Differential Geometry

Lesson 23

The First Fundamental Form and Covariant Derivatives

(be sure to go through this entire document watching all the videos and completing the classwork)

The homework is to repeat the starred classwork with your personal surface that was emailed to you on Wed April 22. You have also been emailed a point p and a chart Y s.t. Y(0)=p to use.

You may complete the homework at your own pace.   All homework completed for the rest of the course should be kept in a single online document that I have access to (for example a google document) with headers for each lesson and photos of your work inserted below that lesson in order. You may work together and submit a document together if you wish. Share the document with me when you want feedback.

This lesson has two halves with a break in the middle.

The First Fundamental Form of a Submanifold or Surface:

In Video FFFPart1 we review:

In Linear Algebra you learned the notion of a Vector Space which is a collection of vectors closed under vector addition (with four rules about the vector addition) and closed under scalar multiplication (with four rules about scalar multiplication).

Recall that the basis $v_1 \ldots v_m$ of a vector space is a linearly independant collection of vectors whose span is the vector space. So any vector V can be written uniquely as a linear combination of the basis vectors $V=a_1v_1+a_2v_2+\ldots+a_mv_m$
The dimension of the vector space is the number of vectors in the basis.

Example the tangent space T_pM at a point in an m dim submanifold is a vector space of dim m and given a chart Y about p with Y(0)=p the columns of DY_p form a basis. A different chart will give a different basis. It is ok if we find a chart where Y(u)=p, we can just use DY_{u} and get another basis.

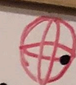
Classwork 1ab: M is the sphere of radius 5 and p is the point (3,4,0). The tan space at p is the plane perpendicular to the normal at p. Find two different charts using the implicit function theorem and two different basis for TM_p. Check your work by verifying that the

span is the same vector space, which is the plane perpendicular to the normal. (for HW if your given p does not work to give two charts for your M, then choose a different p in M that will give two charts for 1ab)

In Video [FFFPart2] we explain the solution of this classwork:

Classwork 1: $M^2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 + y^2 + z^2 = 5^2 \right\}$

$T_pM = \{ \vec{v} \in \mathbb{R}^3 \text{ s.t } \vec{v} \perp \vec{n} \}$ where $\vec{n} = DF = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$ $\quad p = (3,4,0)$



Find two charts at p using Implicit Funct Thm & two different basis for T_pM

$F(x,y,z) = x^2 + y^2 + z^2 - 5^2 = 0$

$x^2 = 5^2 - y^2 - z^2$

$x = \pm\sqrt{5^2 - y^2 - z^2}$

$DF_p = (2x \;\; 2y \;\; 2z)_p = (6 \;\; 8 \;\; 0)$

Chart where x is a leader!

$x = \sqrt{5^2 - y^2 - z^2}$

$Y\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{5^2 - y^2 - z^2} \\ y \\ z \end{pmatrix}$

$U = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} : y^2 + z^2 < 5^2 \right\}$ includes $(4,0)$ ✓

$Y\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \quad \vec{u} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad Y(\vec{u}) = p$

