1.

Fig. 1 below shows the rotor blades for the propeller of a boat.

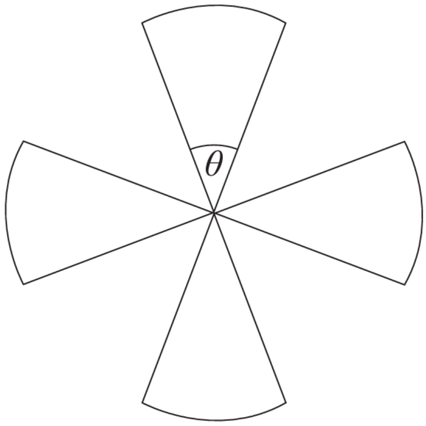


Fig. 1

They are made up of four equal sectors of a circle of radius 30 cm.

(i) If the total area of the blades is 300π cm^2, find the exact value of θ in radians. [3]

$$\begin{aligned} \text{area} &= 4 \times \frac{1}{2} r^2 \theta \\ 300\pi &= 2(30)^2 \theta \\ \theta &= \frac{\pi}{6} \end{aligned}$$

(ii) Hence find the total perimeter of the blades. [3]

$$\begin{aligned} \text{perimeter} &= 4 \times r\theta + 8r \\ &= 4(30)\left(\frac{\pi}{6}\right) + 8(30) \\ &= (20\pi + 240)\ \text{cm (3sf)} \end{aligned}$$

2.

(a) State whether the following sequences diverge, converge or oscillate:

(i) $\dfrac{n}{n^2+1}$ [1]

Converge to 0

(ii) $\sin\left(\dfrac{n\pi}{6}\right)$ [1]

Oscillate about 0

(b) (i) Use the Binomial Theorem to expand

$$\sqrt{1-3x}$$

in ascending powers of x up to and including the term in x^3 [5]

$$\begin{aligned}\sqrt{1-3x} &= (1-3x)^{\frac{1}{2}}\\ &= 1+\left(\frac{1}{2}\right)(-3x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-3x)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-3x)^3\\ &= 1-\frac{3}{2}x-\frac{9}{8}x^2-\frac{27}{16}x^3\end{aligned}$$

(ii) State the range of values of x for which the expansion is valid. [1]

$$|3x|<1, \quad -\frac{1}{3}<x<\frac{1}{3}$$

3.

(a) Use the Trapezium Rule with 3 ordinates to find an approximate value for

$$\int_2^4 x\cos x\,dx$$ [5]

$$h = \frac{4-2}{2} = 1$$

x	2	3	4
y	-0.8323	-2.9700	-2.6146

$$\begin{aligned}\int_2^4 x\cos x\,\mathrm{d}x &\approx \frac{1}{2}(1)[-0.8323 - 2.6146 + 2(-2.9700)] \\ &\approx -4.69\ (3\text{sf})\end{aligned}$$

(b) Part of a logo for a ski company can be modelled by the area between the curves $y = 4x - x^2$ and $y = x^2 - 6$ as shown in **Fig. 2** below.

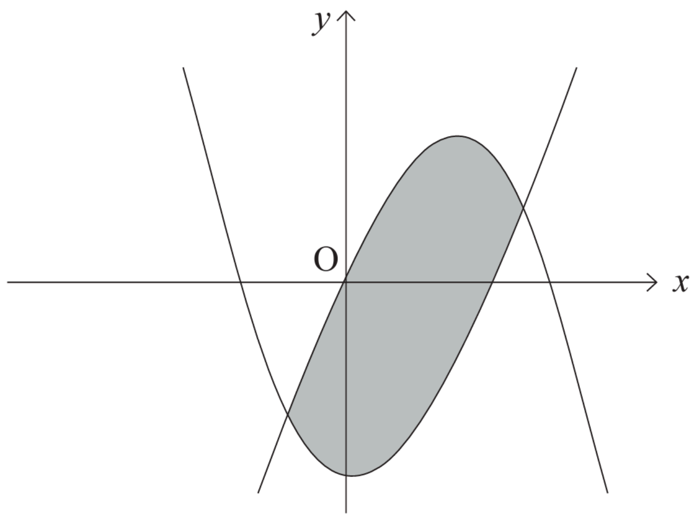


Fig. 2

Use calculus to find this area. [7]

