01 | Polynomial representations

TEACHER EXPLANATION & KEY

Today I am learning to rewrite polynomial and rational functions. So I can choose a form that reveals useful graph features. I'll know I have learned it when I can find zeros, restrictions, asymptotes and expand binomials. Essential question: How does changing a function's form reveal its behavior?

1. Warm-up. Factor $x^2 - 9$ and $x^2 + 5x + 6$. Expand $(x + 2)^2$.
Difference of squares: $(x - 3)(x + 3)$. Find numbers with product 6 and sum 5: $(x + 2)(x + 3)$. Multiply $(x + 2)(x + 2) = x^2 + 4x + 4$.

2. Vocabulary. Standard form reveals degree and leading coefficient. Factored form reveals _____ and their multiplicities. A zero with even multiplicity usually makes the graph _____ the x-axis; odd multiplicity makes it cross.
zeros; touch (bounce). The degree is the highest exponent after expansion. A polynomial has domain all real numbers.

3. I DO. Analyze $f(x) = 2x^3 - 16x^2 + 32x$. Factor, find degree, end behavior, zeros, y-intercept and where $f(x) \geq 0$.
Factor out 2x: $2x(x^2 - 8x + 16) = 2x(x - 4)^2$. Degree 3, positive leading coefficient: left down, right up. Zeros 0 (odd), 4 (even); y-intercept (0,0). Since the square is nonnegative, $f(x) \geq 0$ on $[0,\infty)$.

4. WE DO. $g(x) = x^3 - 2x^2 - 15x = x(x - ___)(x + ___)$. Find zeros and signs on the intervals they create.
5, 3. Zeros −3, 0, 5. All simple. Signs from left to right: −, +, −, +. Degree 3, left down/right up. y-intercept (0,0).

5. YOU DO. Expand $2(x - 4)^2$. Find degree, zeros, y-intercept and where it is nonnegative.
$2x^2 - 16x + 32$. Degree 2; zero 4 of multiplicity 2; (0,32). Nonnegative everywhere because 2 times a square is nonnegative. Both ends rise.

6. Check. Which form reveals the y-intercept directly? Why does a double zero not change sign?
Standard form: constant term. An even power stays nonnegative on both sides of its zero.

02 | Rational expressions: factor before analyzing

TEACHER EXPLANATION & KEY

KEY IDEA: A rational function is a quotient of polynomials. Record every original denominator zero BEFORE canceling. A canceled factor does not restore an excluded input.

7. Rule box. Domain excludes values making the original denominator _____. A factor that completely disappears from the denominator produces a _____. A factor remaining in the denominator produces a _____.
zero; hole (removable discontinuity); vertical asymptote. A zero must be in the original domain. Find the hole's y-coordinate using the reduced expression.

8. I DO. Analyze $\frac{x^2-4}{x^2+7x+10}$. Factor numerator and denominator; find domain, hole, vertical and horizontal asymptotes, zeros and intercepts.
$\frac{(x-2)(x+2)}{(x+2)(x+5)} = \frac{x-2}{x+5}$, $x \neq -2, -5$. Hole $(-2,-\frac{4}{3})$; VA $x = -5$; HA $y = 1$ (equal degrees); zero 2; y-intercept $(0,-\frac{2}{5})$. Domain $(-\infty,-5) \cup (-5,-2) \cup (-2,\infty)$.

9. WE DO. Analyze $\frac{x^2+7x+12}{x^2+x-6}$. Factor: numerator (x+___)(x+___); denominator (x+___)(x−___). Find all features as in #8.
3, 4; 3, 2. Reduced $\frac{x+4}{x-2}$, $x \neq -3,2$. Hole $(-3,-\frac{1}{5})$; zero −4; VA x=2; HA y=1; y-intercept $(0,-2)$. Domain $(-\infty,-3) \cup (-3,2) \cup (2,\infty)$.

10. YOU DO. Convert $\frac{2(x+3)(x-3)}{(x+2)(x+3)}$ to general form. Identify all features as in #8.
$\frac{2x^2-18}{x^2+5x+6}$. Reduced $\frac{2(x-3)}{x+2}$, exclusions −3,−2. Hole (−3,12); VA x=−2; HA y=2; zero 3; y-intercept $(0,-3)$. Domain $(-\infty,-3) \cup (-3,-2) \cup (-2,\infty)$.

11. Check. Is x = −3 a zero in #10? Can a hole lie on the x-axis?
No; it is excluded. Yes, if the reduced numerator is zero there but the original function is undefined.