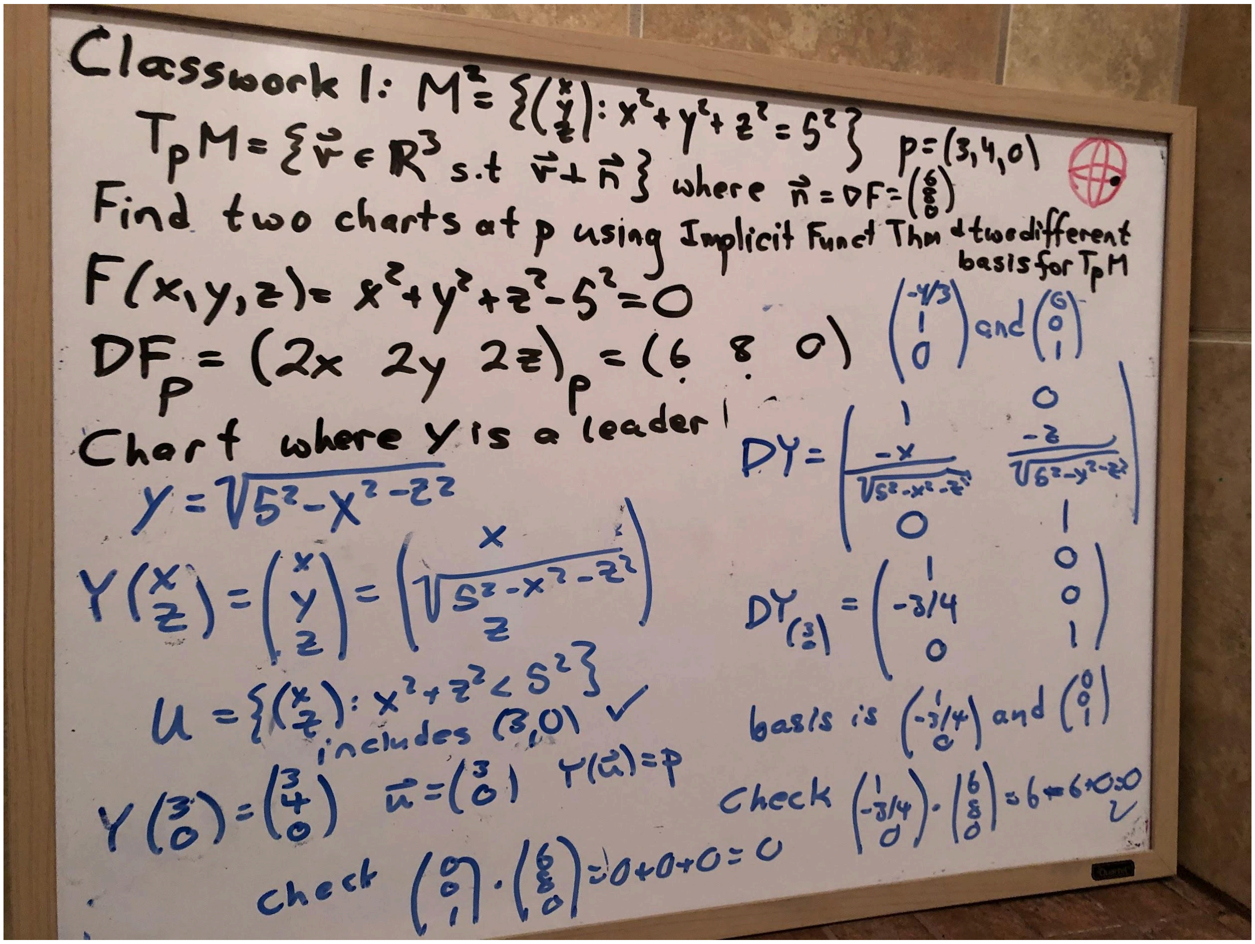
$DY_{\vec{u}} = \begin{pmatrix} \frac{-y}{\sqrt{5^2 - y^2 - z^2}} & \frac{-z}{\sqrt{5^2 - y^2 - z^2}} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are a basis for T_pM

Check: $\perp$ to $\vec{n}$

$\begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix} = -8 + 8 + 0 = 0$ ✓

