Where

$$\tilde{\mathrm{c}} = \bar{c} - \text{Risk Discount}$$

$$\tilde{\mathrm{a}} = \bar{a} + \text{Risk Discount}$$

$$\tilde{\mathrm{b}} = \tilde{b} - \text{Risk Discount}$$

E- V

Maximize E(income) - RAP * Variance(income)

Expected utility

Maximize Sum(p,Probability(p)*U[Wealth(p)])
S.T. Wealth(p)=InitWealth + Income(p) for all p
Income(p)=C(p)*X for all p

Safety First based

Maximize Sum(p,Probability(p)*Income(p))
S.T. Income(p)=C(p)*X for all p
Income(p)≥safety for all p

Including Non Adaptive Risk

First Risk Model

Markowitz mean-variance portfolio choice formulation

Given Problem

Max sum(invest, moneyinvest(invest)*avgreturn(invest))
s.t   sum(invest,moneyinvest(invest)*price(invest)) ■ funds

Markowitz observed not all money in highest valued stock

Inconsistent with LP formulation
Why? Not a basic solution

Markowitz posed the hypothesis that average returns and the variance of returns were important

Including Non Adaptive Risk

EV Formulation – Statistical Background

Given a linear objective function

$$Z = c_1 X_1 + c_2 X_2$$

where X_1, X_2 are decision variables and c_1 , c_2 are uncertain parameters distributed with means $\bar{c}_1$ and $\bar{c}_2$ as well as variances s_{11} , s_{22}, and covariance s_{12}; then Z is distributed with mean

$$\bar{Z} = \bar{c}_1 X_1 + \bar{c}_2 X_2$$

and variance $\sigma_Z^2 = s_{11} X_1^2 + s_{22} X_2^2 + 2 s_{21} X_1 X_2.$

in terms of correlation

$s_{12} = \rho_{12} \sigma_1 \sigma_2$

In matrix terms the mean and variance of Z are

$$(\bar{C}X, X'SX)$$

where in the two by two case

$$\bar{C} = \begin{bmatrix} \bar{c}_1 & \bar{c}_2 \end{bmatrix} \qquad S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}.$$

Including Non Adaptive Risk

EV Formulation – Statistical Background

Defining terms

s_{ii} **is the variance of the objective function coefficient of X_i, which is calculated using the formula**

s_{ik} = ◄ $(c_{ik} - \bar{c}_i)^2/N$ where c_{ik} is the k^{th} observation on the objective value of X_i and N is the number of observations, assuming an equally likely probability (1/N) of occurrence.[1]

s_{ij} **for $i \neq j$ is the covariance of the objective function coefficients between c_i and c_j, calculated by the formula**

$s_{ij} = \sum (c_{ik} - \bar{c}_i)(c_{jk} - \bar{c}_j)/N$. Note $s_{ij} = s_{ji}$.

$\bar{c}_i$ **is the mean value of the objective function coefficient c_i, calculated by $\bar{c}_i = _{sum}\, c_{ik}/N$. (Assuming an equally likely probability of occurrence.)**

---

1 One could also use the divisor N-1 when working with a sample.

Including Non Adaptive Risk

E-V Model Commonly Used Formulation

Markowitz Formulation

$$\text{Min } \sigma_Z^2$$

$$\text{s.t. } E = K$$

or

Freund Formulation

$$Max \quad E - \phi\, \sigma_Z^2$$

or Max E - RAP * Variance

Why Use Later – reuse of RAP and Transferability

Commonly Used Freund Formulation

$$Max \sum_j \bar{c}_j X_j \; - \; \varphi \sum_j \sum_k s_{jk} X_j X_k \quad s.t. \; \sum_j p_j X_j \; \le funds \quad X_j \; \ge 0 \; \forall j$$

Where

E = **expected value of risky c times choice of x**

Var = **sum over j of var (s_{jj}) times the square of the x variables (x_j^2) minus sum over j and k where j≠k of twice the covariance(s_{jj}) times x_j times x_k**

∅ = **risk aversion parameter**

Risk Modeling and Stochastic Programming

EV Model – Example

Assume an investor wishes to develop a stock portfolio given the stock annual returns information shown in Table 14.1. The investor has 500 dollars to invest.

Table 14.1. Data for E-V Example -- Returns by Stock and Event

	----Stock Returns by Stock and Event----			
	Stock1	Stock2	Stock3	Stock4
Event1	7	6	8	5
Event2	8	4	16	6
Event3	4	8	14	6
Event4	5	9	-2	7
Event5	6	7	13	6
Event6	3	10	11	5
Event7	2	12	-2	6
Event8	5	4	18	6
Event9	4	7	12	5
Event10	3	9	-5	6
	Stock1	Stock2	Stock3	Stock4
Price	22	30	28	26

Risk Modeling and Stochastic Programming

EV Model – Example

Table 14.2. Mean Returns and Variance Parameters for Stock Example

	Stock1	Stock2	Stock3	Stock4
Mean Returns	4.70	7.60	8.30	5.80

Variance-Covariance Matrix

	Stock1	Stock2	Stock3	Stock4
Stock1	3.21	-3.52	6.99	0.04
Stock2	-3.52	5.84	-13.68	0.12
Stock3	6.99	-13.68	61.81	-1.64
Stock4	0.04	0.12	-1.64	0.36

Risk Modeling and Stochastic Programming

EV Model – Example

In turn the objective function is

$$Max\ [4.70\ \ 7.60\ \ 8.30\ \ 5.80]\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} - \Phi\ [X_1\ \ X_2\ \ X_3\ \ X_4]\begin{bmatrix} +3.21 & -3.52 & +6.99 & +0.04 \\ -3.52 & +5.84 & -13.68 & +0.12 \\ +6.99 & -13.68 & +61.81 & -1.64 \\ +0.04 & +0.13 & -1.64 & +0.36 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

or, in scalar notation

$$Max\quad 4.70\ X_1\ +7.60\ X_2\ +8.30\ X_3\ +5.80\ X_4$$

$$-\Phi\begin{pmatrix} +3.21\ X_1^2 & -3.52\ X_1X_2 & +6.99\ X_1X_3 & +0.04\ X_1X_4 \\ -3.52\ X_2X_1 & +5.84\ X_2^2 & -13.68\ X_2X_3 & +0.12\ X_2X_4 \\ +6.99\ X_3X_1 & -13.68\ X_3X_2 & +61.81\ X_3^2 & -1.64\ X_3X_4 \\ +0.04\ X_4X_1 & +0.12\ X_4X_2 & -1.64\ X_4X_3 & +0.36\ X_2^4 \end{pmatrix}$$

This objective function is maximized subject to a constraint on investable funds:

$$22X_1\ +\ 30X_2\ +\ 28X_3\ +\ 26X_4\ \leq\ 500$$

and non-negativity conditions on the variables.