$$\begin{aligned}4x - x^2 &= x^2 - 6 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x &= -1, 3\end{aligned}$$

$$\begin{aligned}\text{area} &= \int_{-1}^{3} \left[\left(4x - x^2\right) - \left(x^2 - 6\right)\right]\mathrm{d}x \\ &= \int_{-1}^{3} \left(4x - 2x^2 + 6\right)\mathrm{d}x \\ &= \left[2x^2 - \frac{2}{3}x^3 + 6x\right]_{-1}^{3} \\ &= 18 - \left(-\frac{10}{3}\right) = \frac{64}{3}\ \text{units}^2\end{aligned}$$

4.

(a) Find $\frac{dy}{dx}$ for each of the following:

(i) $y = \sec^5(2x)$ [4]

$$\begin{aligned}\frac{dy}{dx} &= 5\sec^4 2x(2\sec 2x \tan 2x)\\ &= 10\sec^5 2x \tan 2x\end{aligned}$$

(ii) $y = \frac{\cot x}{e^{4x}}$ [4]

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(-\operatorname{cosec}^2 x\right)\left(e^{4x}\right) - (\cot x)\left(4e^{4x}\right)}{e^{8x}}\\ &= \frac{-\operatorname{cosec}^2 x - 4\cot x}{e^{4x}}\end{aligned}$$

(b) A vase is modelled by rotating the curve

$$y = \sqrt{x} + 2$$

between the lines $x = 0$ and $x = 4$ through 2π radians about the x-axis, as shown in **Fig. 3** below.

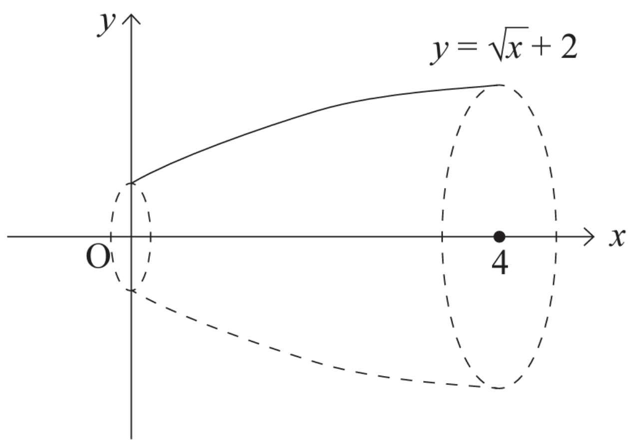


Fig. 3

Find the exact volume of the vase. [7]

$$\begin{aligned}\text{volume} &= \pi\int_0^4 \left(\sqrt{x}+2\right)^2 \mathrm{d}x\\ &= \pi\int_0^4 \left(x+4x^{\frac{1}{2}}+4\right)\mathrm{d}x\\ &= \pi\left[\frac{x^2}{2}+\frac{8}{3}x^{\frac{3}{2}}+4x\right]_0^4\\ &= \frac{136\pi}{3}\text{ units}^3\end{aligned}$$

5.

(i) Write the following expression in partial fractions.

$$\frac{3x^2-10x+5}{(x+1)(x-2)^2}$$ [6]

$$\begin{aligned}\frac{3x^2-10x+5}{(x+1)(x-2)^2} &= \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\\ 3x^2-10x+5 &= A(x-2)^2+B(x+1)(x-2)+C(x+1)\\ x &= -1\\ 18 &= A(9) \quad \Rightarrow A=2\\ x &= 2\\ -3 &= C(3) \quad \Rightarrow C=-1\\ x &= 0\\ 5 &= 2(4)+B(-2)-1(1) \quad \Rightarrow B=1\\ \frac{3x^2-10x+5}{(x+1)(x-2)^2} &= \frac{2}{x+1}+\frac{1}{x-2}-\frac{1}{(x-2)^2}\end{aligned}$$

