Linear Algebra MAT313 Spring 2023
Professor Sormani
Lesson 3 Augmented Matrices and the Algorithm

In this lesson we will learn a specific algorithm that we will use all semester for solving many different kinds of problems. It is necessary to use this specific algorithm and not another one taught in other courses because I will be designing homework and quiz problems that can be solved quickly using this algorithm. If you have difficulty learning this algorithm, I will help with feedback and you can resubmit your work.

Warning: do not start this lesson until you have completed and corrected Lesson 2 following the solutions and submitted all the notes, classwork, and homework for that lesson.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313S23-lesson1-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You can use your Lehman id and hand instead of your face in your selfie. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

This Lesson has two parts and each has its own playlist:

Part I Augmented Matrices

Part II Algorithm

There are 9 homework problems.

Part I Augmented Matrices

Solving a linear system using augmented matrices, reducing to Echelon form, converting back into a system, solving for leaders, subbing up, and writing a solution set. Watch the [Playlist 313F23-L3-P1-1to3]()

Converting a system into an Augmented Matrix

(especially useful if you have many equations and many unknowns)

$$\begin{aligned} 2x + 3y + 4z &= 8 \\ 5x + 6y + 8z &= 9 \\ 4x + 5y + 7z &= 8 \end{aligned} \longrightarrow \underbrace{\left[\begin{array}{ccc|c} 2 & 3 & 4 & 8 \\ 5 & 6 & 8 & 9 \\ 4 & 5 & 7 & 8 \end{array}\right]}_{\text{coefficients on the right}} \leftarrow \text{Augmented matrix}$$

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 &= d_1 \\ a_{2,1}x_1 + a_{2,2}x_2 &= d_2 \end{aligned} \longrightarrow \left[\begin{array}{cc|c} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \end{array}\right]$$

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 8x_4 + 9x_5 + 10x_6 &= 0 \\ 3x_1 + 2x_2 + 0x_3 + 5x_4 + 2x_5 + 8x_6 &= 1 \end{aligned} \longrightarrow \left[\begin{array}{cccccc|c} 2 & 3 & 4 & 8 & 9 & 10 & 0 \\ 3 & 2 & 0 & 5 & 2 & 8 & 1 \end{array}\right]$$

$$\begin{aligned} 2x + 3y + 4z &= 8 \\ x - y + z &= 0 \\ x + 5z &= 10 \end{aligned} \xrightarrow{\text{add in 1 and 0 as needed}} \begin{aligned} 2x + 3y + 4z &= 8 \\ 1x - 1y + 1z &= 0 \\ 1x + 0y + 5z &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 3 & 4 & 8 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 5 & 10 \end{array}\right]$$

Converting an Augmented Matrix Back into a system and writing the solution set.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{array}\right] \end{array} \qquad \begin{aligned} \boxed{1x} + 4y + 0z &= 0 \\ 0x + \boxed{1y} + 2z &= 4 \\ 0x + 0y + 0z &= 5 \end{aligned}$$

Solve for leaders

$$\begin{aligned} x &= 0 - 4y - 0z \\ y &= 4 - 2z \\ 0 &= 5 \leftarrow \text{No solution} \end{aligned}$$

$$\emptyset$$

Converting an Augmented Matrix back into a system:

box the leaders — count the columns: five variables

$$\left[\begin{array}{ccccc|c} \boxed{1} & 2 & 3 & 0 & 0 & 6 \\ 0 & \boxed{1} & 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon form

$$\boxed{\begin{array}{l} \boxed{1x_1}+2x_2+3x_3+0x_4+0x_5=6 \\ 0x_1+\boxed{1x_2}+2x_3+0x_4+0x_5=5 \\ 0x_1+0x_2+0x_3+\boxed{1x_4}+0x_5=0 \\ 0x_1+0x_2+0x_3+0x_4+\boxed{1x_5}=6 \\ 0x_1+0x_2+0x_3+0x_4+0x_5=0 \end{array}}$$

box the system

last line $0=0$ is OK — box the leaders

Solve for leaders and sub up

$1x_1 = 6 - 2x_2 - 3x_3 = 6-2(5-2x_3)-3x_3 = 6-10+4x_3-3x_3$
$= -4 + x_3$

$1x_2 = 5 - 2x_3$

$1x_4 = 0$ — $x_3 = x_3$ is free

$1x_5 = 6$

$0=0$ OK still a solution

Find the solution set and write it as a line

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4+x_3 \\ 5-2x_3 \\ x_3 \\ 0 \\ 6 \end{pmatrix} \middle| \; x_3 \in \mathbb{R} \right\}$$

to write as a line

note: $x_3 = 0 + 1x_3$; $0 = 0 + 0x_3$; $6 = 6 + 0x_3$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4+1x_3 \\ 5-2x_3 \\ 0+1x_3 \\ 0+0x_3 \\ 6+0x_3 \end{pmatrix} \middle| \; x_3 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 0 \\ 0 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \middle| \; x_3 \in \mathbb{R} \right\}$$

The position comes from constants (not multiplied by free variable)

direction comes from coefficients of the free variable.

Part II Algorithm

Watch [Playlist 313S23-L3-P2-1to10]

Our Algorithm to Reduce an Augmented Matrix to Echelon Form
(there are many algorithms but we will use this one in this course)