$\begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 + 0 + 0 = 0$ ✓



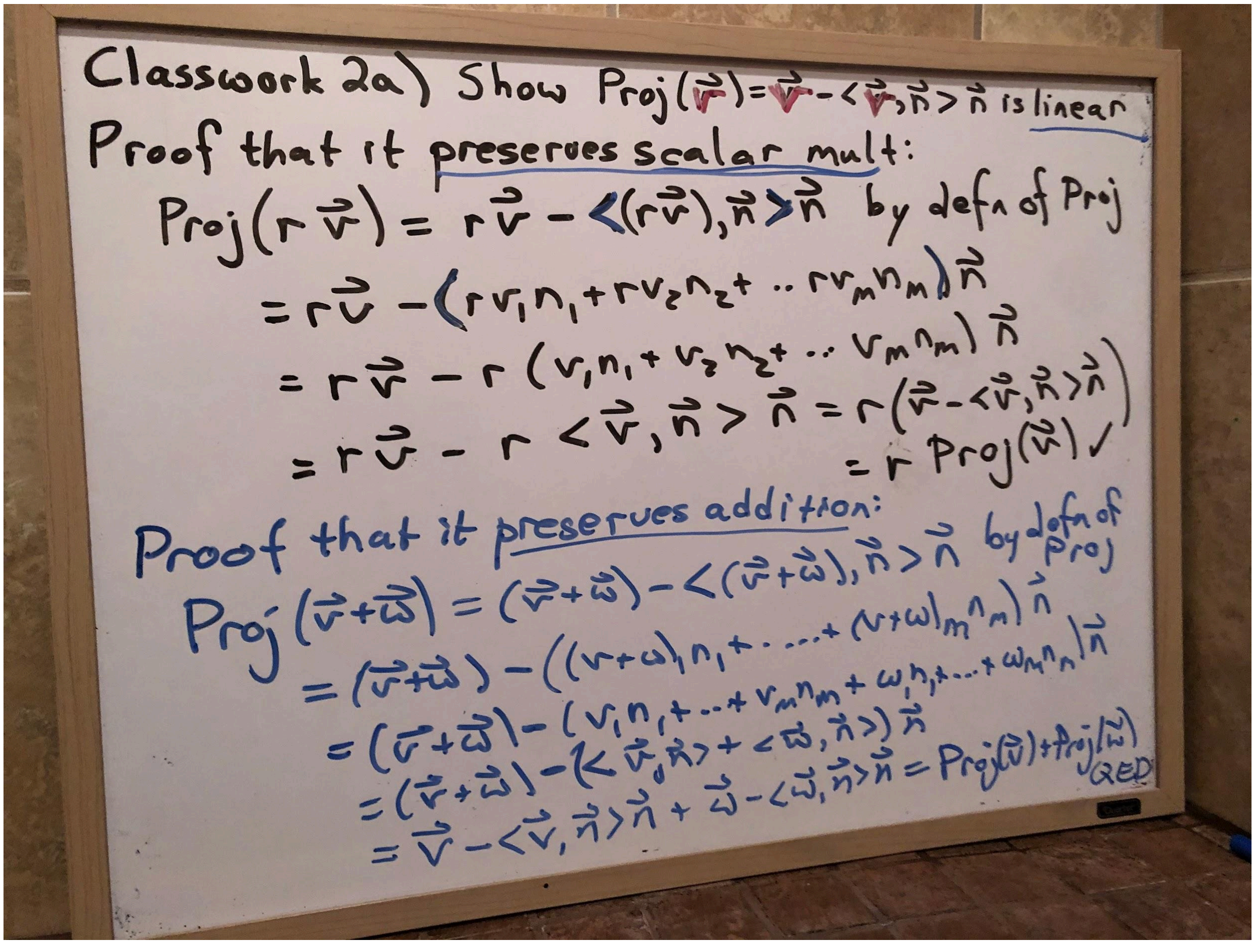
In Video FFFPart3 we go over the following:

Recall a Linear Map, F, between vector spaces:
preserves scalar multiplication: F(rV)=rF(V)
And also preserves addition: F(V+W)=F(V)+F(W)
So in fact: F(tV+sW)=tF(V)+sF(W)
So F is determined completely by the values of F on a basis of the domain:
F(v)=F(a_1v_1+...+a_mv_m)=a_1F(v_1)+...+a_mF(v_m)