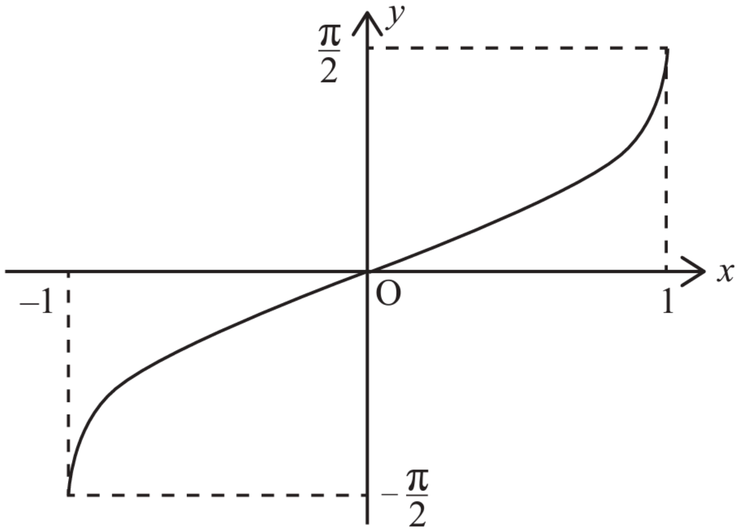
(ii) Hence show that

$$\int_0^1 \frac{3x^2-10x+5}{(x+1)(x-2)^2}\,\mathrm{d}x = \ln 2-\frac{1}{2}$$ [6]

$$\begin{aligned}\int_0^1 \frac{3x^2-10x+5}{(x+1)(x-2)^2}\mathrm{d}x &= \left[2\ln(x+1)+\ln(x-2)+\frac{1}{(x-2)}\right]_0^1\\ &= 2\ln 2+\ln|-1|+\frac{1}{(-1)}-\left(2\ln 1+\ln|-2|+\frac{1}{(-2)}\right)\\ &= \ln 2-\frac{1}{2}\end{aligned}$$

6.

(a) Sketch the graph of $y = \sin^{-1}(x)$ on the axes below for $-1 \leqslant x \leqslant 1$ [2]



(b) Express

$$\cos x - \sqrt{3}\sin x$$

in the form $R\cos(x + \alpha)$ where R is an integer and $0 < \alpha < 90°$ [7]

$$\begin{aligned}\cos x - \sqrt{3}\sin x &= R\cos x\cos\alpha - R\sin x\sin\alpha \\ 1 &= R\cos\alpha \\ \sqrt{3} &= R\sin\alpha \quad \Rightarrow \tan\alpha = \sqrt{3} \quad \Rightarrow 60^\circ \\ R &= \sqrt{1+3} = 2 \\ \cos x - \sqrt{3}\sin x &= 2\cos\ (x + 60^\circ)\end{aligned}$$

(c) Starting with the appropriate compound angle formula, prove that

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$$ [5]

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta) \\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\ &= 2\sin\theta\cos^2\theta + \left(1 - 2\sin^2\theta\right)\sin\theta \\ &= 2\sin\theta\left(1 - \sin^2\theta\right) + \sin\theta - 2\sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

7.

(a) Solve the inequality $|2x-1|>5$ [4]

$$2x-1=5$$
$$x=3$$
$$1-2x=5$$
$$x=-2$$
$$x>3 \quad \cup\, x<-2$$

(b) The graph of the function $y = \mathrm{f}(x)$ is sketched below in **Fig. 4**
The graph cuts the x-axis at A$(2, 0)$ and has an asymptote of $x = 1$