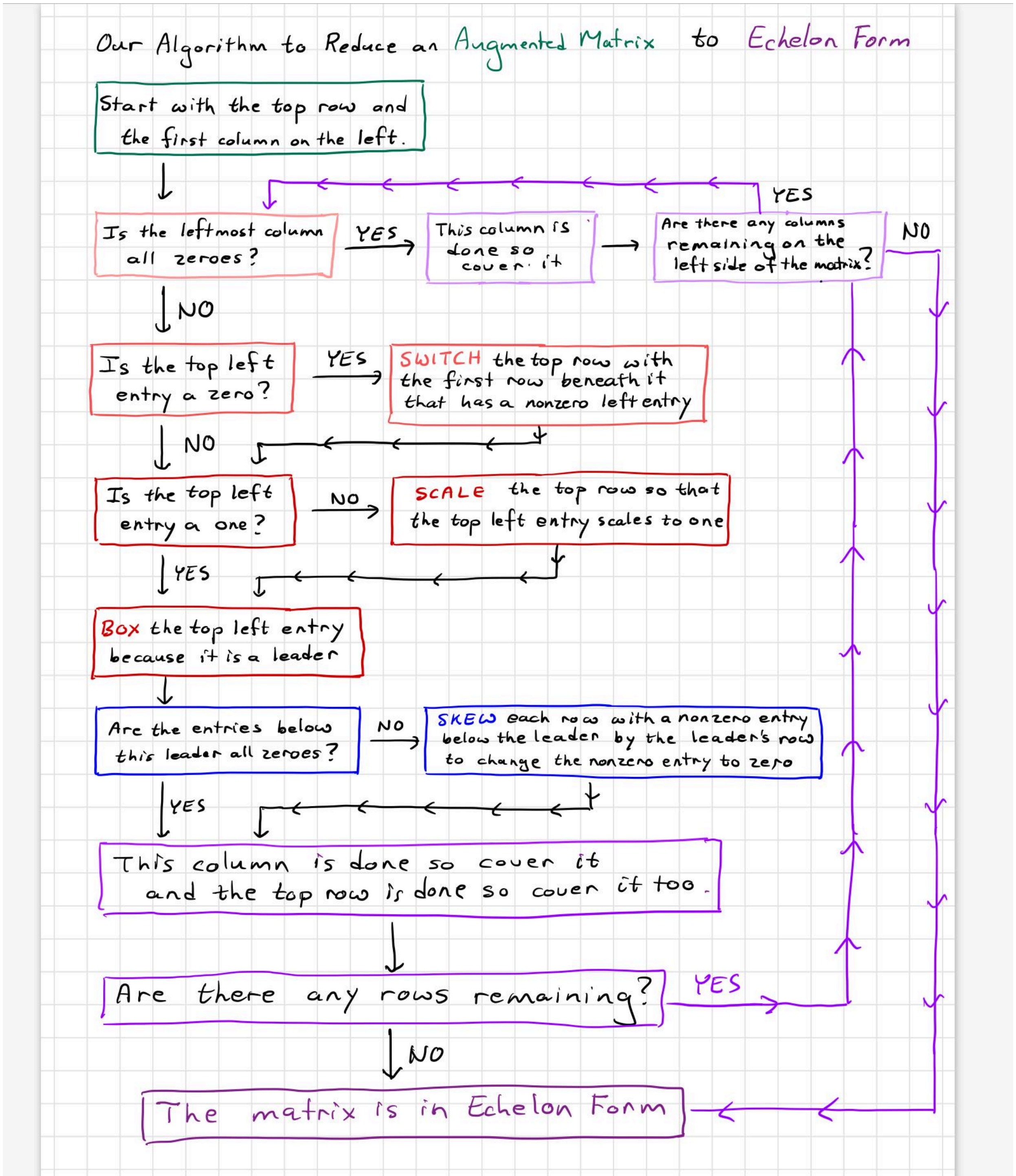
Definition: a matrix is in Echelon Form if
it has zeroes to the left and below all leaders

and all the leaders are ones and
all rows without leaders are at the bottom.

Definition: a leader is the first nonzero entry of a row appearing on the left side of an augmented matrix.

Method: Start on the upper left.
Step 1: Find a leader and make it one using row actions.
Step 2: Get zeroes below the leader using row actions.
Go to the next row and column and repeat Step 1 and Step 2.
When we run out of rows, the matrix is in Echelon Form:
it has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

A quick glance at our algorithm:



Echelon Form Classwork: (use graph paper and a multicolor pen)

Classwork: Which of the following matrices is in Echelon Form?

Definition: a matrix is in **Echelon Form** if it has **zeroes to the left and below all leaders** and **all the leaders are ones** and **all rows without leaders are at the bottom.**

Definition: a **leader** is the first nonzero entry of a row appearing on the left side of an augmented matrix.

(a) Yes in Echelon Form

$$\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & \boxed{1} & 3 & 5 & 7 & 9 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{array}\right]$$

box each leader and check for zeroes and check the leader is a one

← last row has no leaders because the 8 is on the right side. The only row with no leaders is at the bottom.

(b)

$$\left[\begin{array}{cccccc|c} 1 & 0 & 5 & 3 & 7 & 2 & 8 \\ 0 & 0 & 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 2 & 3 \end{array}\right]$$

(c)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \end{array}\right]$$

(d)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(e)

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

Remember you are handing in this classwork!

Try the rest.

Classwork: Which of the following matrices is in Echelon Form?

Definition: a matrix is in **Echelon Form** if it has **zeroes to the left and below all leaders** and **all the leaders are ones** and **all rows without leaders are at the bottom.**

Definition: a **leader** is the first nonzero entry of a row appearing on the left side of an augmented matrix.

(a)

$$\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & \boxed{1} & 3 & 5 & 7 & 9 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{array}\right]$$

← Yes this is in Echelon Form

(b)

$$\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 5 & 3 & 7 & 2 & 8 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 2 & 3 \end{array}\right]$$

This 1 is to the left and below the second leader!

Not in Echelon Form

(c)

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 4 \end{array}\right]$$

← No leader in the second row. The 3 is on the right side so not a leader

← A leader in the third row. The second row without a leader should be below the third row because the third row has a leader!

Not in Echelon Form!

(d)

$$\left[\begin{array}{ccccc|c} \boxed{1} & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & \boxed{7} & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

← The leader in the second row is not one so this is not in Echelon Form!

(e)

$$\left[\begin{array}{cccccc|c} 0 & \boxed{1} & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

← first leader has zeroes to left and below and is a one.

← no leader in the second row which is at the bottom.

This is in Echelon Form.