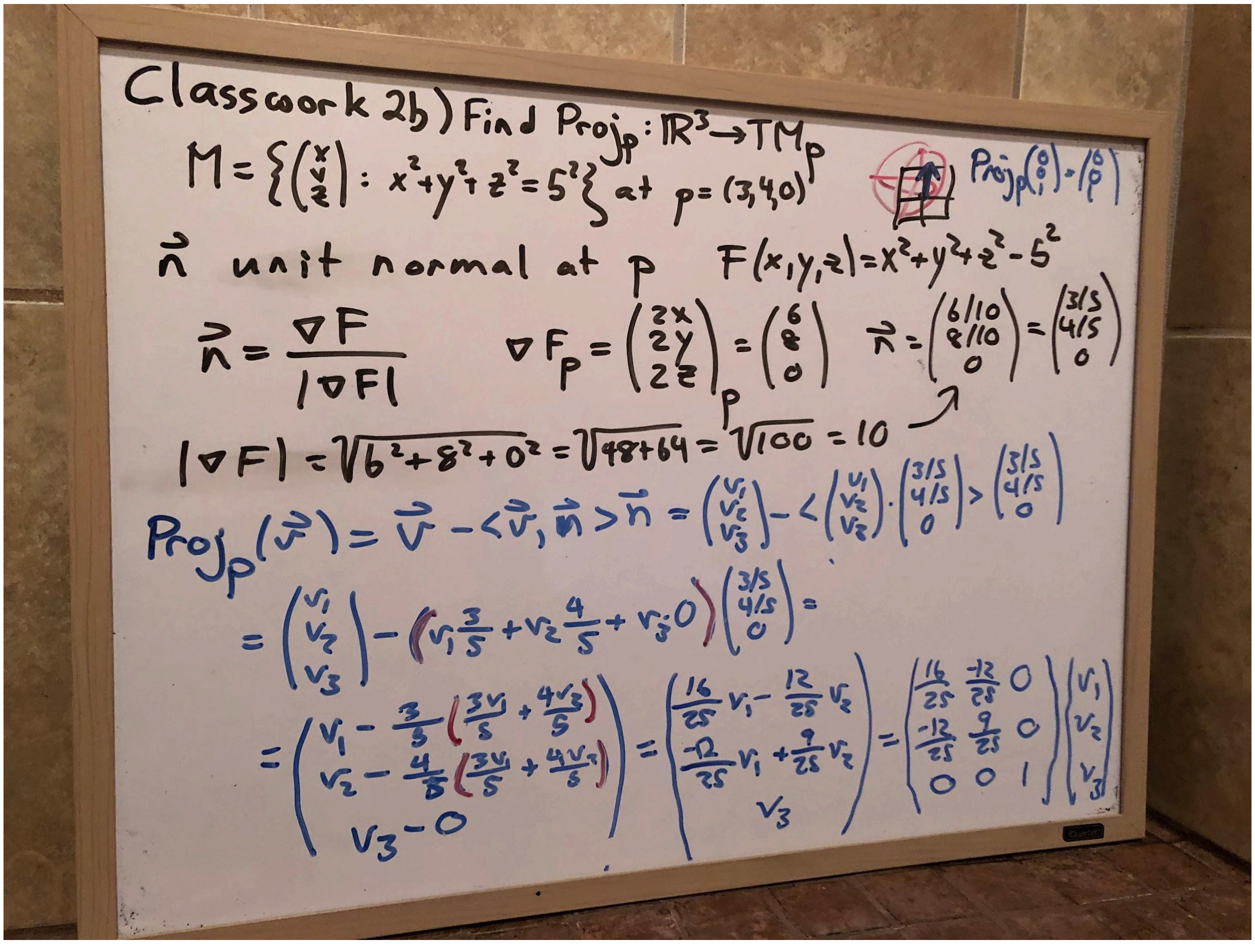
Example: The projection map to a plane perpendicular to a unit normal n:
Proj(v)= v - <n,v>n where <n,v> is the dot product of n and v

Classwork 2ab: (use your original p for HW)
2a) Verify that this projection map is a linear map.
2b) Find the projection map Proj_p from R^3 to TM_p where M is the sphere of radius 5 and p is the point (3,4,0). Write the projection map as a matrix.

In Video FFFPart4a we complete the solution of this classwork 2a as follows:



In Video FFFPart4b we complete the solution of classwork 2b as follows:



In Video FFFPart5a we introduce the following:

There is another notion called a Bilinear Form, B(X,Y), which takes in two vectors and produces a real number that is bilinear:

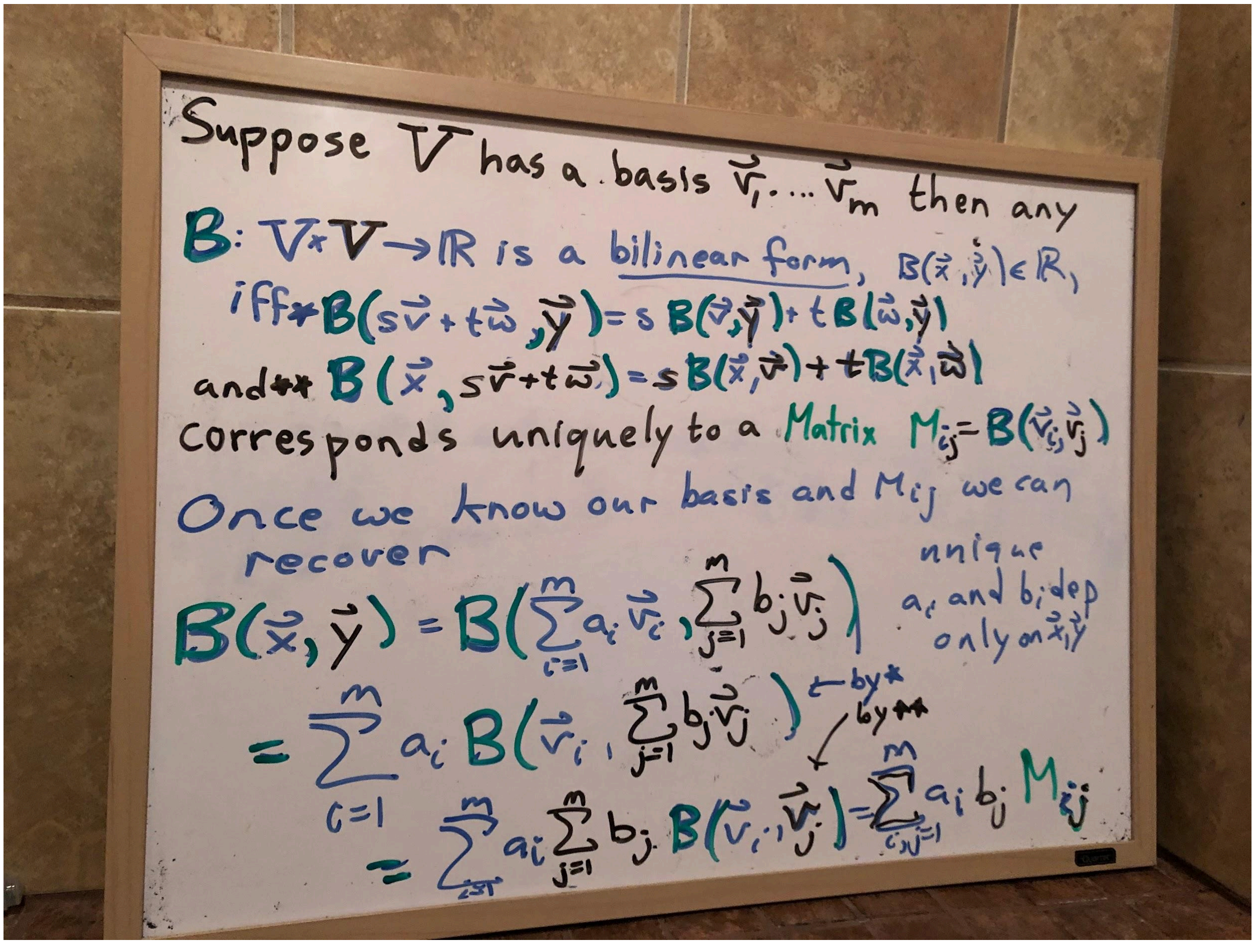
B(tV+sW,Y)=tB(V,Y)+sB(W,Y)
and
B(X, tV+sW)=tB(X,V,)+sB(X,W)

It is symmetric if B(X,Y)=B(Y,X)
and antisymmetric if B(X,Y)=-B(Y,X)

Example the inner product <V,W>
which is the dot product of V and W
Is a symmetric bilinear form.

Example: Det of a 2x2 matrix is an antisymmetric bilinear form

In Video [FFFPart5b]() we introduce the following:



In Video [FFFPart6a]() we introduce the following:

The First Fundamental Form (also called Riemannian metric tensor) in Differential Geometry is a bilinear form denoted I_p (or g_p) which takes in tangent vectors at p and gives a real number as follows:

I_p(V,W)=<V,W> where <V,W> is the dot product of the vectors as Euclidean vectors.