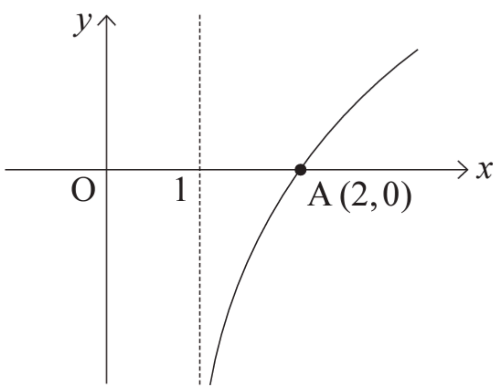
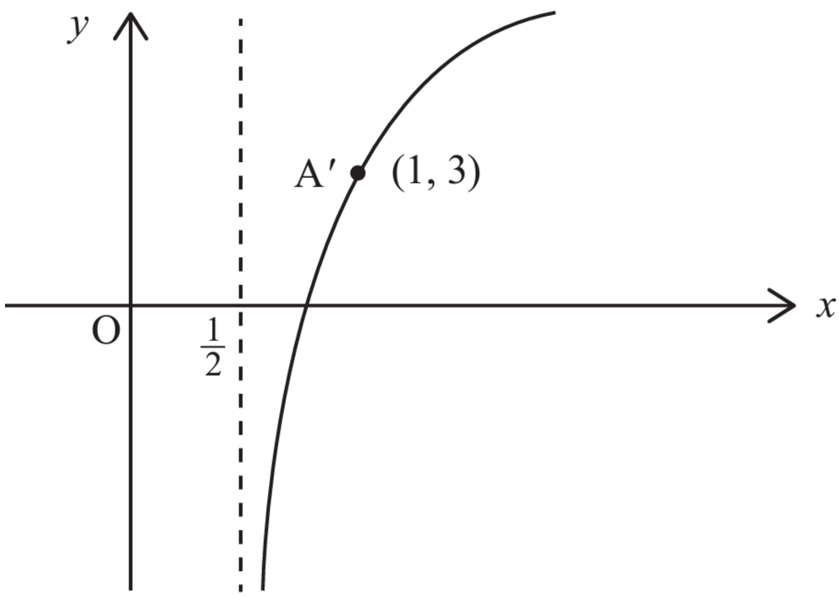


Fig. 4

(i) On the axes below, sketch the graph of

$$y = \mathrm{f}(2x) + 3$$

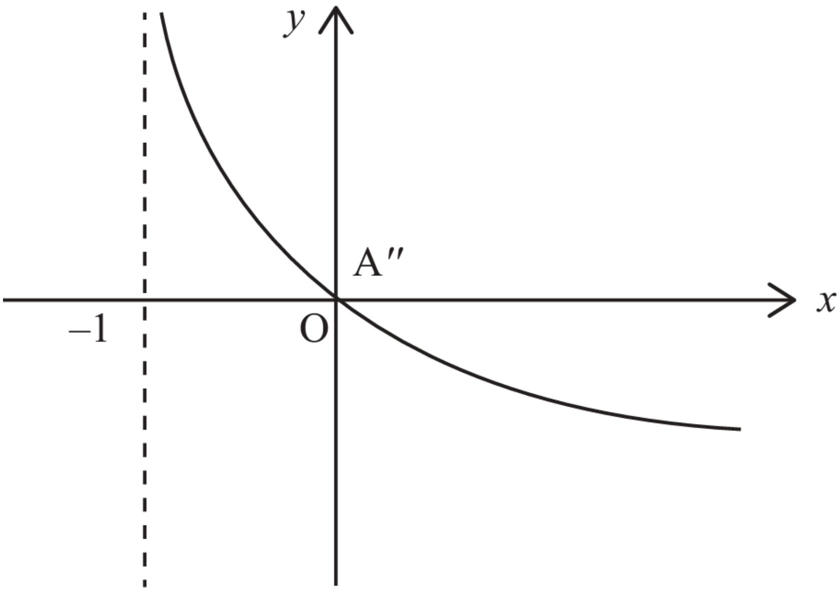
and clearly label the image of A and the asymptote. [3]