GAMS Formulation

```
SETS        STOCKS  POTENTIAL INVESTMENTS / BUYSTOCK1*BUYSTOCK4 /
            EVENTS  EQUALLY LIKELY RETURN STATES OF NATURE
                                          /EVENT1*EVENT10 / ;
ALIAS (STOCKS,STOCK);
PARAMETERS     PRICES(STOCKS) PURCHASE PRICES OF THE STOCKS
                             / BUYSTOCK1   22, BUYSTOCK2   30
                               BUYSTOCK3   28, BUYSTOCK4   26 / ;
SCALAR      FUNDS    TOTAL INVESTABLE FUNDS / 500 / ;
TABLE RETURNS(EVENTS,STOCKS) RETURNS BY STATE OF NATURE EVENT
          BUYSTOCK1   BUYSTOCK2   BUYSTOCK3   BUYSTOCK4
  EVENT1      7           6           8           5
  EVENT2      8           4          16           6
  EVENT3      4           8          14           6
  EVENT4      5           9          -2           7
  EVENT5      6           7          13           6
  EVENT6      3          10          11           5
  EVENT7      2          12          -2           6
  EVENT8      5           4          18           6
  EVENT9      4           7          12           5
  EVENT10     3           9          -5           6
PARAMETERS
   MEAN (STOCKS)       MEAN RETURNS TO X(STOCKS)
   COVAR(STOCK,STOCKS) VARIANCE COVARIANCE MATRIX;
MEAN(STOCKS) = SUM(EVENTS , RETURNS(EVENTS,STOCKS) / CARD(EVENTS) );
COVAR(STOCK,STOCKS)
     = SUM (EVENTS ,(RETURNS(EVENTS,STOCKS) - MEAN(STOCKS))
                 *(RETURNS(EVENTS,STOCK)- MEAN(STOCK)))/CARD(EVENTS);
DISPLAY MEAN , COVAR ;
SCALAR RAP   RISK AVERSION PARAMETER / 0.0 / ;
POSITIVE VARIABLES    INVEST(STOCKS)  MONEY INVESTED IN EACH STOCK
VARIABLE              OBJ             NUMBER TO BE MAXIMIZED ;
EQUATIONS             OBJJ            OBJECTIVE FUNCTION
                      INVESTAV        INVESTMENT FUNDS AVAILABLE;
OBJJ..OBJ =E=   SUM(STOCKS, MEAN(STOCKS) * INVEST(STOCKS))
        - RAP*(SUM(STOCK, SUM(STOCKS,
                    INVEST(STOCK)* COVAR(STOCK,STOCKS)*invest(stocks));
INVESTAV..     SUM(STOCKS, PRICES(STOCKS) * INVEST(STOCKS)) =L= FUNDS ;
MODEL EVPORTFOL /ALL/ ;
SCALAR VAR  THE VARIANCE ;
SET RAPS   RISK AVERSION PARAMETERS /R0*R25/
PARAMETER RISKAVER(RAPS) RISK AVERSION COEFICIENT BY RISK AVERSION
             /R0   0.00000,  R1   0.00025,  R2   0.00050,  R3  0.00075,
              R4   0.00100,  R5   0.00150,  R6   0.00200,  R7  0.00300,
              R20  5.00000,  R21  10.0000,  R22  15.    ,  R23  20.
              R24  40.    ,  R25  80./
PARAMETER OUTPUT(*,RAPS) RESULTS FROM MODEL RUNS WITH VARYING RAP
LOOP (RAPS,RAP=RISKAVER(RAPS);
           SOLVE EVPORTFOL USING NLP MAXIMIZING OBJ ;
           VAR = SUM(STOCK, SUM(STOCKS,
                INVEST.L(STOCK)* COVAR(STOCK,STOCKS) * INVEST.L(STOCKS)
         OUTPUT("RAP",RAPS)=RAP;
         OUTPUT(STOCKS,RAPS)=INVEST.L(STOCKS);
         OUTPUT("OBJ",RAPS)=OBJ.L;
         OUTPUT("MEAN",RAPS)=SUM(STOCKS, MEAN(STOCKS)*invest.l(stock));
         OUTPUT("VAR",RAPS) = VAR;
         OUTPUT("STD",RAPS)=SQRT(VAR);
         OUTPUT("SHADPRICE",RAPS)=INVESTAV.M;
         OUTPUT("IDLE",RAPS)=FUNDS-INVESTAV.L); );
DISPLAY OUTPUT;
```

Risk Modeling and Stochastic Programming

EV Model – Example

Table 14.4. E-V Example Solutions for Alternative Risk Aversion Parameters

RAP or φ	0	0.00025	0.0005	0.00075	0.001
BUYSTOCK2			1.263	5.324	7.355
BUYSTOCK3	17.857	17.857	16.504	12.152	9.977
OBJ	148.214	143.287	138.444	135.688	134.245
MEAN	148.214	148.214	146.581	141.331	138.705
VAR	19709.821	19709.821	16274.764	7523.441	4460.478
STD	140.392	140.392	127.573	86.738	66.787
SHADPRICE	0.296	0.277	0.261	0.260	0.260

RAP or φ	0.0015	0.002	0.003	0.005	0.010
BUYSTOCK2	9.386	10.401	11.416	12.229	12.838
BUYSTOCK3	7.801	6.713	5.625	4.755	4.102
OBJ	132.671	131.753	130.575	129.005	125.999
MEAN	136.080	134.767	133.454	132.404	131.617
VAR	2272.647	1506.907	959.949	679.907	561.764
STD	47.672	38.819	30.983	26.075	23.702
SHADPRICE	0.259	0.257	0.255	0.251	0.241

RAP or φ	0.011	0.012	0.015	0.025	0.050
BUYSTOCK1			1.273	4.372	4.405
BUYSTOCK2	12.893	12.960	12.420	11.070	8.188
BUYSTOCK3	4.043	3.972	3.550	2.561	1.753
BUYSTOCK4					4.168
OBJ	125.441	124.614	123.380	120.375	116.805
MEAN	131.545	131.459	129.839	125.939	121.656
VAR	554.929	547.587	430.560	222.576	97.026
STD	23.557	23.401	20.750	14.919	9.850
SHADPRICE	0.239	0.236	0.234	0.230	0.224

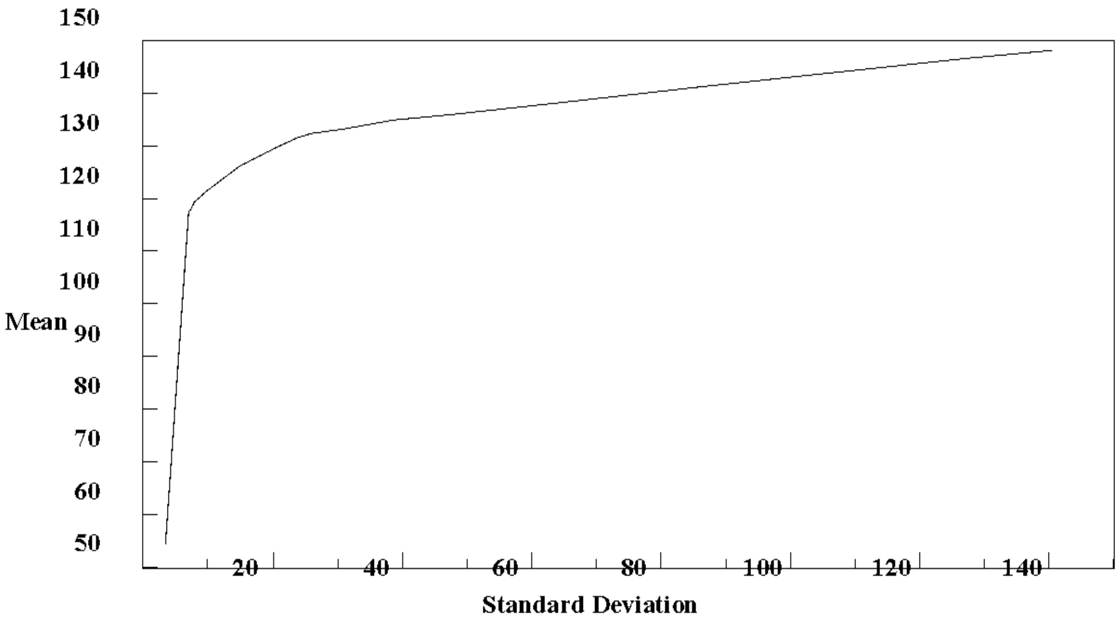
RAP or φ	0.100	0.300	0.500	1.000	2.500
BUYSTOCK1	4.105	3.905	3.865	3.835	1.777
BUYSTOCK2	6.488	5.354	5.128	4.958	2.289
BUYSTOCK3	1.340	1.064	1.009	0.968	0.446
BUYSTOCK4	6.829	8.602	8.957	9.223	4.296
OBJ	113.118	102.254	92.010	66.674	27.185
MEAN	119.327	117.774	117.463	117.230	54.370
VAR	62.086	51.734	50.905	50.556	10.874
STD	7.879	7.193	7.135	7.110	3.298
SHADPRICE	0.214	0.173	0.133	0.032	0
IDLE FUNDS					268.044

Risk Modeling and Stochastic Programming

EV Model – Example

Efficient Frontier:

14.1. E-V Model Efficient Frontier



Risk Modeling and Stochastic Programming

Characteristics of E-V Model Optimal Solutions

Properties of optimal E-V solutions may be examined via the Kuhn-Tucker conditions.