Given a basis of the tangent space at p we can define the mxm matrix:

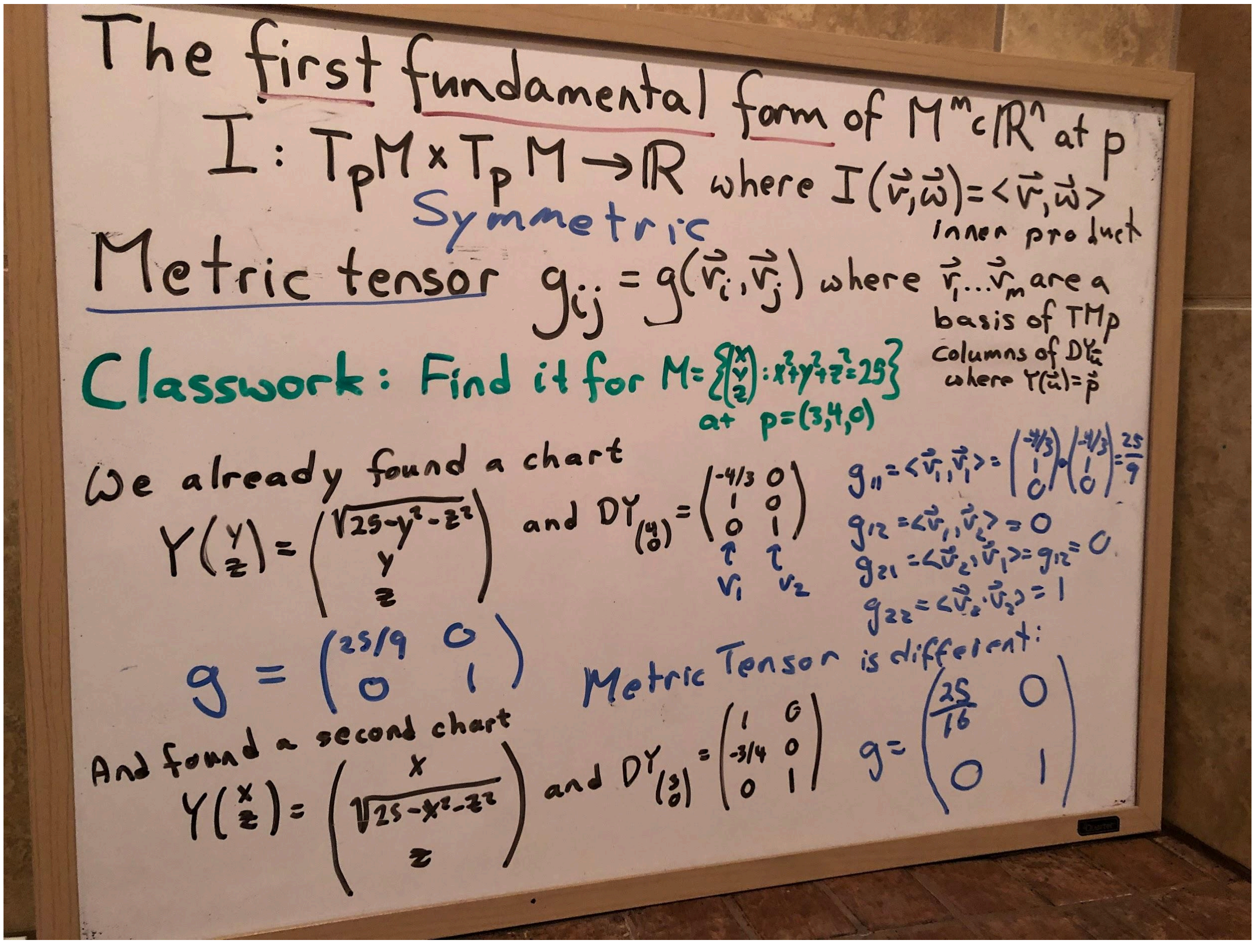
g_{i,j}=<v_i,v_j>

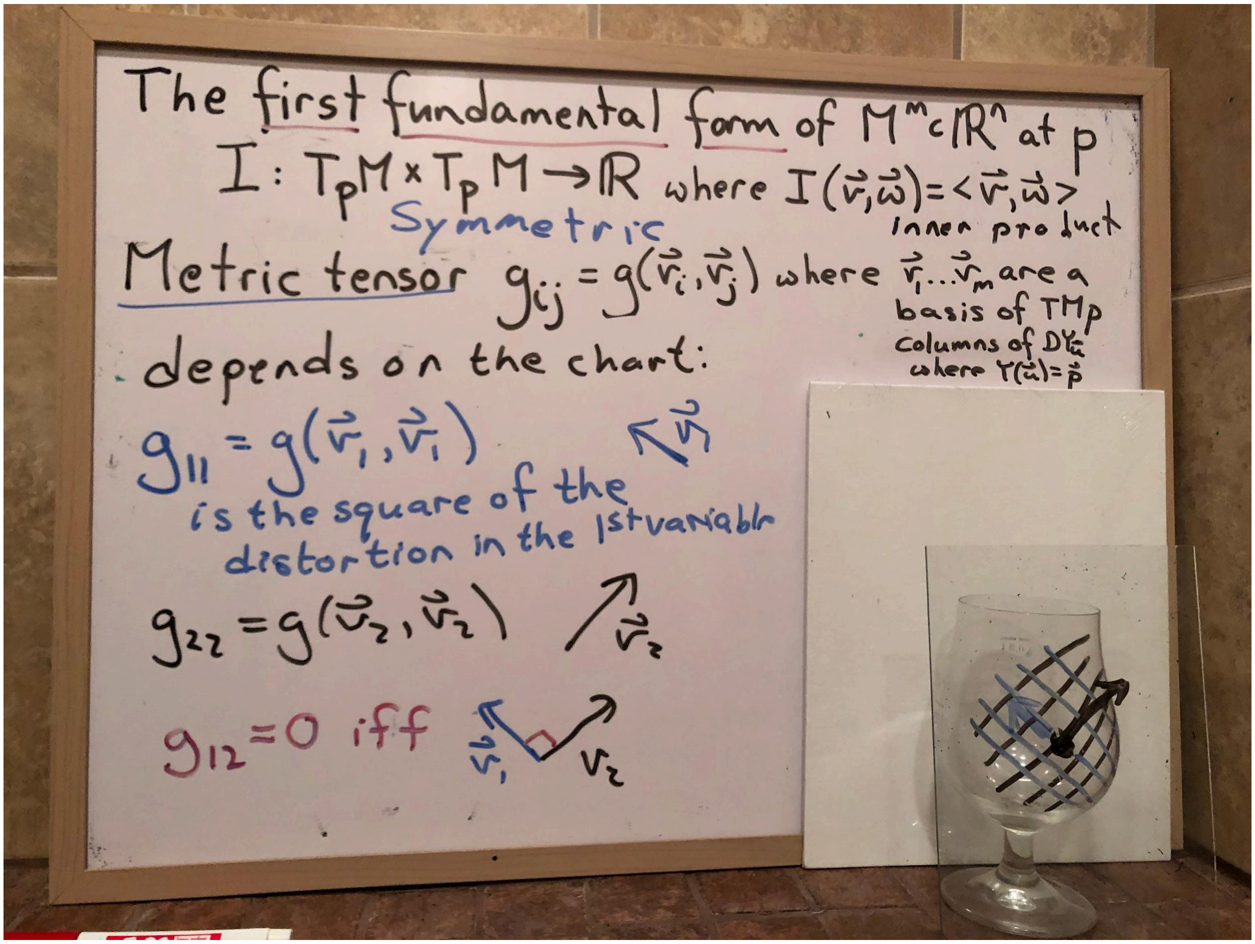
The values of this matrix depends on the basis which depends on the chart

Classwork 3:

3) Find the first fundamental form of the tangent space to the sphere of radius 5 at (3,4,0) using the chart.

In Video FFFPart6b we complete the solution of this classwork as in the photo:





The textbook has more information about how the metric tensor changes when you change the chart using the transition functions between the charts.

The Homework is to complete Classwork 1-3 for up to 3 stars using your given M and your given p and your given chart Y such that Y(0)=p. There is no particular deadline. Upload your solution to a google doc with your first name and course in the title and share the document with me so I can give feedback as you proceed when you request it.