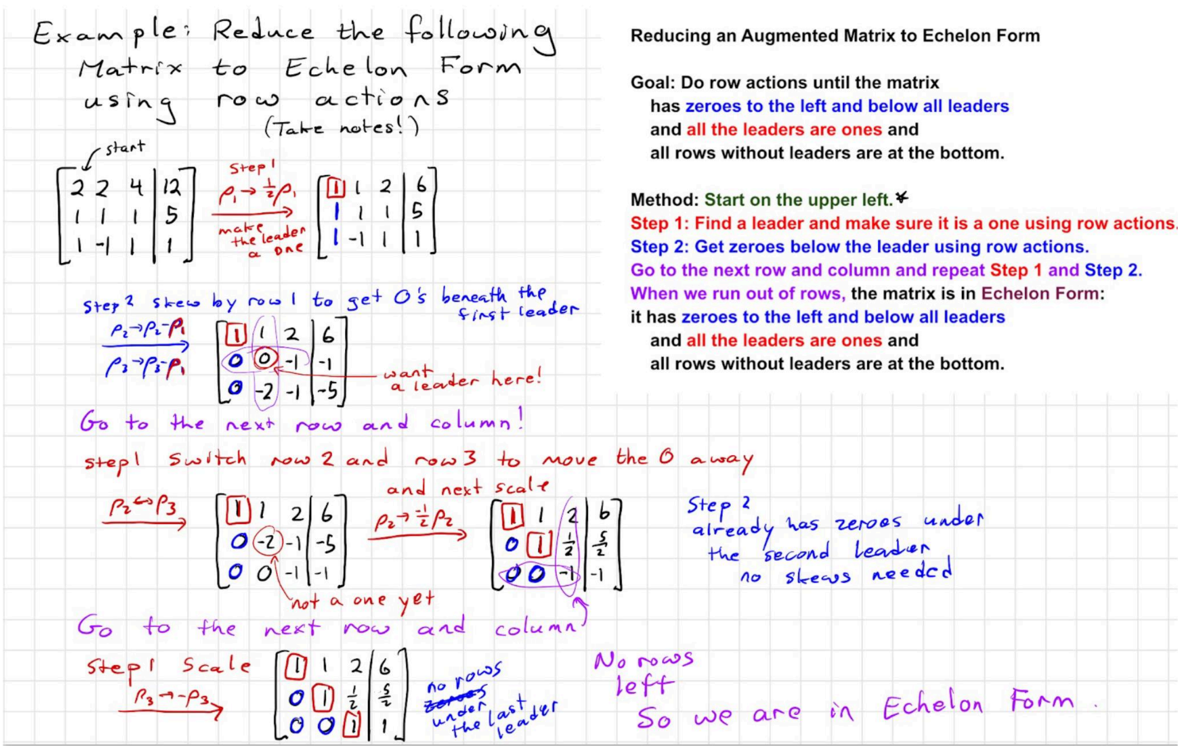
Reducing an Augmented Matrix to Echelon Form

Goal: Do row actions until the matrix

has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Method: Start on the upper left.
Step 1: Find a leader and make sure it is a one using row actions.
Step 2: Get zeroes below the leader using row actions.
Go to the next row and column and repeat Step 1 and Step 2.
When we run out of rows, the matrix is in **Echelon Form:**
it has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Example:


Example: Reduce the following Matrix to Echelon Form using row actions (Take notes!)

start

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 12 \\ 1 & 1 & 1 & 5 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[\text{make the leader a one}]{\text{Step 1: } \rho_1 \to \frac{1}{2}\rho_1} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \\ 1 & -1 & 1 & 1 \end{array}\right]$$

Step 2 skew by row 1 to get 0's beneath the first leader

$$\xrightarrow[\rho_3 \to \rho_3 - \rho_1]{\rho_2 \to \rho_2 - \rho_1} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 6 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & -1 & -5 \end{array}\right]$$

want a leader here!

Go to the next row and column!

step 1 switch row 2 and row 3 to move the 0 away and next scale

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 6 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{\rho_2 \to -\frac{1}{2}\rho_2} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 6 \\ 0 & \boxed{1} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & -1 & -1 \end{array}\right]$$

not a one yet

Step 2 already has zeroes under the second leader no skews needed

Go to the next row and column

Step 1 Scale

$$\xrightarrow{\rho_3 \to -\rho_3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 6 \\ 0 & \boxed{1} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

no rows ~~zeroes~~ under the last leader

No rows left So we are in Echelon Form.

Reducing an Augmented Matrix to Echelon Form

Goal: Do row actions until the matrix
has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Method: Start on the upper left.
Step 1: Find a leader and make sure it is a one using row actions.
Step 2: Get zeroes below the leader using row actions.
Go to the next row and column and repeat Step 1 and Step 2.
When we run out of rows, the matrix is in Echelon Form:
it has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Classwork:

Classwork: Reduce the matrix to Echelon Form.

$$\left[\begin{array}{cccc|c} 2 & 2 & 2 & 2 & 4 \\ 4 & 4 & 6 & 6 & 6 \\ 5 & 5 & 4 & 3 & 2 \end{array}\right]$$

Pause + Try!

Reducing an Augmented Matrix to Echelon Form

Goal: Do row actions until the matrix
has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Method: Start on the upper left.
Step 1: Find a leader and make sure it is a one using row actions.
Step 2: Get zeroes below the leader using row actions.
Go to the next row and column and repeat Step 1 and Step 2.
When we run out of rows, the matrix is in Echelon Form:
it has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Classwork: Reduce the matrix to Echelon Form.

$$\left[\begin{array}{cccc|c} 2 & 2 & 2 & 2 & 4 \\ 4 & 4 & 6 & 6 & 6 \\ 5 & 5 & 4 & 3 & 2 \end{array}\right]$$

step 1 Turn this 2 into a 1

$$\xrightarrow{\rho_1 \to \frac{1}{2}\rho_1} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 2 \\ 4 & 4 & 6 & 6 & 6 \\ 5 & 5 & 4 & 3 & 2 \end{array}\right]$$

box the leader!

Step 2 Turn this 4 and 5 into zeroes using skews by the leader's row.

$$\xrightarrow[\rho_3 \to \rho_3 - 5\rho_1]{\rho_2 \to \rho_2 - 4\rho_1} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & -1 & -2 & -8 \end{array}\right]$$

Next row and column is zeroes so go to the next

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & -1 & -2 & -8 \end{array}\right]$$

Step 1 turn this 2 into a 1

$$\xrightarrow{\rho_2 \to \frac{1}{2}\rho_2} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & -1 & -2 & -8 \end{array}\right]$$

Step 2 turn -1 into a zero because under the second leader using a skew by second leader's row

$$\xrightarrow{\rho_3 \to \rho_3 + \rho_2} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & -1 & -9 \end{array}\right]$$

Next row and column

Step 1 -1 to one

$$\xrightarrow{\rho_3 \to -\rho_3} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & \boxed{1} & 9 \end{array}\right]$$

Last row so we are in Echelon Form!

