Flipping Physics Lecture Notes:

Maximum Energy in Simple Harmonic Motion
http://www.flippingphysics.com/shm-maximum-energy.html

Let's look at the mechanical energy in a horizontal mass-spring system with a massless and frictionless spring on a frictionless surface.

The spring will have elastic potential energy:

$$U_e = \frac{1}{2}kx^2 = \frac{1}{2}k\left[A\cos(\omega t + \phi)\right]^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

We already derived the velocity equation for simple harmonic motion from the position equation:[1]

$$x(t) = A\cos(\omega t + \phi) \Rightarrow v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

Because the spring is massless, the kinetic energy of the mass-spring system will be of the mass only.

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left[-A\omega\sin(\omega t + \phi)\right]^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$$

The total mechanical energy of the mass-spring system then is:

$$ME_t = KE + U_e = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

Previously we derived the equation for angular frequency of a mass-spring system:[2]

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow \omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 \;\;\&\;\; ME_t = \frac{1}{2}kA^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

We can use the trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow ME_t = \frac{1}{2}kA^2\left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right] \Rightarrow ME_t = \frac{1}{2}kA^2$$

The total mechanical energy of the mass-spring system is equal to the amount of elastic potential energy of the system when the mass is located at the maximum distance from equilibrium, A, the amplitude. This is because the mass is temporarily at rest at this location and all the mechanical energy is elastic potential energy.

The total mechanical energy of the mass-spring system can also be expressed in terms of the kinetic energy when the mass is located at the equilibrium position. Because the mass-spring system has zero elastic potential energy at this location, all the mechanical energy when the mass is at the equilibrium position is kinetic energy. We can set those two total mechanical energies equal to one another and solve for the maximum velocity of the mass:

$$ME_t = \frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2 \Rightarrow mv_{max}^2 = kA^2 \Rightarrow v_{max} = A\sqrt{\frac{k}{m}} \Rightarrow v_{max} = A\omega$$

And we can then substitute in angular frequency:

$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

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[1] "Simple Harmonic Motion - Velocity and Acceleration Equation Derivations" https://www.flippingphysics.com/shm-velocity-acceleration.html
[2] "Simple Harmonic Motion Derivations using Calculus (Mass-Spring System)" https://www.flippingphysics.com/SHM-derivation-mass-spring.html

Resulting in the equation for maximum velocity we had derived earlier using the equation for velocity as a function of time of an object in simple harmonic motion.

$$v_{max} = A\omega$$