(ii) On the axes below, sketch the graph of

$$y = -f(x+2)$$

and clearly label the image of A and the asymptote. [3]



(c) The functions f and g are defined as:

$$f(x) = \frac{10}{x-1} \qquad x \in \mathbb{R}, x > 1$$

$$g(x) = e^{3x} \qquad x \in \mathbb{R}, x > 0$$

(i) State the range of f(x) and the range of g(x). [2]

Range of $f(x) : y > 0$ Range of $g(x) : y > 1$

(ii) Find the inverse $f^{-1}(x)$, stating its domain. [4]

$$y = \frac{10}{x-1}$$
$$x - 1 = \frac{10}{y}$$
$$f^{-1}(x) = \frac{10}{x} + 1$$
$$\text{domain of } f^{-1}(x) = \text{range of } f(x)$$
$$\text{domain of } f^{-1}(x) = x > 0$$

(iii) Find the composite function fg(x), stating its domain. [3]

$$fg(x) = \frac{10}{e^{3x} - 1}$$
$$\text{domain} = x > 0$$

8.

Find the equation of the tangent to the curve

$$x^2 + 5y\ln x - xy^2 = 0$$

at the point (1, –1). [11]

$$2x + 5\frac{dy}{dx}\ln x + \frac{5y}{x} - y^2 - x\left(2y\frac{dy}{dx}\right) = 0$$
$$2 + 0 + \frac{5(1)}{(1)} - (-1)^2 - 1\left(-2\frac{dy}{dx}\right) = 0$$
$$\frac{dy}{dx} = -3$$
$$y + 1 = -3(x-1)$$
$$y = -3x + 2$$

9.

(a) Find

$$\int \ln x \, dx$$ [5]

$$\int \ln x \, dx = \int 1 \times \ln x \, dx$$
$$= x\ln x - \int x\left(\frac{1}{x}\right) dx$$
$$= x\ln x - x + c$$

(b) Using the substitution $x = \frac{2}{3}\sin\theta$ evaluate

$$\int_0^{\frac{1}{3}} \sqrt{4 - 9x^2}\, dx$$ [12]

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{2}{3}\cos\theta$$

$$\int_0^{\frac{1}{3}} \sqrt{4-9x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sqrt{4-9\left(\frac{4}{9}\sin^2\theta\right)}\left(\frac{2}{3}\cos\theta\,\mathrm{d}\theta\right)$$

$$= \int_0^{\frac{\pi}{6}} 2\sqrt{1-\sin^2\theta}\left(\frac{2}{3}\cos\theta\,\mathrm{d}\theta\right)$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{6}} \cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{6}} (1+\cos 2\theta)\mathrm{d}\theta$$

$$= \frac{2}{3}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{2}{3}\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

10.

(i) Find the four values of x for which the curve

$$y = \sqrt{2}\,\mathrm{e}^{-x}\cos x$$

has stationary points in the range $0 < x < 4\pi$

Write your values in ascending order. [8]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{2}\mathrm{e}^{-x}\cos x - \sqrt{2}\mathrm{e}^{-x}\sin x = 0$$

$$\mathrm{e}^{-x} \neq 0$$

$$-\cos x - \sin x = 0$$

$$\sin x = -\cos x$$

$$\tan x = -1$$

$$x = -\frac{\pi}{4} + \pi n$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

(ii) Show that the corresponding y values of the stationary points found in **(i)** form the first four terms of a geometric progression. [4]