Given the problem

$$
\begin{array}{llcl}
Max & \bar{C}X - \Phi\, X'SX & & \\
s.t. & AX & \leq & b \\
 & X & \geq & 0
\end{array}
$$

Its Lagrangian function is

$$\mathcal{L}(X,\mu) = \bar{C}X - \Phi X'SX - \mu(AX - b)$$

and the Kuhn-Tucker conditions are

$$
\begin{array}{rclcl}
\partial\mathcal{L}/\partial X & = & \bar{C} - 2\Phi X'S - \mu A & \leq & 0 \\
(\partial\mathcal{L}/\partial X)X & = & (\bar{C} - 2\Phi X'S - \mu A)X & = & 0 \\
X & & & \geq & 0 \\
\partial\mathcal{L}/\partial \mu & = & -(AX - b) & \geq & 0 \\
\mu(\partial\mathcal{L}/\partial \mu) & = & \mu(AX - b) & = & 0 \\
\mu & & & \geq & 0
\end{array}
$$

Risk Modeling and Stochastic Programming

Unified Model

Expected Income = sum (k,prob(k)*Income(k))

Variance = sum (k,prob(k)*(income(k)-Expected Income)**2)

$$Max \quad \overline{inc} - \Phi \left\{ \sum_k p_k \left[\left(d_k^+\right)^2 + \left(d_k^-\right)^2 \right] \right\}^{0.5}$$

$$s.t. \quad \sum_j a_{ij} X_j \leq b_i \quad for\ all\ i$$

$$\sum_j c_{kj} X_j - Inc_k = 0 \quad for\ all\ k$$

$$\sum_k p_k\ Inc_k - \overline{Inc} = 0$$

$$Inc_k - \overline{Inc} - d_k^+ + d_k^- = 0 \quad for\ all\ k$$

$$X_j, \quad d_k^+, \quad d_k^- \geq 0 \quad for\ all\ j,k$$

$$Inc_k, \quad \overline{Inc} \quad \begin{matrix} \leq \\ \overline{\phantom{<}} \\ > \end{matrix} \quad 0 \quad for\ all\ k$$

Risk Modeling and Stochastic Programming

Unified Model- GAMS Formulation

```
SETS        STOCKS   POTENTIAL INVESTMENTS / BUYSTOCK1*BUYSTOCK4 /
            EVENTS   EQUALLY LIKELY RETURN STATES OF NATURE
                                          /EVENT1*EVENT10 / ;
PARAMETERS      PRICES(STOCKS) PURCHASE PRICES OF THE STOCKS
                             / BUYSTOCK1   22
                               BUYSTOCK2   30
                               BUYSTOCK3   28
                               BUYSTOCK4   26 / ;

SCALAR       FUNDS     TOTAL INVESTABLE FUNDS / 500 / ;
TABLE RETURNS(EVENTS,STOCKS) RETURNS BY STATE OF NATURE EVENT

          BUYSTOCK1   BUYSTOCK2   BUYSTOCK3   BUYSTOCK4
  EVENT1      7           6            8           5
  EVENT2      8           4           16           6
  EVENT3      4           8           14           6
  EVENT4      5           9           -2           7
  EVENT5      6           7           13           6
  EVENT6      3          10           11           5
  EVENT7      2          12           -2           6
  EVENT8      5           4           18           6
  EVENT9      4           7           12           5
  EVENT10     3           9           -5           6

SCALAR RAP   RISK AVERSION PARAMETER / 0.0 / ;

POSITIVE VARIABLES  INVEST(STOCKS) MONEY INVESTED IN EACH STOCK
                    POSDEV(EVENTS) POSITIVE DEVIATIONS FROM MEAN INCOME
                    NEGDEV(EVENTS) NEGATIVE DEVIATIONS FROM MEAN INCOME

VARIABLES             OBJ               NUMBER TO BE MAXIMIZED
                      RETURN(EVENTS)    RETURNS BY EVENT
                      MEAN              MEAN RETURNS ;

EQUATIONS             OBJJ              OBJECTIVE FUNCTION
                      RETURNDEF(EVENTS) RETURNS DEFINITION
                      AVRET             AVERAGE RETURNS
                      INVESTAV          INVESTMENT FUNDS AVAILABLE
                      DEVIATION(EVENTS) DEVIATIONS FROM MEAN INCOME ;
OBJJ..
     OBJ =E= MEAN
   - RAP*(SUM(EVENTS,(POSDEV(EVENTS)+NEGDEV(EVENTS))**2)/CARD(EVENTS));

INVESTAV..     SUM(STOCKS, PRICES(STOCKS) * INVEST(STOCKS)) =L= FUNDS ;

RETURNDEF(EVENTS)..SUM(STOCKS, RETURNS(EVENTS,STOCKS) * INVEST(STOCKS))
                   - RETURN(EVENTS) =E= 0 ;

AVRET..          SUM(EVENTS,1/CARD(EVENTS)*RETURN(EVENTS)) - MEAN=E= 0 ;

DEVIATION(EVENTS)..RETURN(EVENTS)-MEAN -POSDEV(EVENTS) + NEGDEV(EVENTS)
                   =E= 0 ;
MODEL EVPORTFOL /ALL/ ;

SOLVE EVPORTFOL USING NLP MAXIMIZING OBJ ;
```

Risk Modeling and Stochastic Programming

Motad Model Development

Formally, the total absolute deviation of income from mean income under the k^{th} state of nature (D_k) is

$$D_k = \left| \left(\sum_j c_{kj} X_j \right) - \left(\sum_j \bar{c}_j X_j \right) \right|$$

which can be rewritten as

$$D_k = \left| \sum_j \left(c_{kj} - \bar{c}_j \right) X_j \right|$$

Total absolute deviation (TAD) is the sum of D_k across the states of nature. Now introducing deviation variables to depict positive and negative deviations we get

$$TAD = \sum_k D_k = \sum_k \left(d_k^+ + d_k^- \right)$$

$$where \sum_j \left(c_{kj} - \bar{c}_j \right) X_j - d_k^+ + d_k^- = 0 \quad for\ all\ k$$

The final Minimization of Total Absolute Deviations (MOTAD) formulation is

$$\begin{array}{llll}
Max & \sum_j \bar{c}_j X_j - \Psi \sum_k \left(d_k^+ + d_k^- \right) & & \\
s.t. & \sum_j \left(c_{kj} - \bar{c}_j \right) X_j - d_k^+ + d_k^- & = 0 & for\ all\ k \\
 & \sum_j a_{ij} X_j & \leq b_i & for\ all\ i \\
 & X_j, \quad d_k^+, \quad d_k^- & \geq 0 & for\ all\ j, k
\end{array}$$

Risk Modeling and Stochastic Programming

Motad Model Development

Model considering only negative deviations from the mean

$$
\begin{array}{llll}
Max & \sum_j \bar{c}_j X_j - \Theta \sum_k d_k^- & & \\
s.t. & \sum_j \left(c_{kj} - \bar{c}_j\right) X_j + d_k^- & \geq 0 & \text{for all } k \\
 & \sum_j a_{ij} X_j & \leq b_i & \text{for all } i \\
 & X_j, \quad d_k^- & \geq 0 & \text{for all } j, k
\end{array}
$$

the standard error of a normally distributed population can be estimated given sample size N, by multiplying mean absolute deviation (MAD), total absolute deviation (TAD), or total negative deviation (TND) by appropriate constraints. Thus,

$$
\sigma \approx \left|\frac{\pi N}{2(N-1)}\right|^{0.5} MAD = \left|\frac{\pi N}{2(N-1)}\right|^{0.5} \frac{TAD}{N} = \left|\frac{\pi}{2N(N-1)}\right|^{0.5} TAD = \left|\frac{2\pi}{N(N-1)}\right|^{0.5} TND
$$

This transformation is commonly used in MOTAD formulations such as:

$$
\begin{array}{llll}
Max & \sum_j \bar{c}_j X_j - \gamma \sigma & & \\
s.t. & \sum_j \left(c_{kj} - \bar{c}_j\right) X_j + d_k^- & \geq 0 & \text{for all } k \\
 & \sum_j a_{ij} X_j & \leq b_i & \text{for all } i \\
 & -TND + \sum_k d_k^- & = 0 & \\
 & \left(\frac{2\pi}{N(N-1)}\right)^{0.5} TND - \sigma & = 0 & \\
 & X_j, \quad TND, \quad d_k^-, \quad \sigma & \geq 0 & \text{for all } j, k
\end{array}
$$

Risk Modeling and Stochastic Programming

Motad Example

This example uses the same data as in the E-V Portfolio example.