03 | Repeated factors and end behavior

TEACHER EXPLANATION & KEY

Compare the exponent of a shared factor in numerator (m) and denominator (n). If n > m, a vertical asymptote remains. If 0 < n ≤ m, there is a hole. If m > n > 0, the hole lies on the x-axis.

12. I DO. Compare $\frac{x^3+2x}{x^2(x-3)}$ and $\frac{(x+3)^2(x-2)}{(x+3)(x+1)}$. Identify holes and vertical asymptotes.

First: $\frac{x^2+2}{x(x-3)}$, x≠0,3. VA x=0,3; no holes. HA y=1. Second: $\frac{(x+3)(x-2)}{x+1}$, x≠−3,−1. Hole (−3,0); VA x=−1; no HA; slant y=x (remainder −6).

13. WE DO. $\frac{2x(x+4)(x-1)}{x^2-2x+1}$. Factor denominator. Is x=1 a hole or VA? Explain.

Denominator $(x-1)^2$. Cancel one factor: $\frac{2x(x+4)}{x-1}$. One factor remains, so x=1 is a VA, no hole. Slant y=2x+10.

14. YOU DO. $\frac{(x-2)^3(x+1)}{(x-2)^2(x+4)}$. Find domain, hole, zero and VA.

Reduced $\frac{(x-2)(x+1)}{x+4}$, exclusions 2,−4. Hole (2,0); zero −1; VA x=−4. Domain $(-\infty,-4) \cup (-4,2) \cup (2,\infty)$.

15. End-behavior rules. Compare degrees p (numerator) and q (denominator): p<q → y=___; p=q → ratio of leading coefficients; p=q+1 → _____; p>q+1 → neither horizontal nor slant.

0; slant asymptote. A higher-degree quotient may describe polynomial end behavior, but is not a horizontal or slant line.

16. Check. Classify the end behavior: A $\frac{3x-7}{x^2+4x+8}$; B $\frac{5x^2+3x-2}{2x^2-3}$; C $\frac{4x^2-8}{2x+1}$; D $\frac{x^4+1}{x^2+1}$.

A: HA y=0; B: HA y=$\frac{5}{2}$; C: slant (degree difference 1), y=2x−1; D: neither (degree difference 2).

04 | Polynomial long division and slant asymptotes

TEACHER EXPLANATION & KEY

Division algorithm: P(x) = D(x)Q(x) + R(x), or $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$. The remainder has a smaller degree than the divisor. Divide → multiply → subtract → bring down → repeat.

17. Warm-up. 675 ÷ 21. Find the quotient and remainder.

21 × 32 = 672, so quotient 32, remainder 3. Check: 21(32)+3=675.

18. I DO. Divide $(6x^2 + 7x + 5)$ by $(2x + 1)$.

$6x^2 \div 2x = 3x$. Multiply: $6x^2+3x$; subtract: $4x+5$. Divide $4x \div 2x = 2$. Multiply: $4x+2$; subtract: 3. Quotient $3x+2$; remainder 3. Result $3x+2+\frac{3}{2x+1}$. Slant $y=3x+2$.

19. WE DO. Divide $(6x^2 + x + 5)$ by $(2x + 1)$. First quotient term = ____. After subtracting, remainder expression = ____. Finish.

$3x$; $-2x+5$. Next quotient term -1; multiply $-2x-1$; subtract to obtain 6. Result $3x-1+\frac{6}{2x+1}$. Slant $y=3x-1$.

20. YOU DO. Analyze $\frac{x^3+3x^2-10x}{x^2+8x+15}$. Factor; find all asymptotes and any hole.

Factor $\frac{x(x+5)(x-2)}{(x+5)(x+3)}$. Hole at $x=-5$; reduced value $-\frac{35}{2}$. VA $x=-3$. Divide x^2-2x by $x+3$: quotient $x-5$, remainder 15. Slant $y=x-5$; no HA.

21. Check. Why is the quotient the slant line? What should you insert when a power is missing?

The remainder divided by the divisor tends to zero as x approaches $\pm\infty$. Insert a zero coefficient for the missing power to align terms.

