9-2 The Binomial Theorem

Many important mathematical discoveries have begun with the study of patterns. Now we will introduce an important polynomial theorem called the Binomial Theorem....

If you expand (*a* + *b*)*ⁿ* for *n* = 0, 1, 2, 3, 4, and 5, here is what you get:

$$
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= 1a^1b^0 + 1a^0b^1\\
(a+b)^2 &= 1a^2b^0 + 2a^1b^1 + 1a^0b^2\\
(a+b)^3 &= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3\\
(a+b)^4 &= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4\\
(a+b)^5 &= 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5
\end{aligned}
$$

If the signs, powers and variables are eliminated, the “triangular” array of binomial coefficients are known as Pascal’s Triangle.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\cdot^{\cdot^{\cdot}} & & & & & \vdots & & & & & \ddots
\end{array}
$$

Pascal’s Triangle is very rich in patterns…

Hockey Stick pattern:

If a diagonal of numbers of any length is selected starting at any of the 1's bordering the sides of the triangle and ending on any number inside the triangle on that diagonal, the sum of the numbers inside the selection is equal to the number below the end of the selection that is not on the same diagonal itself.

Magic 11's

If a row is made into a single number by using each element as a digit of the number (carrying over when an element itself has more than one digit), the number is equal to 11 to the n*th* power or 11^n when n is the number of the row the multi-digit number was taken from.

Row #	**Formula**	**=**	**Multi-Digit number**	**Actual Row**
Row 0	11^0	=	1	1
Row 1	11^1	=	11	1 1
Row 2	11^2	=	121	1 2 1
Row 3	11^3	=	1331	1 3 3 1
Row 4	11^4	=	14641	1 4 6 4 1

Properties of Pascal's Triangle...

The top "1" is row zero, thus "1   1" is row 1 etc…

The sum of the entries in any row is 2^n

The outer numbers are always 1

To find the next row in the triangle you add the 2 numbers immediately above.

Any number in the triangle is a Combination of it's position.

Binomial Coefficient

The binomial coefficients that appear in the expansion of $(a+b)^n$ are the values of ${}_nC_r$ for $r = 0,1,2,3,...,n.$

A classical notation for ${}_nC_r$, especially in the context of binomial coefficients, is $\binom{n}{r}$.

Both notations are read "n choose r."

Example: Using ${}_nC_r$ to Expand a Binomial

Expand $(a + b)^4$ using a calculator to find the binomial coefficients.

Expand $(a + b)^5$ using a calculator to find the binomial coefficients.

Example: Computing binomial Coefficients

Find the coefficient of x^{10} in the expansion of $(x + 2)^{15}$

Find the coefficient of x^{12} in the expansion of $(x + 4)^{15}$

The Binomial Theorem

For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n,$$

where $\binom{n}{r} = {}_nC_r = \dfrac{n!}{r!(n-r)!}$.

Example: Expanding a Binomial

Expand $(2x - y^2)^4$

Expand $(x + 3y^2)^5$

Basic Factorial Identities

For any integer $n \geq 1$, $n! = n(n-1)!$

For any integer $n \geq 0$, $(n+1)! = (n+1)n!$

Example: Proving an identity with Factorials

Prove that $\binom{n+1}{2} - \binom{n}{2} = n$ for all integers $n \geq 2$.