Table 14.1. Data for E-V Example -- Returns by Stock and Event

	----Stock Returns by Stock and Event----			
	Stock1	Stock2	Stock3	Stock4
Event1	7	6	8	5
Event2	8	4	16	6
Event3	4	8	14	6
Event4	5	9	-2	7
Event5	6	7	13	6
Event6	3	10	11	5
Event7	2	12	-2	6
Event8	5	4	18	6
Event9	4	7	12	5
Event10	3	9	-5	6
	Stock1	Stock2	Stock3	Stock4
Price	22	30	28	26

Table 14.5. Deviations from the Mean for Portfolio Example

	Stock1	Stock2	Stock3	Stock4
Event1	2.3	-1.6	-0.3	-0.8
Event2	3.3	-3.6	7.7	0.2
Event3	-0.7	0.4	5.7	0.2
Event4	0.3	1.4	-10.3	1.2
Event5	1.3	-0.6	4.7	0.2
Event6	-1.7	2.4	2.7	-0.8
Event7	-2.7	4.4	-10.3	0.2
Event8	0.3	-3.6	9.7	0.2
Event9	-0.7	-0.6	3.7	-0.8
Event10	-1.7	1.4	-13.3	0.2

Risk Modeling and Stochastic Programming

Motad Example

Table 14.6. Example MOTAD Model Formulation

$$
\begin{array}{lrlrlrlrlllll}
\text{Max} & 4.70 X_1 & + & 7.60 X_2 & + & 8.30 X_3 & + & 5.80 X_4 & & & -\gamma\ \sigma & & \\
\text{s.t.} & 22 X_1 & + & 30 X_2 & + & 28 X_3 & + & 26 X_4 & & & & \leq & 500 \\
+ & 2.300 X_1 & - & 1.600 X_2 & - & 0.300 X_3 & - & 0.800 X_4 & + d_1^- & & & \geq & 0 \\
+ & 3.300 X_1 & - & 3.600 X_2 & + & 7.700 X_3 & + & 0.200 X_4 & + d_2^- & & & \geq & 0 \\
- & 0.700 X_1 & + & 0.400 X_2 & + & 5.700 X_3 & + & 0.200 X_4 & + d_3^- & & & \geq & 0 \\
+ & 0.300 X_1 & + & 1.400 X_2 & - & 10.300 X_3 & + & 1.200 X_4 & + d_4^- & & & \geq & 0 \\
+ & 1.300 X_1 & - & 0.600 X_2 & + & 4.700 X_3 & + & 0.200 X_4 & + d_5^- & & & \geq & 0 \\
- & 1.700 X_1 & + & 2.400 X_2 & + & 2.700 X_3 & - & 0.800 X_4 & + d_6^- & & & \geq & 0 \\
- & 2.700 X_1 & + & 4.400 X_2 & - & 10.300 X_3 & + & 0.200 X_4 & + d_7^- & & & \geq & 0 \\
+ & 0.300 X_1 & - & 3.600 X_2 & + & 9.700 X_3 & + & 0.200 X_4 & + d_8^- & & & \geq & 0 \\
- & 0.700 X_1 & - & 0.600 X_2 & + & 3.700 X_3 & - & 0.800 X_4 & + d_9^- & & & \geq & 0 \\
- & 1.700 X_1 & + & 1.400 X_2 & - & 13.300 X_3 & + & 0.200 X_4 & + d_{10}^- & & & \geq & 0 \\
 & & & & & & & & \sum_k d_k^- & - \text{TND} & & = & 0 \\
 & & & & & & & & & \Delta\ \text{TND} & -\ \sigma & = & 0
\end{array}
$$

Risk Modeling and Stochastic Programming

Motad Example GAMS Formulation

```
SETS        STOCKS  POTENTIAL INVESTMENTS / BUYSTOCK1*BUYSTOCK4 /
            EVENTS  EQUALLY LIKELY RETURN STATES OF NATURE
                                    /EVENT1*EVENT10 / ;
PARAMETERS     PRICES(STOCKS) PURCHASE PRICES OF THE STOCKS
                           / BUYSTOCK1   22
                             BUYSTOCK2   30
                             BUYSTOCK3   28
                             BUYSTOCK4   26 / ;
SCALAR       FUNDS    TOTAL INVESTABLE FUNDS / 500 /
             N        SAMPLE SIZE
             PI       /3.141716/
            TRAN     TRANSFORMATION COEF MAD TO STD ERROR ;
        N=CARD(EVENTS);
        TRAN = ((PI * N)/(2*(N-1)))**0.5  ;
TABLE RETURNS(EVENTS,STOCKS) RETURNS BY OBSERVATION
            BUYSTOCK1   BUYSTOCK2   BUYSTOCK3   BUYSTOCK4
  EVENT1      7           6           8           5
  EVENT2      8           4          16           6
  EVENT3      4           8          14           6
  EVENT4      5           9          -2           7
  EVENT5      6           7          13           6
  EVENT6      3          10          11           5
  EVENT7      2          12          -2           6
  EVENT8      5           4          18           6
  EVENT9      4           7          12           5
  EVENT10     3           9          -5           6
PARAMETERS
   MEAN  (STOCKS)        MEAN RETURNS TO X(STOCKS)
   DEVS(EVENTS,STOCKS)   DEVIATIONS FROM MEAN INCOME  ;
MEAN(STOCKS)  = SUM(EVENTS , RETURNS(EVENTS,STOCKS) / N );
DEVS(EVENTS,STOCKS) = RETURNS(EVENTS,STOCKS) - MEAN(STOCKS) ;
DISPLAY MEAN , DEVS ;
SCALAR RAP   RISK AVERSION PARAMETER / 0.0 / ;
DISPLAY TRAN ;
POSITIVE VARIABLES INVEST(STOCKS)     MONEY INVESTED IN EACH STOCK
                   DEVIATION(EVENTS) DEVIATION OF TOTAL INCOME BY EVENT
                   TRETURN            TOTAL RETURNS
                   APPROXSTDE         STANDARD ERROR AS APPROXIMATED
                   MAD                MEAN ABSOLUTE DEVIATION
VARIABLE           OBJ                NUMBER TO BE MAXIMIZED ;
EQUATIONS          OBJJ               OBJECTIVE FUNCTION
                   INVESTAV           INVESTMENT FUNDS AVAILABLE
                   DEVIATE(EVENTS)    DEVIATION EQUATION FOR EVENTS
                   MADBALANCE         MEAN ABSOLUTE DEVIATION DEFINITION
                   SEBALANCE          STANDARD DEVIATION APPROXIMATION
OBJJ..         OBJ =E= SUM(STOCKS, MEAN(STOCKS) * INVEST(STOCKS))
                         - RAP*APPROXSTDE;
INVESTAV..     SUM(STOCKS, PRICES(STOCKS) * INVEST(STOCKS)) =L= FUNDS ;
DEVIATE(EVENTS).. SUM(STOCKS, DEVS(EVENTS,STOCKS)*INVEST(STOCKS))
                      + DEVIATION(EVENTS) =G= 0. ;
MADBALANCE..      2*SUM(EVENTS, DEVIATION(EVENTS))/N - MAD =E= 0. ;
SEBALANCE..       TRAN*MAD - APPROXSTDE =E= 0. ;
MODEL MOTADPORTF /ALL/ ;
SOLVE MOTADPORTF USING LP MAXIMIZING OBJ ;
```

Risk Modeling and Stochastic Programming

Motad Example

Table 14.7. MOTAD Example Solutions for Alternative Risk Aversion Parameters

RAP		0.050	0.100	0.110	0.120
BUYSTOCK2					11.603
BUYSTOCK3	17.857	17.857	17.857	17.857	5.425
OBJ	148.214	140.146	132.078	130.464	129.390
MEAN	148.214	148.214	148.214	148.214	133.213
MAD	122.143	122.143	122.143	122.143	24.111
STDAPPROX	161.367	161.367	161.367	161.367	31.854
VAR	19709.821	19709.821	19709.821	19709.821	883.113
STD	140.392	140.392	140.392	140.392	29.717
SHADPRICE	0.296	0.280	0.264	0.261	0.259

RAP	0.130	0.150	0.260	0.400	0.500
BUYSTOCK1					2.663
BUYSTOCK2	11.603	11.603	11.916	12.379	10.985
BUYSTOCK3	5.425	5.425	5.090	4.594	3.995
OBJ	129.072	128.435	125.179	121.204	118.606
MEAN	133.213	133.213	132.809	132.210	129.161
MAD	24.111	24.111	22.212	20.827	15.979
STDAPPROX	31.854	31.854	29.345	27.515	21.110
VAR	883.113	883.113	771.228	643.507	455.983
STD	29.717	29.717	27.771	25.367	21.354
SHADPRICE	0.258	0.257	0.250	0.242	0.237

RAP	0.750	1.000	1.250	1.500	1.750
BUYSTOCK1	5.145	7.119	2.817	2.817	2.817
BUYSTOCK2	10.409	9.879	5.617	5.617	5.617
BUYSTOCK3	2.661	1.564	1.824	1.824	1.824
BUYSTOCK4		0.123	8.402	8.402	8.402
OBJ	114.168	111.009	108.372	106.086	103.801
MEAN	125.384	122.240	119.799	119.799	119.799
MAD	11.320	8.501	6.920	6.920	6.920
STDAPPROX	14.955	11.231	9.142	9.142	9.142
VAR	211.996	121.386	83.886	83.886	83.886
STD	14.560	11.018	9.159	9.159	9.159
SHADPRICE	0.228	0.222	0.217	0.212	0.208