05 | Graph analysis: from equation to graph

TEACHER EXPLANATION & KEY

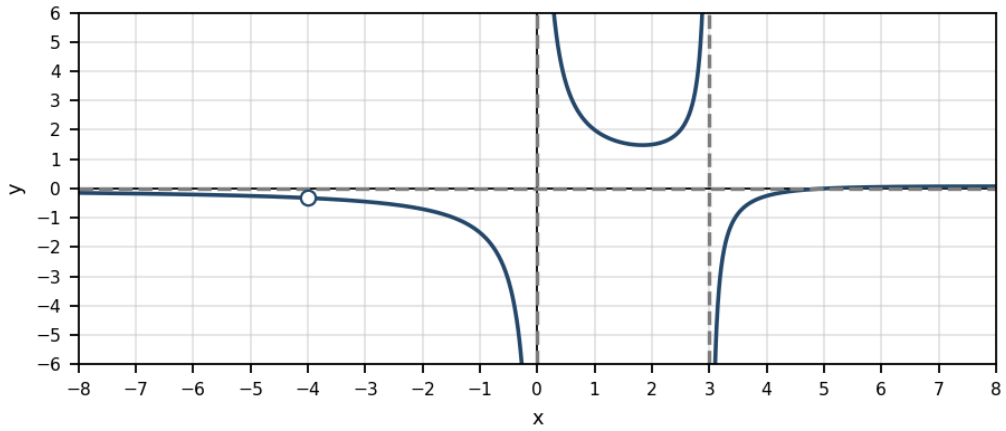
Graphing checklist: factor; record domain; find intercepts and holes; locate asymptotes; test symmetry using $f(-x)$; make sign chart including all excluded inputs; plot and connect branches without crossing vertical asymptotes.

22. I DO. Analyze $\frac{x^2-x-20}{x^3+x^2-12x}$. Find reduced form, domain, intercepts, hole, asymptotes, symmetry and signs.

$\frac{(x-5)(x+4)}{x(x+4)(x-3)} \longrightarrow \frac{x-5}{x(x-3)}$, $x\neq-4,0,3$. Hole $(-4,-\frac{9}{28})$; zero 5; no y-intercept; VAs $x=0,3$; HA $y=0$. Neither symmetry. Signs on intervals split at $-4,0,3,5$: $-,-,+,-,+$. Near 0: left $-\infty$, right $+\infty$; near 3: left $+\infty$, right $-\infty$.

23. WE DO. Use the grid to sketch #22. Mark dashed asymptotes, open circle and intercept.

Use signs and one-sided behavior from #22. The curve approaches y=0 at both ends. The hole interrupts the negative branch left of x=0.



24. Check. Why must a calculator graph be checked algebraically? Can a graph cross a horizontal asymptote?

A calculator may not display a single missing point or may draw across a discontinuity. Yes, a horizontal asymptote describes end behavior only.

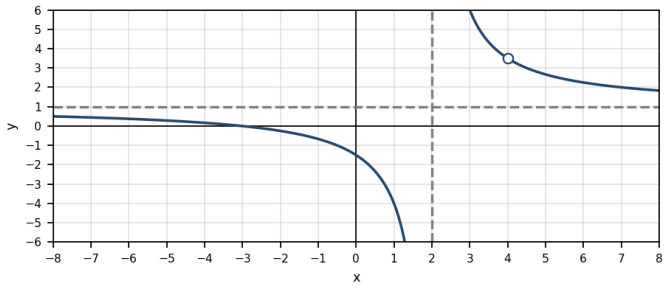
06 | From graphs and limits to equations

TEACHER EXPLANATION & KEY

A zero suggests a numerator factor, a vertical asymptote suggests a denominator factor, and a hole requires a common factor. A scale factor can be determined using a known point or the horizontal asymptote.

25. I DO. Use the displayed graph to write a possible factored equation. Label zero, hole and asymptotes.

Zero −3, VA x=2, HA y=1, hole at x=4. Start with $\frac{x+3}{x-2}$; multiply numerator and denominator by (x−4). Result $\frac{(x+3)(x-4)}{(x-2)(x-4)}$. Hole $(4,\frac{7}{2})$.

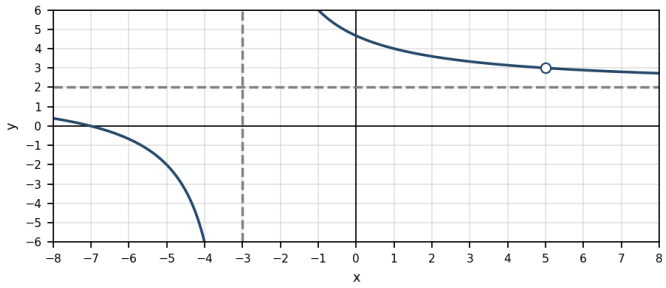


26. WE DO. A graph has zero 1, VAs x=0 and x=3, hole x=2, HA y=0 and passes through (−1,1). Write a possible equation.

Start $\frac{A(x-1)(x-2)}{x(x-3)(x-2)}$. At −1, 1=A(−2)/4, so A=−2. Hole (2,1).