Reducing an Augmented Matrix to Echelon Form

Goal: Do row actions until the matrix
has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Method: Start on the upper left.
Step 1: Find a leader and make sure it is a one using row actions.
Step 2: Get zeroes below the leader using row actions.
Go to the next row and column and repeat Step 1 and Step 2.
When we run out of rows, the matrix is in Echelon Form:
it has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Classwork

Classwork: Reduce the matrix to Echelon Form using our method,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right]$$

Reducing an Augmented Matrix to Echelon Form

Goal: Do row actions until the matrix
has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Method: Start on the upper left.
Step 1: Find a leader and make sure it is a one using row actions.
Step 2: Get zeroes below the leader using row actions.
Go to the next row and column and repeat Step 1 and Step 2.
When we run out of rows, the matrix is in Echelon Form:
it has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Classwork: Reduce the matrix to Echelon Form using our method,

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right]$$

Step 1 already a 1 so done

Step 2 change row 4 skewing by leader's row

Reducing an Augmented Matrix to Echelon Form

Goal: Do row actions until the matrix
has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

Method: Start on the upper left.
Step 1: Find a leader and make sure it is a one using row actions.
Step 2: Get zeroes below the leader using row actions.
Go to the next row and column and repeat Step 1 and Step 2.
When we run out of rows, the matrix is in Echelon Form:
it has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

$$\xrightarrow{\rho_4 \to \rho_4 - \rho_1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Go to the next row and column

look in upper left

Step 1 already a 1 so done

Step 2 already 0s under so done

Go to next row and column

covering columns + rows that are done

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

zero!

Step 1 turn into a 1 by switching with row 4

$$\xrightarrow{\rho_3 \leftrightarrow \rho_4} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

step 2 done

Next row + column

up to here

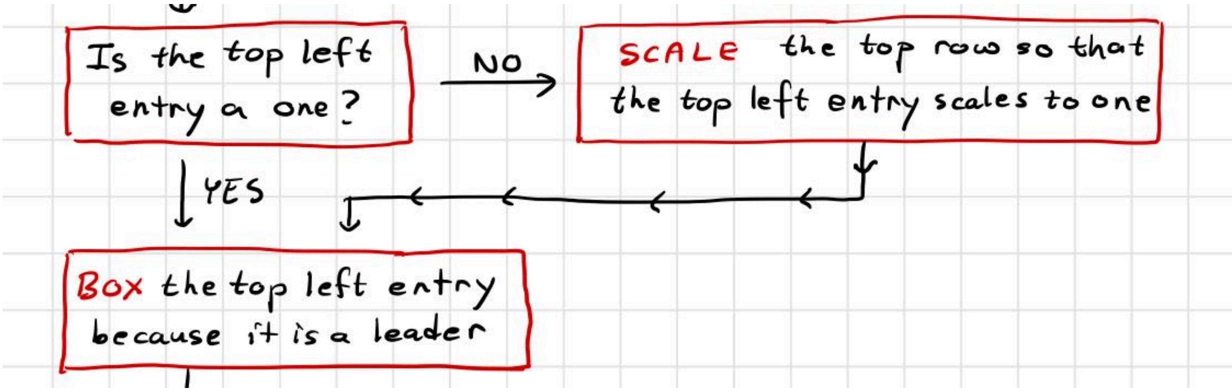
Step 1 change 4 to 1

$$\xrightarrow{\rho_4 \to \frac{1}{4}\rho_4} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

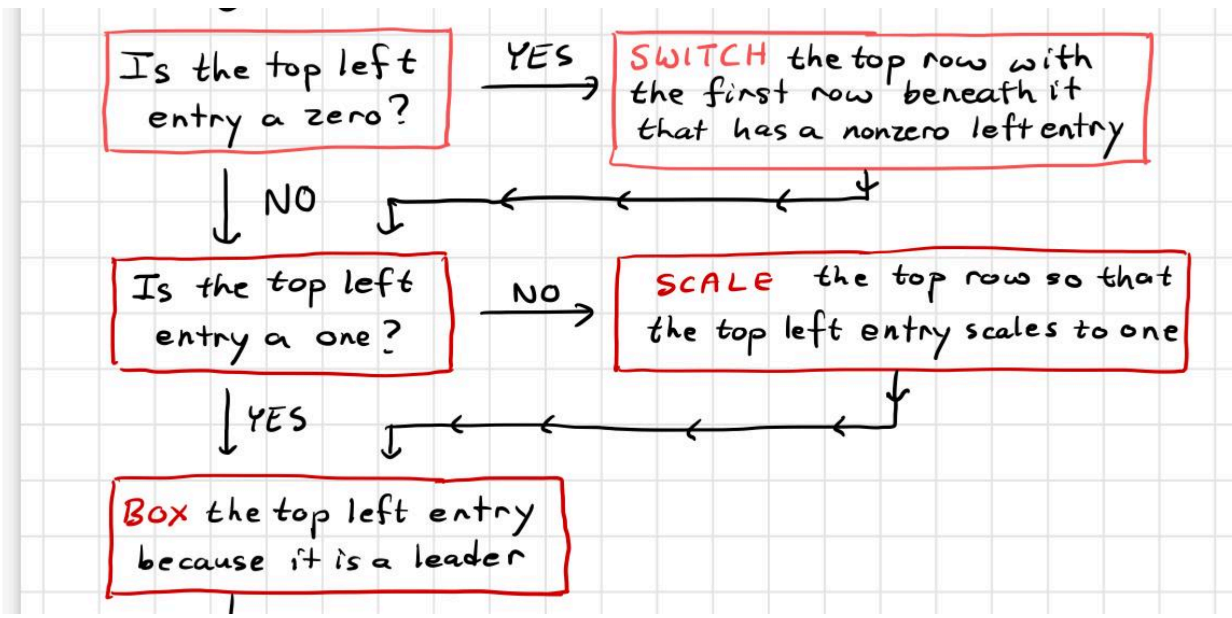
Creating Our Algorithm:

Method: Start on the upper left.
Step 1: Find a leader and make sure it is a one using row actions.
Step 2: Get zeroes below the leader using row actions.
Go to the next row and column and repeat Step 1 and Step 2.
When we run out of rows, the matrix is in Echelon Form:
it has zeroes to the left and below all leaders
and all the leaders are ones and
all rows without leaders are at the bottom.