$$x_1 = \frac{3\pi}{4},\ y_1 = \sqrt{2}\mathrm{e}^{-\frac{3\pi}{4}}\left(-\frac{1}{\sqrt{2}}\right) = -\mathrm{e}^{-\frac{3\pi}{4}}$$
$$x_2 = \frac{7\pi}{4}, y_2 = \sqrt{2}\mathrm{e}^{-\frac{7\pi}{4}}\left(\frac{1}{\sqrt{2}}\right) = \mathrm{e}^{-\frac{7\pi}{4}}$$
$$x_3 = \frac{11\pi}{4}, y_3 = \sqrt{2}\mathrm{e}^{-\frac{11\pi}{4}}\left(-\frac{1}{\sqrt{2}}\right) = -\mathrm{e}^{-\frac{11\pi}{4}}$$
$$x_4 = \frac{15\pi}{4}, y_2 = \sqrt{2}\mathrm{e}^{-\frac{15\pi}{4}}\left(\frac{1}{\sqrt{2}}\right) = \mathrm{e}^{-\frac{15\pi}{4}}$$
$$\frac{y_2}{y_1} = \frac{\mathrm{e}^{-\frac{7\pi}{4}}}{-\mathrm{e}^{-\frac{3\pi}{4}}} = -\mathrm{e}^{-\pi}$$
$$\frac{y_3}{y_2} = \frac{-\mathrm{e}^{-\frac{11\pi}{4}}}{\mathrm{e}^{-\frac{7\pi}{4}}} = -\mathrm{e}^{-\pi}$$
$$\frac{y_4}{y_3} = \frac{\mathrm{e}^{-\frac{15\pi}{4}}}{-\mathrm{e}^{-\frac{11\pi}{4}}} = -\mathrm{e}^{-\pi}$$
$$a = -\mathrm{e}^{-\frac{3\pi}{4}},\ r = -\mathrm{e}^{-\pi}$$

(iii) For the geometric progression defined in **(ii)**, find an exact value for the sum to infinity. [2]

$$S_\infty = \frac{-\mathrm{e}^{-\frac{3\pi}{4}}}{1-(-\mathrm{e}^{-\pi})} = \frac{-\mathrm{e}^{\frac{\pi}{4}}}{\mathrm{e}^{\pi}+1}$$

11.

The path of a train on a model railway can be represented by the parametric equations

$$x = 2t^2 + 4t \quad \text{and} \quad y = \sin 2t$$

(i) Find the co-ordinates of the stationary point on this curve for $0 < t < \frac{\pi}{2}$ [6]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$
$$= \frac{2\cos 2t}{4t+4} = 0$$
$$\cos 2t = 0$$
$$2t = \frac{\pi}{2}, \frac{3\pi}{2}$$
$$t = \frac{\pi}{4}$$
$$x = 2\left(\frac{\pi^2}{16}\right) + 4\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8} + \pi$$
$$y = 1$$

(ii) Using $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ classify this stationary point. [6]

$$\frac{d^2y}{dx^2} = \frac{-4\sin 2t(4t+4) - (2\cos 2t)(4)}{(4t+4)^2}\frac{dx}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{-16\sin 2t(t+1) - 8\cos 2t}{(4t+4)^2}(4t+4)$$

$$f''\left(\frac{\pi}{4}\right) = -5.12 < 0, \text{ maximum point.}$$

12.

Water leaks out of a small hole in the bottom of a paddling pool.

The rate at which the water leaves the pool is proportional to the square root of the volume of water that remains in the pool.

The initial volume of water is V_0

If $\frac{5}{9}$ of the initial volume leaks out in the first hour, find how long it takes for the pool to empty. [10]

$$-\frac{dV}{dt} = k\sqrt{V}$$

$$\int V^{-\frac{1}{2}}dV = -k\int dt$$

$$2V^{\frac{1}{2}} = -kt + c$$

$$t = 0,\ V = V_0$$

$$2\sqrt{V_0} = +c$$

$$2V^{\frac{1}{2}} = -kt + 2\sqrt{V_0}$$

$$t = 1,\ V = \frac{4}{9}V_0$$

$$2\sqrt{\frac{4}{9}V_0} = -k + 2\sqrt{V_0}$$

$$k = \frac{2}{3}\sqrt{V_0}$$

$$2V^{\frac{1}{2}} = -\frac{2}{3}\sqrt{V_0}t + 2\sqrt{V_0}$$

$$0 = -\frac{t}{3}\sqrt{V_0} + \sqrt{V_0}$$

$$t = 3 \text{ hours}$$