27. YOU DO. Sketch a graph with end limits 2, left limit at −3 equal to −∞, right limit +∞, and a hole (5,3).

One possibility: $\frac{(2x+14)(x-5)}{(x+3)(x-5)}$. Reduced $2+\frac{8}{x+3}$. At 5, reduced value 3. The positive numerator 8 makes left branch fall and right branch rise near −3.



28. Check. If both sides approach −∞ at x=−1, what multiplicity is suggested? Give a function with left end +∞ and right end −∞.

Even denominator multiplicity, e.g. $-x-\frac{1}{(x+1)^2}$. The linear term controls end behavior; the negative reciprocal square controls the VA.

07 | Pascal's triangle and binomial expansion

TEACHER EXPLANATION & KEY

A binomial has two terms. Pascal's triangle supplies coefficients. Each inside number equals the sum of the two above it. Row numbering begins at zero.

29. Complete Pascal's triangle (rows 0–8). Row 0: 1; row 1: 1 1; row 2: 1 2 1; row 3: 1 3 3 1.

Rows 4–8: 1 4 6 4 1; 1 5 10 10 5 1; 1 6 15 20 15 6 1; 1 7 21 35 35 21 7 1; 1 8 28 56 70 56 28 8 1.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

30. Formula box. $(a+b)^n = \Sigma\, C(n,r)a^{n-r}b^r$, r=0 to n. There are ___ terms. The exponent of a decreases; the exponent of b _____. Optional: $C(n,r)=\frac{n!}{r!(n-r)!}$.

n+1; increases. Factorial formula is enrichment; Pascal's triangle is sufficient for these examples. C(6,2)=15, C(9,4)=126, C(3,3)=1.

31. I DO. Expand $(x+3)^3$ and $(x+3)^2$.

Row 3: $x^3+3x^2(3)+3x(3^2)+3^3 = x^3+9x^2+27x+27$. Row 2: $x^2+2x(3)+3^2=x^2+6x+9$.

32. WE DO. Expand $(2x-1)^4$. Coefficients: 1, ___, ___, ___, 1.

4,6,4. Terms $(2x)^4$, $4(2x)^3(-1)$, $6(2x)^2(-1)^2$, $4(2x)(-1)^3$, $(-1)^4$. Result $16x^4-32x^3+24x^2-8x+1$.

33. YOU DO. Expand $(x-3)^5$.

Row 5: $x^5-15x^4+90x^3-270x^2+405x-243$. Use powers of −3; signs alternate. Check x=0 gives −243.

34. Check. Is C(6,2) the second term's coefficient? Why do subtraction signs alternate?

No, it is the third Pascal coefficient (r=2). Odd powers of a negative term are negative; even powers are positive.

08 | Find a specific term or coefficient

TEACHER EXPLANATION & KEY

Use $T_{r+1} = C(n,r)a^{n-r}b^r$. For the kth term, r = k−1. To locate a power of x, solve for the exponent before computing the coefficient.

35. I DO. Find the third term of $(2x+3)^6$.
Third term means r=2. C(6,2)=15. $15(2x)^4(3)^2 = 15 \cdot 16 \cdot 9\, x^4 = 2160x^4$.

36. WE DO. Find the coefficient of x^4 in $(x+5)^6$. Set 6−r=4, so r=___.
2. C(6,2)·5^2=15·25=375. The full term is $375x^4$; coefficient is only 375.

37. YOU DO. Find the coefficient of x^3 in $(x-3)^8$.
8−r=3 → r=5. C(8,5)=56. $56(-3)^5=-13{,}608$. Term $-13{,}608x^3$.

38. Check. Find the last term in $(2x+3)^4$. Evaluate C(6,2) using its row and position.
Last term uses r=4, $3^4=81$. Row 6, element indexed 2 (third position) gives 15.

39. AP-style MCQ. What is the coefficient of the fourth term of $(x-2)^6$? A −160 B −120 C 120 D 160. Explain.
A. r=3; C(6,3)=20; $20(-2)^3=-160$.

40. AP-style MCQ. Second term of $(2a-b)^4$? A $-32a^3b$ B $-4a^2b^2$ C $-16ab^3$ D $-ab^4$.
A. r=1; $4(2a)^3(-b)=-32a^3b$.

09 | Practice: basic and application

TEACHER EXPLANATION & KEY

Show work in each box. Use exact values and interval notation. Level 1: recall and apply. Level 2: select a representation and justify.