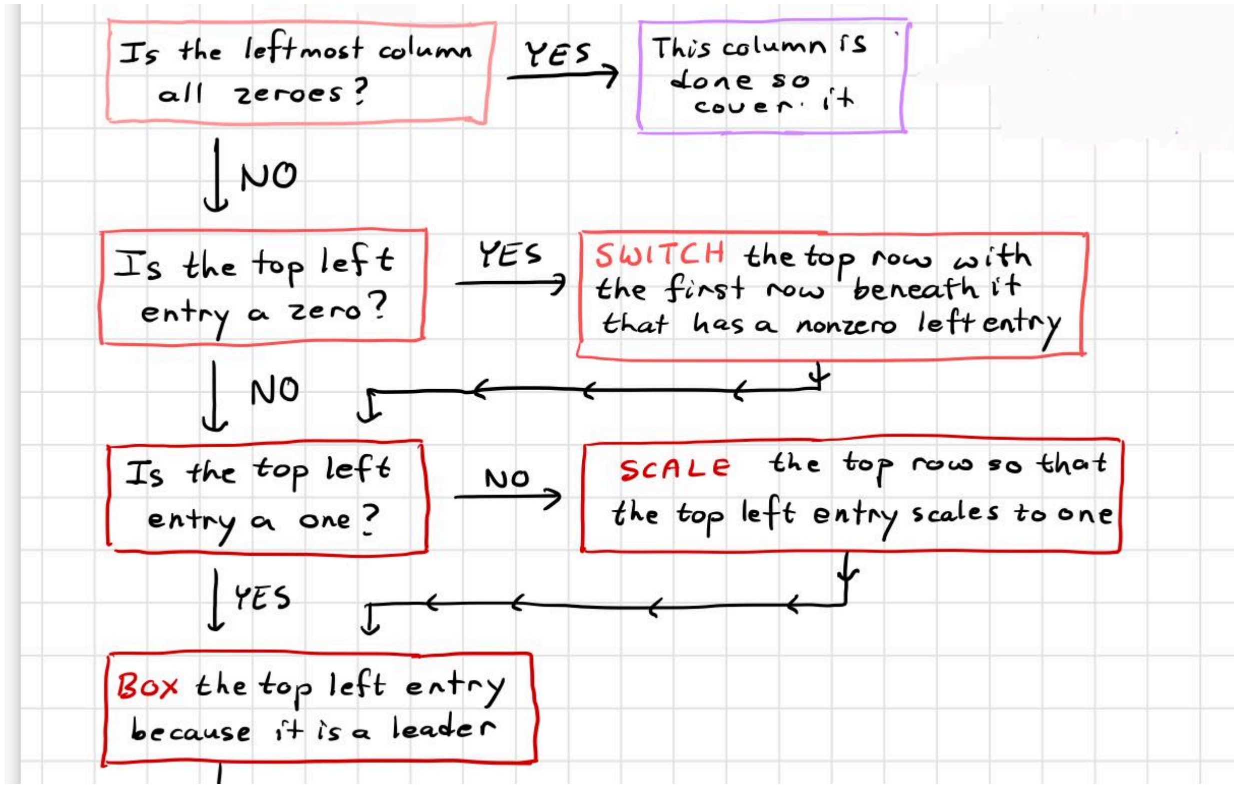
Method: Start on the upper left.
Step 1: Find a leader and make sure it is a one using row actions.



However, what if the top left entry is zero? Then we cannot scale it! Let's eliminate that first by adding this question on top:



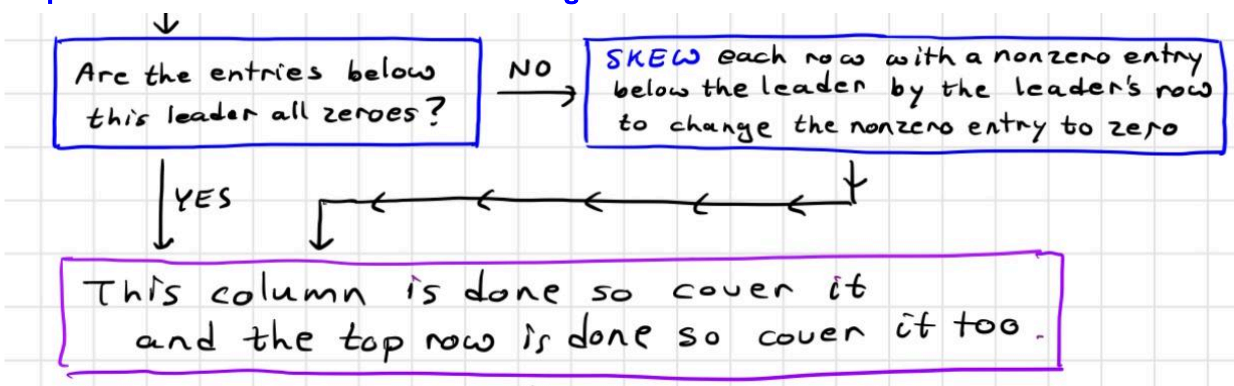
However, what if everything under it is zero too? Then we cannot switch it! Let's eliminate that first by adding this question on top:



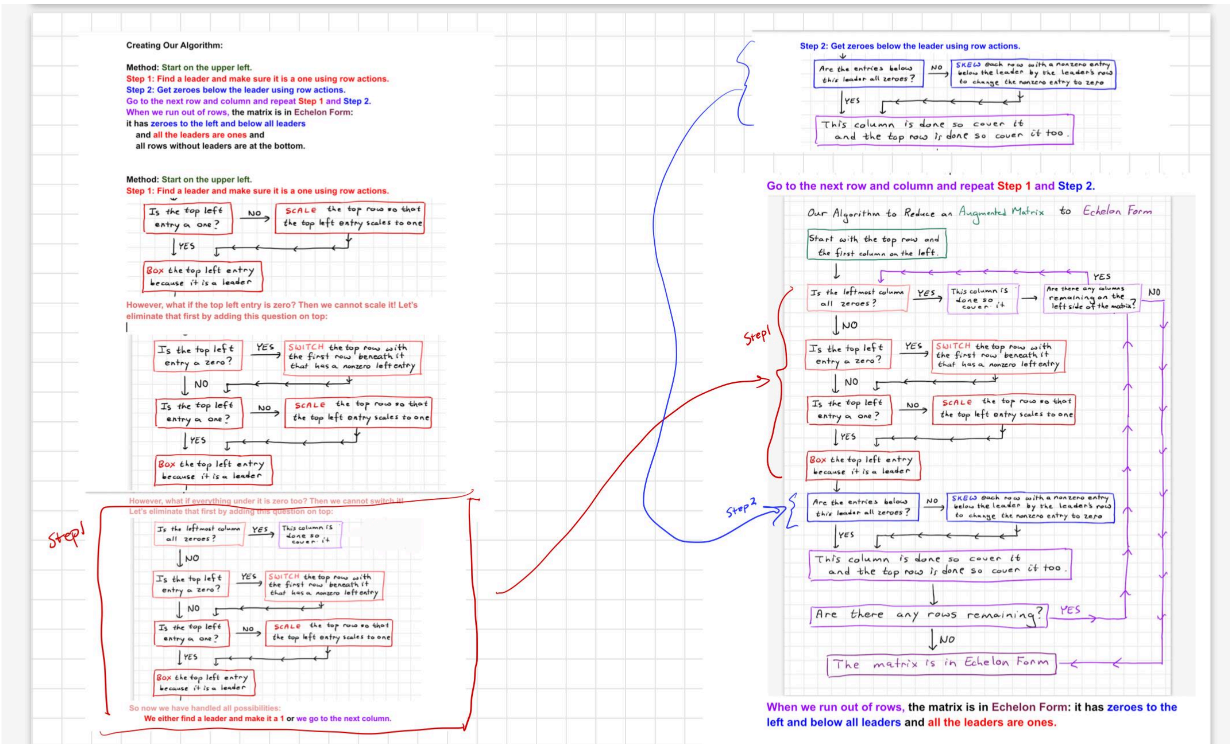
So now we have handled all possibilities:

We either find a leader and make it a 1 or we go to the next column.

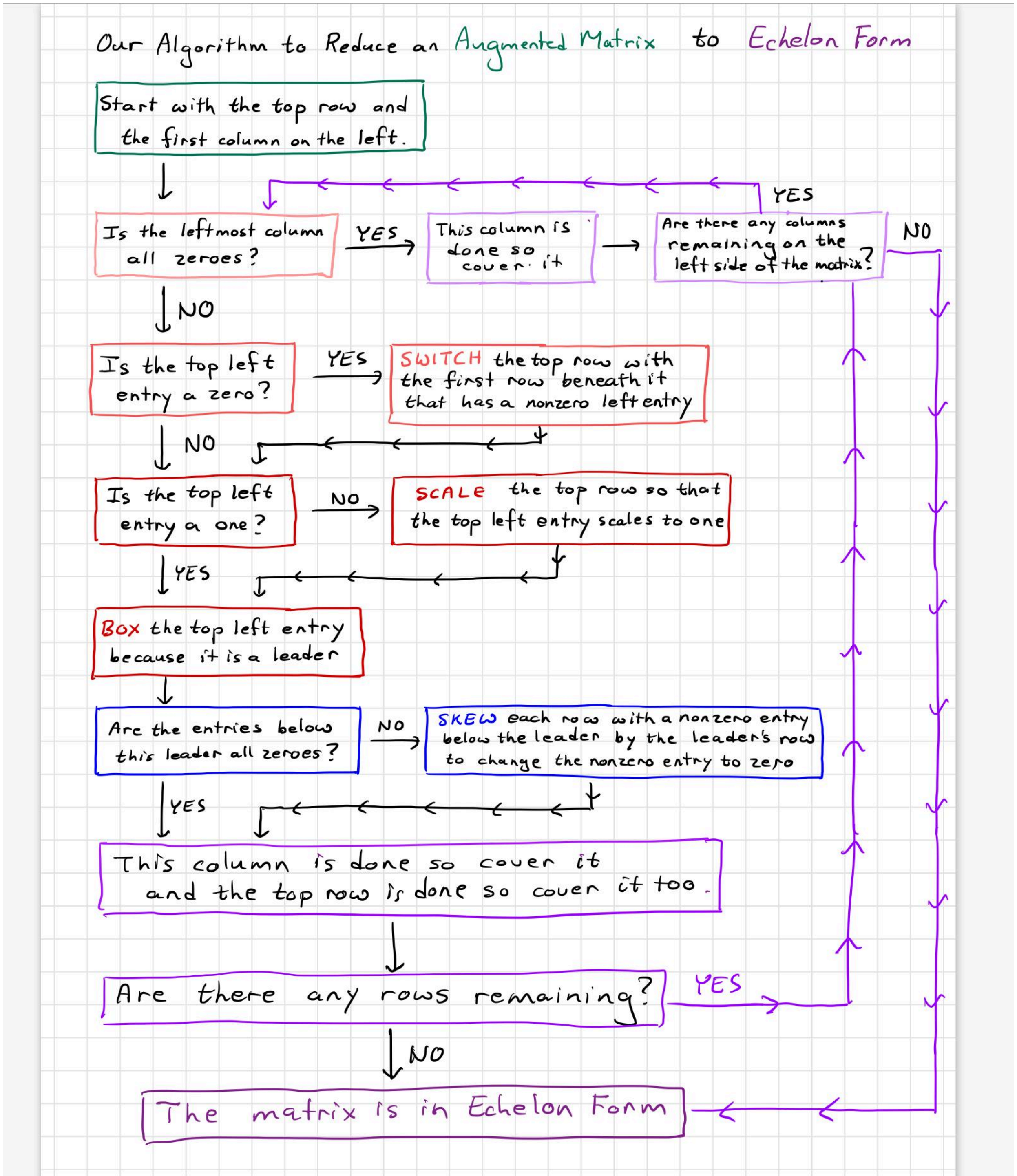
Step 2: Get zeroes below the leader using row actions.



Go to the next row and column and repeat Step 1 and Step 2.



Our Algorithm:



When we run out of rows, the matrix is in Echelon Form: it has zeroes to the left and below all leaders and all the leaders are ones.