RAP	2.000	2.500	5.000	10.000	12.500
BUYSTOCK1	2.817	2.817	2.858	2.858	2.858
BUYSTOCK2	5.617	5.617	4.178	4.178	4.178
BUYSTOCK3	1.824	1.824	1.242	1.242	1.242
BUYSTOCK4	8.402	8.402	10.654	10.654	10.654
OBJ	101.515	96.944	76.540	35.790	15.415
MEAN	119.799	119.799	117.289	117.289	117.289
MAD	6.920	6.920	6.169	6.169	6.169
STDAPPROX	9.142	9.142	8.150	8.150	8.150
VAR	83.886	83.886	57.695	57.695	57.695
STD	9.159	9.159	7.596	7.596	7.596
SHADPRICE	0.203	0.194	0.153	0.072	0.031

Risk Modeling and Stochastic Programming

Motad Example

Risk Modeling and Stochastic Programming

E-V – Nasty Assumptions

Form of Distribution

- **Normality**
- **Location and Scale**

Form of Utility

- **Exponential**
- **Taylor Series Approximation**

Risk Modeling and Stochastic Programming

Motad – Finding a Risk Aversion Parameter

The link between the E-standard error and E-V risk aversion parameters is as follows:

Consider the models

$$\begin{array}{llll} Max\ cX - \psi\ \sigma^2(X) & & & \\ s.t. & AX & \leq & b \\ & X & \geq & 0 \end{array} \quad versus \quad \begin{array}{llll} Max\ cX - \xi\ \sigma(X) & & & \\ s.t. & AX & \leq & b \\ & X & \geq & 0 \end{array}$$

The first order conditions assuming X is nonzero are

$$c - 2\ \psi\ \sigma(X)\frac{\partial \sigma(X)}{\partial X} - \lambda A = 0 \qquad c - \xi\ \frac{\partial \sigma(X)}{\partial X} - \lambda A = 0$$

For these two solutions to be identical in terms of X and ▶▶, then

$$\psi = \frac{\xi}{2\ \sigma(X)}$$

Risk Modeling and Stochastic Programming

Safety First

The Safety First model assumes that decision makers will choose plans to first assure a given safety level for income.

$$\sum_j c_{kj} X_j \quad \geq \quad S \quad \text{for all } k$$

The overall problem then becomes

$$\begin{aligned}
Max \quad & \sum_j \overline{c_j} X_j \\
s.t. \quad & \sum_j a_{ij} X_j \quad \leq \quad b_i \quad \text{for all } i \\
& \sum_j c_{kj} X_j \quad \geq \quad S \quad \text{for all } k \\
& X_j \quad \geq \quad 0 \quad \text{for all } j
\end{aligned}$$

Safety First – Example Formulation

Table 14.8. Example Formulation of Safety First Problem

$$
\begin{array}{lrlrlrlrlcl}
Max & 4.70 X_1 & + & 7.60 X_2 & + & 8.30 X_3 & + & 5.80 X_4 & & \\
s.t. & 22 X_1 & + & 30 X_2 & + & 28 X_3 & + & 26 X_4 & \leq & 500 \\
 & 7 X_1 & + & 6 X_2 & + & 8 X_3 & + & 5 X_4 & \geq & S \\
 & 8 X_1 & + & 4 X_2 & + & 16 X_3 & + & 6 X_4 & \geq & S \\
 & 4 X_1 & + & 8 X_2 & + & 14 X_3 & + & 6 X_4 & \geq & S \\
 & 5 X_1 & + & 9 X_2 & - & 2 X_3 & + & 7 X_4 & \geq & S \\
 & 6 X_1 & + & 7 X_2 & + & 13 X_3 & + & 6 X_4 & \geq & S \\
 & 3 X_1 & + & 10 X_2 & + & 11 X_3 & + & 5 X_4 & \geq & S \\
 & 2 X_1 & + & 12 X_2 & - & 2 X_3 & + & 6 X_4 & \geq & S \\
 & 5 X_1 & + & 4 X_2 & + & 18 X_3 & + & 6 X_4 & \geq & S \\
 & 4 X_1 & + & 7 X_2 & + & 12 X_3 & + & 5 X_4 & \geq & S \\
 & 3 X_1 & + & 9 X_2 & - & 5 X_3 & + & 6 X_4 & \geq & S
\end{array}
$$

Solutions

Table 14.9. Safety First Example Solutions for Alternative Safety Levels

RUIN	-100.000	-50.000	0.0	25.000	50.000
BUYSTOCK2	0.0	2.736	6.219	7.960	9.701
BUYSTOCK3	17.857	14.925	11.194	9.328	7.463
OBJ	148.214	144.677	140.174	137.923	135.672
MEAN	148.214	144.677	140.174	137.923	135.672
VAR	19709.821	12695.542	6066.388	3717.016	2011.116
STD	140.392	112.674	77.887	60.967	44.845
SHADPRICE	0.296	0.280	0.280	0.280	0.280

Note: The abbreviations are the same as in the previous example solutions with RUIN giving the safety level.

Safety First – GAMS Formulation

```
SETS STOCKS  POTENTIAL INVESTMENTS / BUYSTOCK1*BUYSTOCK4 /
     EVENTS  EQUALLY LIKELY RETURN STATES OF NATURE/EVENT1*EVENT10 / ;
PARAMETERS      PRICES(STOCKS) PURCHASE PRICES OF THE STOCKS
                           / BUYSTOCK1    22, BUYSTOCK2    30
                             BUYSTOCK3    28, BUYSTOCK4    26 / ;
SCALAR      FUNDS    TOTAL INVESTABLE FUNDS / 500 / ;
TABLE RETURNS(EVENTS,STOCKS) RETURNS BY STATE OF NATURE EVENT
          BUYSTOCK1    BUYSTOCK2    BUYSTOCK3    BUYSTOCK4
  EVENT1      7            6            8            5
  EVENT2      8            4           16            6
  EVENT3      4            8           14            6
  EVENT4      5            9           -2            7
  EVENT5      6            7           13            6
  EVENT6      3           10           11            5
  EVENT7      2           12           -2            6
  EVENT8      5            4           18            6
  EVENT9      4            7           12            5
  EVENT10     3            9           -5            6
SCALAR RUIN  LEVEL OF SAFETY
       N      SAMPLE SIZE;
       N = CARD(EVENTS) ;
PARAMETER STANDERROR(STOCKS)  STANDARD ERROR OF STOCKS
          AVRET(STOCKS)       AVERAGE RETURN BY STOCK        ;
          AVRET(STOCKS)      = SUM(EVENTS,RETURNS(EVENTS,STOCKS))/N;
          STANDERROR(STOCKS) = SQRT(SUM(EVENTS,
                         (RETURNS(EVENTS,STOCKS)-AVRET(STOCKS))*
                         (RETURNS(EVENTS,STOCKS)-AVRET(STOCKS))/(N-1)));
POSITIVE VARIABLES     INVEST(STOCKS)     MONEY INVESTED IN EACH STOCK
                       ART                ARTIFICIAL VARIABLE
VARIABLES              OBJ                NUMBER TO BE MAXIMIZED
                       MEAN               MEAN INCOME
EQUATIONS              OBJJ               OBJECTIVE FUNCTION
                       AVRETDEF           AVERAGE RETURNS DEFINITION
                       INVESTAV           INVESTMENT FUNDS AVAILABLE
                       SAFETY(EVENTS)     SAFETY EQUATION ;
OBJJ..                 OBJ =E= MEAN -99999* ART;
AVRETDEF..        MEAN =E= SUM(STOCKS, AVRET(STOCKS) * INVEST(STOCKS));
INVESTAV..        SUM(STOCKS, PRICES(STOCKS) * INVEST(STOCKS)) =L=FUNDS ;
SAFETY(EVENTS)..       SUM(STOCKS,RETURNS(EVENTS,STOCKS)*INVEST(STOCKS))
                                           + ART =G=  RUIN;
MODEL SAFETYPORT /ALL/ ;
SET RUNS   ALTERNATIVE RUIN LEVELS /R0*R6/
PARAMETER RUINLEVEL(RUNS) SAFETY LEVELS
             /R0  -100,    R1  -50,  R2   0,  R3   25,
              R4    50,    R5   75 , R6 100/
PARAMETER OUTPUT(*,RUNS) RESULTS FROM MODEL RUNS WITH VARYING RUIN LEVEL
           RETURN(EVENTS) RETURN BY EVENTS
           VAR            VARIANCE;
LOOP (RUNS,RUIN=RUINLEVEL(RUNS);
            SOLVE SAFETYPORT USING LP MAXIMIZING OBJ ;
  RETURN(EVENTS) =SUM(STOCKS, RETURNS(EVENTS,STOCKS)*INVEST.L(STOCKS));
   VAR = SUM(EVENTS,(RETURN(EVENTS)-MEAN.L)*(RETURN(EVENTS)-MEAN.L))/N;
            OUTPUT("RUIN",RUNS)=RUIN;
            OUTPUT(STOCKS,RUNS)=INVEST.L(STOCKS);
            OUTPUT("OBJ",RUNS)=OBJ.L;
            OUTPUT("MEAN",RUNS)=MEAN.L;
            OUTPUT("VAR",RUNS) = VAR;
            OUTPUT("STD",RUNS)=SQRT(VAR);
            OUTPUT("SHADPRICE",RUNS)=INVESTAV.M;
            OUTPUT("IDLE",RUNS)=FUNDS-INVESTAV.L);
```