41. Level 1. Factor $-3x^3+27x$. Find zeros, end behavior and where $g(x) \geq 0$.
$-3x(x-3)(x+3)$. Zeros −3,0,3; left up/right down. Signs +,−,+,−; nonnegative $(-\infty,-3] \cup [0,3]$.

42. Level 1. Expand $(x+2)^4$ and $(2x+1)^3$.
Row 4: $x^4+8x^3+24x^2+32x+16$. Row 3: $8x^3+12x^2+6x+1$.

43. Level 2. Analyze $\frac{t^3-4t^2}{t^2+2t-15}$: factored form, domain, zeros, holes, all asymptotes, y-intercept.

$\frac{t^2(t-4)}{(t+5)(t-3)}$. Domain excludes $-5,3$; zeros 0 (double), 4; no holes; VAs $t=-5,3$; y-intercept $(0,0)$. Divide: quotient $t-6$, remainder $27t-90$. Slant $y=t-6$; no HA.

44. Level 2. Expand $(3x-2)^4$. Find the fourth term of $(x-2)^7$.

$81x^4-216x^3+216x^2-96x+16$. Fourth term $r=3$: $35x^4(-2)^3=-280x^4$.

45. Level 2. Find the fifth term of $(3x-1)^8$. Find the coefficient of x^5 in $(x+3)^7$.

$r=4$: $70(3x)^4(-1)^4=5670x^4$. For x^5, $r=2$: $21\cdot 9=189$.

10 | AP-level challenge and exit ticket

TEACHER EXPLANATION & KEY

Level 3: explain your reasoning. Original AP-style practice, not official College Board exam questions.

46. FRQ. Let $\frac{x^2-x-12}{x^3+x^2-20x}$. Find the factored form, domain, zeros, hole, VAs, HA, and y-intercept. Explain why a canceled zero is excluded.

$\frac{(x-4)(x+3)}{x(x+5)(x-4)} \rightarrow \frac{x+3}{x(x+5)}$. Domain excludes $-5,0,4$. Zero -3; hole $(4,\frac{7}{36})$; VAs $x=-5,0$; HA $y=0$; no y-intercept. $x=4$ is excluded because the original denominator is zero.

47. MCQ. Which has a right-end limit of $+\infty$? A $\frac{(x+3)^3}{x(x^3+1)}$ B $-2(x-1)^3(x+5)$ C $\frac{x(x+3)^3}{x(x^3+1)}$ D $x(x+2)^3(x-1)$. Justify.

D. A approaches 0; B has negative leading coefficient and even degree, so tends to $-\infty$; C approaches 1; D degree 5 with positive leading coefficient tends to $+\infty$.

48. Summary. Factoring reveals _____. Original denominator zeros are _____. Long division reveals the _____ asymptote. Pascal's triangle gives _____.

zeros and factors; excluded from domain; slant (when degree difference is one); binomial coefficients.

49. Exit ticket — concept. When does a common factor create a hole instead of a VA?

When all copies of that factor cancel from the denominator, leaving a finite reduced value at the original excluded input.

S0. Exit ticket — application. Find the third term in $(x+5)^5$.
r=2: $C(5,2)x^3 \cdot 5^2 = 10 \cdot 25x^3 = 250x^3$.

S1. Exit ticket — reasoning. Explain why $\frac{x^2-1}{x-1}$ and x+1 do not have identical graphs.
For $x \neq 1$, factor $(x-1)(x+1)$ and cancel. Original undefined at x=1, so it has a hole (1,2); the polynomial includes that point.

Teacher resources & teaching notes

Video links supplied by the instructor (titles and content could not be independently verified). Preview before assigning.

https://www.youtube.com/watch?v=J7NAFRaouv8&t=2s

https://www.youtube.com/watch?v=-6-fYzEOivA

https://www.youtube.com/watch?v=w2AEMJ5OhBQ

https://www.youtube.com/watch?v=uOdyOODTtQc

Instructional sources

The Algebros / FlippedMath, 1.11A packet (5 pages); Bryan Passwater, Topic 1.11 notes (6 pages); Jean Adams / Flamingo Math, Lessons 11-A and 11-B (7 pages). Examples and practice adapted and consolidated. The division identity on page 3 of the Passwater notes has been corrected to P = DQ + R, or P/D = Q + R/D.

Teaching sequence: pages 1–3 factoring and rational features; pages 4–6 division and graph reasoning; pages 7–8 binomial expansion; pages 9–10 review and exit ticket. Ask students to record exclusions before cancellation and distinguish a coefficient from a full term.