Redo Classwork Marking the path

Step 1   Step 2

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{\rho_4 \to \rho_4 - \rho_1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

cover

step 1 done

Step 2 done

covered again

all zeroes no

$$\xrightarrow{\rho_3 \leftrightarrow \rho_4} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

Is top left a 1? Yes

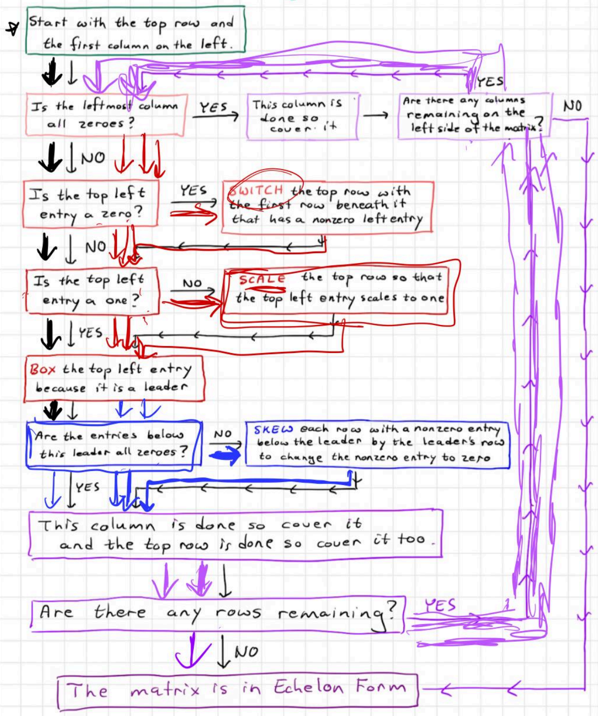
Yes

cover again

$$\xrightarrow{\rho_4 \to \frac{1}{4}\rho_4} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

Now rows remain!

Our Algorithm to Reduce an Augmented Matrix to Echelon Form



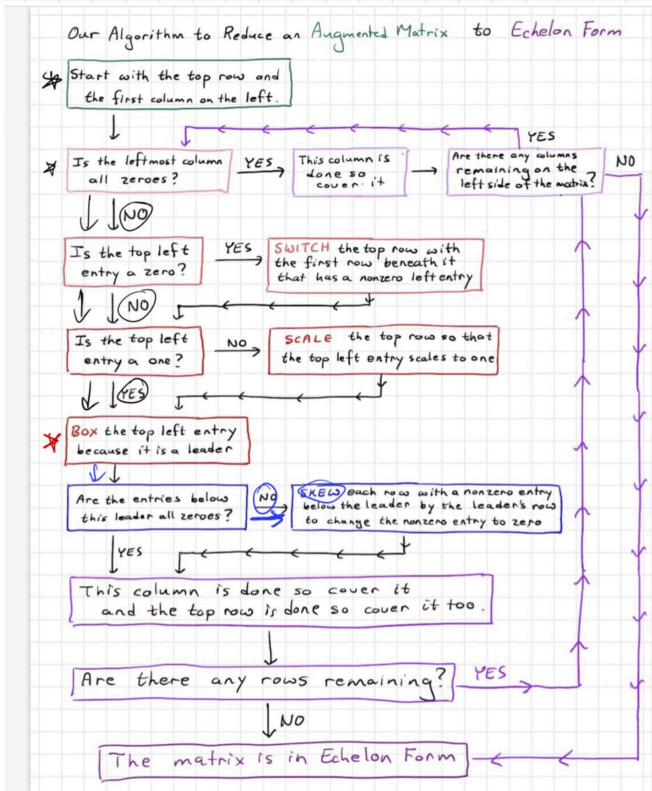
Classwork: Do the next row action for the following matrices

(a) $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right]$ SKEW BY LEADER'S ROW $\xrightarrow{\rho_3 \to \rho_3 - 2\rho_1}$ $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & 3 & 0 & 5 & 0 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & 0 & 6 & 7 & 8 \end{array}\right]$

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 3 & 2 & 0 & 1 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$



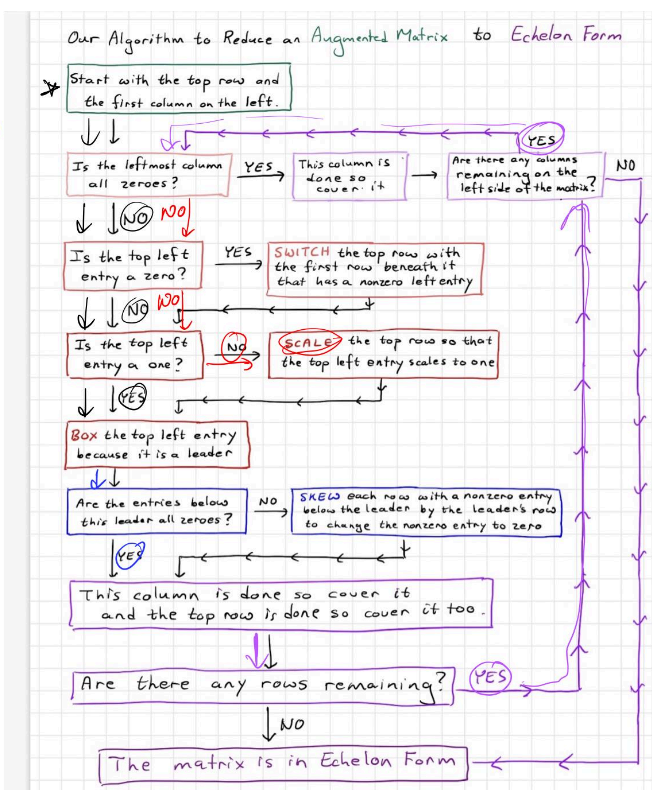
Classwork: Do the next row action for the following matrices

(a) $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right]$ SKEW BY LEADER'S ROW $\xrightarrow{\rho_3 \to \rho_3 - 2\rho_1}$ $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & 3 & 0 & 5 & 0 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & 0 & 6 & 7 & 8 \end{array}\right]$ SCALE $\xrightarrow{\rho_2 \to \frac{1}{4}\rho_2}$ $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & \boxed{1} & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} \\ 0 & 0 & 6 & 7 & 8 \end{array}\right]$

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 3 & 2 & 0 & 1 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$