Risk Modeling and Stochastic Programming

Target Motad

The Target MOTAD formulation incorporates a safety level of income while also allowing negative deviations from that safety level.

$$
\begin{array}{llllll}
Max & \sum_j \overline{c_j} X_j & & & & \\
s.t. & \sum_j a_{ij} X_j & & \leq & b_i & for\ all\ i \\
 & \sum_j c_{kj} X_j & + \quad Dev_k & \geq & T & for\ all\ k \\
 & & \sum_k p_k Dev_k & \leq & \lambda & \\
 & X_j, & Dev_k & \geq & 0 & for\ all\ j,k
\end{array}
$$

Table 14.10. Example Formulation of Target MOTAD Problem

Max	4.70	X_1	+	7.60	X_2	+	8.30	X_3	+	5.80	X_4				
s.t.	22	X_1	+		X_2	+	28	X_3	+	26	X_4			$\leq$	500
	7	X_1	+		X_2	+	8	X_3	+	5	X_4	+	Dev_1	$\geq$	T
	8	X_1	+		X_2	+	16	X_3	+	6	X_4	+	Dev_2	$\geq$	T
	4	X_1	+		X_2	+	14	X_3	+	6	X_4	+	Dev_3	$\geq$	T
	5	X_1	+		X_2	−	2	X_3	+	7	X_4	+	Dev_4	$\geq$	T
	6	X_1	+		X_2	+	13	X_3	+	6	X_4	+	Dev_5	$\geq$	T
	3	X_1	+		X_2	+	11	X_3	+	5	X_4	+	Dev_6	$\geq$	T
	2	X_1	+		X_2	−	2	X_3	+	6	X_4	+	Dev_7	$\geq$	T
	5	X_1	+		X_2	+	18	X_3	+	6	X_4	+	Dev_8	$\geq$	T
	4	X_1	+		X_2	+	12	X_3	+	5	X_4	+	Dev_9	$\geq$	T
	3	X_1	+		X_2	−	5	X_3	+	6	X_4	+	Dev_{10}	$\geq$	T
													$\sum_k Dev_k$	$\leq$	λ

Table 14.11. Target MOTAD Example Solutions for Alternative Deviation Limits

TARGETDEV	120.000	60.000	24.000	12.000	10.800
BUYSTOCK2	0.0	0.0	7.081	10.193	10.516
BUYSTOCK3	17.857	17.857	10.270	6.936	6.590
OBJ	148.214	148.214	139.059	135.037	134.618
MEAN	148.214	148.214	139.059	135.037	134.618
VAR	19709.821	19709.821	4822.705	1646.270	1433.820
STD	140.392	140.392	69.446	40.574	37.866
SHADPRICE	0.296	0.296	0.286	0.295	0.295

TARGETDEV	8.400	7.200	3.600
BUYSTOCK1	0.0	0.0	3.459
BUYSTOCK2	11.259	11.782	11.405
BUYSTOCK3	5.794	5.234	2.919
OBJ	133.659	132.982	127.168
MEAN	133.659	132.982	127.168
VAR	1030.649	816.629	277.270
STD	32.104	28.577	16.651
SHADPRICE	0.298	0.298	0.815

Note: The abbreviations are again the same with TARGETDEV giving the ⏩ value.

Risk Modeling and Stochastic Programming

DEMP Model

Direct Expected Utility Maximizing Nonlinear Programming

$$Max \sum_k p_k U(W_k)$$

$$\begin{aligned}
s.t. \quad & & \sum_j a_{ij} X_j & \leq b_i & & for\ all\ i \\
& W_k & - \sum_j c_{kj} X_j & = W_o & & for\ all\ k \\
& W_k & & \lesseqgtr 0 & & for\ all\ k \\
& & X_j & \geq 0 & &
\end{aligned}$$

where p_k is the probability of the k^{th} state of nature;

W_o is initial wealth;

W_k is the wealth under the k^{th} state of nature; and

c_{kj} is the return to one unit of the j^{th} activity under the k^{th} state of nature.

Risk Modeling and Stochastic Programming

DEMP Model – Example

Table 14.12. Example Formulation of DEMP Problem

$$\text{Max} \quad \sum_k (W_k)^{power}$$

s.t.			22	X_1	+	30	X_2	+	28	X_3	+	26	X_4	$\leq$	500
	W_1	–	7	X_1	–	6	X_2	–	8	X_3	–	5	X_4	=	100
	W_2	–	8	X_1	–	4	X_2	–	16	X_3	–	6	X_4	=	100
	W_3	–	4	X_1	–	8	X_2	–	14	X_3	–	6	X_4	=	100
	W_4	–	5	X_1	–	9	X_2	+	2	X_3	–	7	X_4	=	100
	W_5	–	6	X_1	–	7	X_2	–	13	X_3	–	6	X_4	=	100
	W_6	–	3	X_1	–	10	X_2	–	11	X_3	–	5	X_4	=	100
	W_7	–	2	X_1	–	12	X_2	+	2	X_3	–	6	X_4	=	100
	W_8	–	5	X_1	–	4	X_2	–	18	X_3	–	6	X_4	=	100
	W_9	–	4	X_1	–	7	X_2	–	12	X_3	–	5	X_4	=	100
	W_{10}	–	3	X_1	–	9	X_2	+	5	X_3	–	6	X_4	=	100

DEMP Model – Example

Table 14.13. DEMP Example Solutions for Alternative Utility Function Exponents

POWER	0.950	0.900	0.750	0.500	0.400
BUYSTOCK2			4.560	8.563	9.344
BUYSTOCK3	17.857	17.857	12.972	8.683	7.846
OBJ	186.473	140.169	60.363	15.282	8.848
MEAN	248.214	248.214	242.319	237.144	236.134
VAR	19709.821	19709.821	8903.295	3054.034	2309.233
STD	140.392	140.392	94.357	55.263	48.054
SHADPRICE	0.287	0.277	0.269	0.266	0.265
POWER	0.300	0.200	0.100	0.050	0.030
BUYSTOCK2	9.919	10.358	10.705	10.852	10.907
BUYSTOCK3	7.230	6.759	6.388	6.230	6.171
OBJ	5.127	2.972	1.724	1.313	1.177
MEAN	235.390	234.822	234.374	234.184	234.113
VAR	1843.171	1534.736	1320.345	1236.951	1207.076
STD	42.932	39.176	36.337	35.170	34.743
SHADPRICE	0.264	0.264	0.263	0.263	0.263
POWER	0.020	0.010	0.001	0.0001	
BUYSTOCK2	10.934	10.960	10.960	10.960	
BUYSTOCK3	6.143	6.115	6.115	6.115	
OBJ	1.115	1.056	1.005	1.001	
MEAN	234.079	234.045	234.045	234.045	
VAR	1192.805	1178.961	1178.961	1178.961	
STD	34.537	34.336	34.336	34.336	
SHADPRICE	0.263	0.263	0.263	0	

DEMP Model – GAMS Formulation

```
SETS       STOCKS  POTENTIAL INVESTMENTS / BUYSTOCK1*BUYSTOCK4 /
           EVENTS  EQUALLY LIKELY RETURN STATES OF NATURE
                                /EVENT1*EVENT10 / ;
PARAMETERS     PRICES(STOCKS) PURCHASE PRICES OF THE STOCKS
                             / BUYSTOCK1   22
                               BUYSTOCK2   30
                               BUYSTOCK3   28
                               BUYSTOCK4   26 / ;
SCALAR      FUNDS    TOTAL INVESTABLE FUNDS / 500 / ;
TABLE RETURNS(EVENTS,STOCKS) RETURNS BY STATE OF NATURE EVENT
          BUYSTOCK1   BUYSTOCK2   BUYSTOCK3   BUYSTOCK4
  EVENT1      7           6           8           5
  EVENT2      8           4          16           6
  EVENT3      4           8          14           6
  EVENT4      5           9          -2           7
  EVENT5      6           7          13           6
  EVENT6      3          10          11           5
  EVENT7      2          12          -2           6
  EVENT8      5           4          18           6
  EVENT9      4           7          12           5
  EVENT10     3           9          -5           6
SCALAR   INITWEALTH   INITIAL WEALTH                /100/
         POWER        EXPONENT IN UTILITY FUNCTION  /0.5/;
POSITIVE VARIABLES    INVEST(STOCKS)     MONEY INVESTED IN EACH STOCK
VARIABLES             OBJ                NUMBER TO BE MAXIMIZED
                      WEALTH(EVENTS)     RETURNS BY EVENT
                      MEAN               MEAN RETURNS ;
EQUATIONS             OBJJ               OBJECTIVE FUNCTION
                      WEALTHDEF(EVENTS)  WEALTHS DEFINITION
                      AVWEALTH              AVERAGE WEALTH
                      INVESTAV           INVESTMENT FUNDS AVAILABLE;
OBJJ.. OBJ =E= SUM(EVENTS,(WEALTH(EVENTS)**POWER)/CARD(EVENTS));
INVESTAV..     SUM(STOCKS, PRICES(STOCKS) * INVEST(STOCKS)) =L= FUNDS ;
WEALTHDEF(EVENTS)..  WEALTH(EVENTS)
 - SUM(STOCKS, RETURNS(EVENTS,STOCKS) * INVEST(STOCKS)) =E=INITWEALTH ;
AVWEALTH..         MEAN =E= SUM(EVENTS,1/CARD(EVENTS)*WEALTH(EVENTS));
MODEL EVPORTFOL /ALL/ ;
SET POWERS   UTILITY FUNCTION POWER PARAMETERS /R0*R13/
PARAMETER POWERSET(POWERS) RISK AVERSION COEF BY RISK AVERSION PARAMETER
              /R0   0.95 ,  R1   0.9   ,  R2   0.75,  R3   0.5 ,
               R4   0.4  ,  R5   0.3   ,  R6   0.2 ,  R7   0.1 ,
               R8   0.05 ,  R9   0.03  ,  R10  0.02,  R11  0.01,
               R12  0.001,  R13  0.0001/
PARAMETER VAR        VARIANCE OF RETURNS
PARAMETER OUTPUT(*,POWERS) RESULTS FROM MODEL RUNS WITH VARYING EXPONENT
LOOP  (POWERS,POWER=POWERSET(POWERS);
           SOLVE EVPORTFOL USING NLP MAXIMIZING OBJ ;
 VAR = SUM(EVENTS,(WEALTH.L(EVENTS)-MEAN.L)*(WEALTH.L(EVENTS) -MEAN.L))
           /CARD(EVENTS);
           OUTPUT("OBJ",POWERS)=OBJ.L;
           OUTPUT("POWER",POWERS)=POWER;
           OUTPUT(STOCKS,POWERS)=INVEST.L(STOCKS);
           OUTPUT("MEAN",POWERS)=MEAN.L;
           OUTPUT("VAR",POWERS) = VAR;
           OUTPUT("STD",POWERS)=SQRT(VAR);
           OUTPUT("SHADPRICE",POWERS)$(MEAN.L GT 0.001)=
                      INVESTAV.M/(power*mean.l**(power-1));
           OUTPUT("IDLE",POWERS)=FUNDS-INVESTAV.L
             );
```

Risk Modeling and Stochastic Programming

Chance Constrained Programming

The probability of a constraint being satisfied is greater than or equal to a prespecified value α.

$$P\left(\sum_j a_{ij} X_j \leq b_i\right) \geq \alpha$$

If the average value of the RHS $(\bar{b}_i)$ is subtracted from both sides of the inequality and in turn both sides are divided by the standard deviation of the RHS σ_{b_i} then the constraint becomes

$$P\left[\frac{\sum_j a_{ij} X_j - \bar{b}_i}{\sigma_{b_i}} \leq \frac{(b_i - \bar{b}_i)}{\sigma_{b_i}}\right] \geq \alpha$$

Those familiar with probability theory will note that the term

$$\frac{(b_i - \bar{b}_i)}{\sigma_{b_i}}$$

gives the number of standard errors that b_i is away from the mean. Let Z denote this term. When a particular probability limit (α) is used, then the appropriate value of Z is Z_α and the constraint becomes

$$P\left[\frac{\sum_j a_{ij} X_j - \bar{b}_i}{\sigma_{b_i}} \leq Z_\alpha\right] \geq \alpha$$

Assuming we discount for risk, then the constraint can be restated as

$$\sum_j a_{ij} X_j \leq \bar{b}_i - Z_\alpha \sigma_{b_i}$$

Risk Modeling and Stochastic Programming

Chance Constrained Programming

Table 14.14. Chance Constrained Example Data

Event	Small Lathe	Large Lathe	Carver
1	140	90	120
2	120	94	132
3	133	88	110
4	154	97	118
5	133	87	133
6	142	86	107
7	155	90	120
8	140	94	114
9	142	89	123
10	141	85	123
Mean	140	90	120
Standard Error	9.63	3.69	8.00

Then the resultant chance constrained formulation is

$$
\begin{array}{llllllll}
\text{Max} & 67X_1 & + 66X_2 & + 66.3X_3 & + 80X_4 & + 78.5X_5 & + 78.4X_6 & \\
\text{s.t} & 0.8X_1 & + 1.3X_2 & + 0.2X_3 & + 1.2X_4 & + 1.7X_5 & + 0.5X_6 & \leq 140 - 9.63\ Z_{\alpha} \\
 & 0.5X_1 & + 0.2X_2 & + 1.3X_3 & + 0.7X_4 & + 0.3X_5 & + 1.5X_6 & \leq 90 - 3.69\ Z_{\alpha} \\
 & 0.4X_1 & + 0.4X_2 & + 0.4X_3 & + X_4 & + X_5 & + X_6 & \leq 120 - 8.00\ Z_{\alpha} \\
 & X_1 & + 1.05X_2 & + 1.1X_3 & + 0.8X_4 & + 0.82X_5 & + 0.84X_6 & \leq 125
\end{array}
$$

Risk Modeling and Stochastic Programming

Chance Constrained Programming

Table 14.15. Chance Constrained Example Solutions for Alternative Alpha Levels

Z_α	0.00	1.280	1.654	2.330
PROFIT	10417.291	9884.611	9728.969	9447.647
SMLLATHE	140.000	127.669	124.067	117.554
LRGLATHE	90.000	85.280	83.900	81.407
CARVER	120.000	109.760	106.768	101.360
LABOR	125.000	125.000	125.000	125.000
FUNCTNORM	62.233	78.102	82.739	91.120
FANCYNORM	73.020	51.495	45.205	33.837
FANCYMXLRG	5.180	6.788	7.258	8.108

Note: Z_α is the risk aversion parameter.

Chance Constrained Programming

```
 SET     PROCESS       TYPES OF PRODUCTION PROCESSES
                           /FUNCTNORM , FUNCTMXSML , FUNCTMXLRG
                           ,FANCYNORM , FANCYMXSML , FANCYMXLRG/
         RESOURCE      TYPES OF RESOURCES
                           /SMLLATHE,LRGLATHE,CARVER,LABOR/
         EVENT         STATES OF NATURE   /S1*S10/;
 PARAMETER PRICE(PROCESS)     PRODUCT PRICES BY PROCESS
                  /FUNCTNORM  82, FUNCTMXSML  82, FUNCTMXLRG  82
                  ,FANCYNORM 105, FANCYMXSML 105, FANCYMXLRG 105/
           PRODCOST(PROCESS)      COST BY PROCESS
                  /FUNCTNORM  15, FUNCTMXSML  16  , FUNCTMXLRG  15.7
                  ,FANCYNORM  25, FANCYMXSML  26.5, FANCYMXLRG  26.6/
 TABLE RESORAVAIL(RESOURCE,EVENT) RESOURCE AVAILABLITY
                             S1  S2  S3  S4  S5  S6  S7  S8  S9 S10
                  SMLLATHE  140 120 133 154 133 142 155 140 142 141
                  LRGLATHE   90  94  88  97  87  86  90  94  89  85
                  CARVER    120 132 110 118 133 107 120 114 123 123
                  LABOR     125 125 125 125 125 125 125 125 125 125
 TABLE RESOURUSE(RESOURCE,PROCESS) RESOURCE USAGE
              FUNCTNORM   FUNCTMXSML  FUNCTMXLRG
SMLLATHE         0.80       1.30        0.20
LRGLATHE         0.50       0.20        1.30
CARVER           0.40       0.40        0.40
LABOR            1.00       1.05        1.10
 +            FANCYNORM   FANCYMXSML  FANCYMXLRG
SMLLATHE         1.20       1.70        0.50
LRGLATHE         0.70       0.30        1.50
CARVER           1.00       1.00        1.00
LABOR            0.80       0.82        0.84;
 PARAMETER MEANAVAIL(RESOURCE)   MEAN AMOUNT OF RESOURCE AVAILABILITY
         STDERROR (RESOURCE)   STANDARD ERROR OF RESOURCE AVAILABILITY;
 SCALAR  ZALPHA                CHANCE CONSTRAINT DISCOUNT FACTOR/1.96/;
MEANAVAIL(RESOURCE) = SUM(EVENT,RESORAVAIL(RESOURCE,EVENT))/CARD(EVENT);
STDERROR(RESOURCE) = SQRT(SUM(EVENT,
      SQR(RESORAVAIL(RESOURCE,EVENT)-MEANAVAIL(RESOURCE)))/CARD(EVENT));
 POSITIVE VARIABLES PRODUCTION(PROCESS) ITEMS PRODUCED BY PROCESS;
 VARIABLES          PROFIT              TOTALPROFIT;
 EQUATIONS          OBJT                OBJECTIVE FUNCTION ( PROFIT )
                    AVAILABLE(RESOURCE) RESOURCES AVAILABLE ;
  OBJT..  PROFIT =E=
            SUM(PROCESS,(PRICE(PROCESS)-PRODCOST(PROCESS))
                          * PRODUCTION(PROCESS)) ;
  AVAILABLE(RESOURCE)..
      SUM(PROCESS,RESOURUSE(RESOURCE,PROCESS)*PRODUCTION(PROCESS))
                     =L= MEANAVAIL(RESOURCE)-ZALPHA*STDERROR(RESOURCE);
 MODEL CHANCE /ALL/;
 SET ZALPH  ALTERNATIVE Z VALUE IDENTIFIERS /Z1*Z4/
 PARAMETER OUTPUT(*,ZALPH) RESULTS FROM MODEL RUNS WITH VARYING RAP
           ZALPHS(ZALPH)   ALTERNATIVE Z ALPHA VALUES
                          /Z1 0.0, Z2 1.28, Z3 1.654 , Z4 2.33/
 LOOP (ZALPH,ZALPHA=ZALPHS(ZALPH);
         SOLVE CHANCE USING LP MAXIMIZING PROFIT ;
   OUTPUT(RESOURCE,ZALPH)=MEANAVAIL(RESOURCE)-ZALPHA*STDERROR(RESOURCE);
         OUTPUT("PROFIT",ZALPH)=PROFIT.L;
         OUTPUT("RAP",ZALPH)=ZALPHA;
         OUTPUT(PROCESS,ZALPH)=PRODUCTION.L(PROCESS));
 DISPLAY OUTPUT;
```

Risk Modeling and Stochastic Programming

Technical Coefficient Risk

Merrill's Approach

a constraint containing uncertain coefficients can be rewritten as

$$\sum_j \overline{a_{ij}} X_j \quad + \quad \Phi \sum_j \sum_n X_j X_n \sigma_{inj} \quad \le \quad b_i \quad \text{for all } i$$

or, using standard deviation,

$$\sum_j a_{ij} X_j \quad + \quad \Phi \left(\sum_j \sum_k X_j X_n \sigma_{inj} \right)^{0.5} \quad \le \quad b_i \quad \text{for all } i$$

Wicks-Guise

given that the i^{th} constraint contains uncertain a_{ij}'s, the following constraints may be set up.

$$\sum_j a_{ij} X_j + \Phi D_i \le b_i \qquad \forall\ i \sum_j \left(a_{kij} - \overline{a_{ij}}\right) X_j - d^+_{ki} + d^-_{ki} = 0 \quad \forall\ k, i \sum_k \left(d^+_{ki} + d\right.$$

The general Wicks Guise formulation is

$$
\begin{array}{llllll}
Max & \sum_j c_j X_j & & & & \\
s.t. & \sum_j \overline{a_{ij}} X_j & + \ \Phi \sigma_i & \le & b_i & \text{for all } i \\
\sum_j \left(a_{kij} - \overline{a_{ij}}\right) X_j - d^+_{ki} + d^-_{ki} & & & = & 0 & \text{for all } i, k \\
 & \sum_k \left(d^+_{ki} + d^-_{ki}\right) - D_i & & = & 0 & \text{for all } i \\
 & & \Delta D_i - \sigma_i & = & 0 & \text{for all } i \\
 & X_j, \quad d^+_{ki}, \quad d^-_{ki}, \quad D_i, \quad \sigma_i & & \ge & 0 & \text{for all } j, k, i
\end{array}
$$

Wicks-Guise Example

Table 14.16. Feed Nutrients by State of Nature for Wicks Guise Example

Nutrient	State	CORN	SOYBEANS	WHEAT
ENERGY	S1	1.15	0.26	1.05
ENERGY	S2	1.10	0.31	0.95
ENERGY	S3	1.25	0.23	1.08
ENERGY	S4	1.18	0.28	1.12
PROTEIN	S1	0.23	1.12	0.51
PROTEIN	S2	0.17	1.08	0.59
PROTEIN	S3	0.25	1.01	0.46
PROTEIN	S4	0.15	0.99	0.56

Table 14.17. Wicks Guise Example

	Corn	Soybeans	Wheat	EnDev	EnMAD	Enσ	PrDev	PrMAD	Prσ		
Objective	0.03	0.06	0.04								
Volume	1	1	1							$=$	1
Energy	1.17	0.27	1.05			$-\phi$				$\geq$	0.8
Protein	0.20	1.05	0.53						$-\phi$	$\geq$	0.5
Energys1	-0.02	-0.01	$+0.00$	$-d_{e1}^{+}+d_{e1}^{-}$						$=$	0
Energys2	-0.07	$+0.04$	-0.10	$-d_{e2}^{+}+d_{e2}^{-}$						$=$	0
Energys3	$+0.08$	-0.04	$+0.03$	$-d_{e3}^{+}+d_{e3}^{-}$						$=$	0
Energys4	$+0.01$	$+0.01$	$+0.07$	$-d_{e4}^{+}+d_{e4}^{-}$						$=$	0
EnergyMAD				$\sum_k (d_{ek}^{+}+d_{ek}^{-})/4$	-1					$=$	0
Energyσ					$-\Delta$	$+1$				$=$	0
Proteins1	-0.02	-0.01	$+0.00$				$-d_{p1}^{+}+d_{p1}^{-}$			$=$	0
Proteins2	-0.07	$+0.04$	-0.10				$-d_{p2}^{+}+d_{p2}^{-}$			$=$	0
Proteins3	$+0.08$	-0.04	$+0.03$				$-d_{p3}^{+}+d_{p3}^{-}$			$=$	0
Proteins4	$+0.01$	$+0.01$	$+0.07$				$-d_{p4}^{+}+d_{p4}^{-}$			$=$	0
ProteinMAD							$\sum_k (d_{pk}^{+}+d_{pk}^{-})/4$	-1		$=$	0
Proteinσ								$-\Delta$	$+1$	$=$	0

Note: EnDev is the energy deviation
EnMAD is the energy mean absolute deviation
Enσ is the energy standard deviation approximations
PrDev is the protein deviation
PrMAD is the protein mean absolute deviation
Prσ is the protein standard deviation approximation

Risk Modeling and Stochastic Programming

Wicks-Guise Example Solution

Table 14.18. Results From Example Wicks Guise Model Runs With Varying RAP

RAP		0.250	0.500	0.750	1.000
CORN	0.091	0.046	0.211	0.230	0.221
SOYBEANS			0.105	0.129	0.137
WHEAT	0.909	0.954	0.684	0.641	0.642
OBJ	0.039	0.040	0.040	0.040	0.041
AVGPROTEIN	0.500	0.515	0.515	0.521	0.529
STDPROTEIN	0.054	0.059	0.030	0.028	0.029
AVGENERGY	1.061	1.056	0.993	0.977	0.969
STDENERGY	0.072	0.072	0.061	0.059	0.058
SHADPROT	0.030	0.033	0.036	0.037	0.038

RAP	1.250	1.500	2.000
CORN	0.211	0.200	0.177
SOYBEANS	0.146	0.156	0.176
WHEAT	0.643	0.644	0.647
OBJ	0.041	0.041	0.042
AVGPROTEIN	0.536	0.545	0.563
STDPROTEIN	0.029	0.030	0.031
AVGENERGY	0.961	0.953	0.934
STDENERGY	0.057	0.056	0.055
SHADPROT	0.039	0.040	0.042

Note: RAP gives the risk aversion parameter used
CORN gives the amount of corn used in the solution
SOYBEANS gives the amount of soybeans used in the solution
WHEAT gives the amount of wheat used in the solution
OBJ gives the objective function value
AVGPROTEIN gives the average amount of protein in the diet
STDPROTEIN gives the standard error of protein in the diet
AVGENERGY gives the average amount of energy in the diet
STDENERGY gives the standard error of energy in the diet
SHADPROT gives the shadow price on the protein requirement constraint