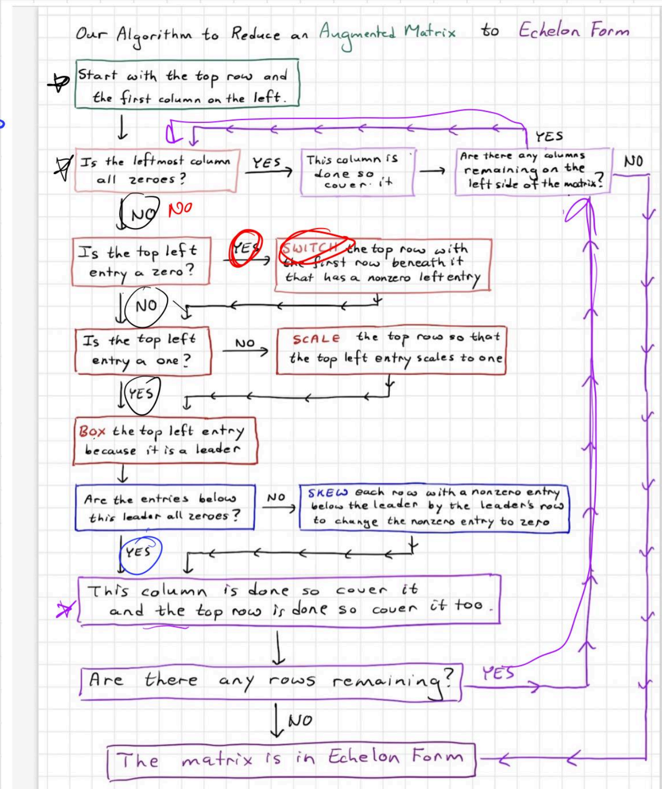
Classwork: Do the next row action for the following matrices

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right] \xrightarrow{\rho_3 \to \rho_3 - 2\rho_1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & 3 & 0 & 5 & 0 \end{array}\right]$ SKEW BY LEADER'S ROW

(b) $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & 0 & 6 & 7 & 8 \end{array}\right] \xrightarrow{\rho_2 \to \frac{1}{4}\rho_2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & \boxed{1} & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} \\ 0 & 0 & 6 & 7 & 8 \end{array}\right]$ SCALE

(c) $\left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\rho_2 \leftrightarrow \rho_3} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 3 & 2 & 0 & 1 \\ 0 & 0 & 5 & 6 & 7 \end{array}\right]$ SWITCH

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$



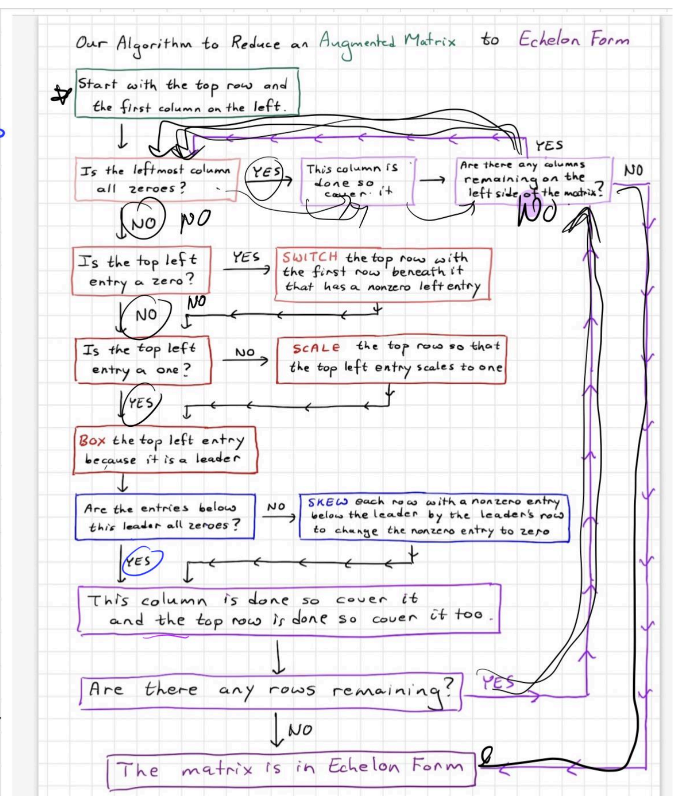
Classwork: Do the next row action for the following matrices

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right] \xrightarrow{\rho_3 \to \rho_3 - 2\rho_1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & 3 & 0 & 5 & 0 \end{array}\right]$ SKEW BY LEADER'S ROW

(b) $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & 0 & 6 & 7 & 8 \end{array}\right] \xrightarrow{\rho_2 \to \frac{1}{4}\rho_2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & \boxed{1} & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} \\ 0 & 0 & 6 & 7 & 8 \end{array}\right]$ SCALE

(c) $\left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\rho_2 \leftrightarrow \rho_3} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 3 & 2 & 0 & 1 \\ 0 & 0 & 5 & 6 & 7 \end{array}\right]$ SWITCH

(d) $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & 0 & 0 \\ 0 & \boxed{1} & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ In Echelon Form Already! No next row action!



The following homework will prepare you for Quiz 1.

Homework: **Check answers to HW1-HW3 at the end of this doc.**

Homework:

Do the next row action for the following matrices:

and count how many times you go back up to the top.

Check answers to HW1 – HW3 before continuing.

HW1 $\left[\begin{array}{cccccc|c} 0 & 0 & 2 & 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 4 & 8 & 0 & 0 \end{array}\right]$

Check answer below before continuing.

HW2 $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

check below

HW3 $\left[\begin{array}{cccccccc|c} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$

Check

HW4 $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$

HW5 $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$

HW6 $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 & 4 \end{array}\right]$

HW1-HW6 above only require one row action each to be completed and students will explain how many times they went to the top of the algorithm for that one action.

HW7-HW9 below require the complete solution including position/direction form when there is one free variable.

In HW7- HW9 below, solve the system by

1. converting into an Augmented Matrix,
2. Reducing to Echelon Form following our algorithm,
3. converting back into a system
4. checking if there is no solution due to 0 = nonzero.
5. solving for leaders and writing free variables equal to themselves
6. subbing up
7. writing the solution set (in position/direction form if 1 free variable)

HW7

$$\begin{aligned} x+y&=2\\ 2x+2y&=4\\ 5x+5y&=10 \end{aligned}$$

HW8

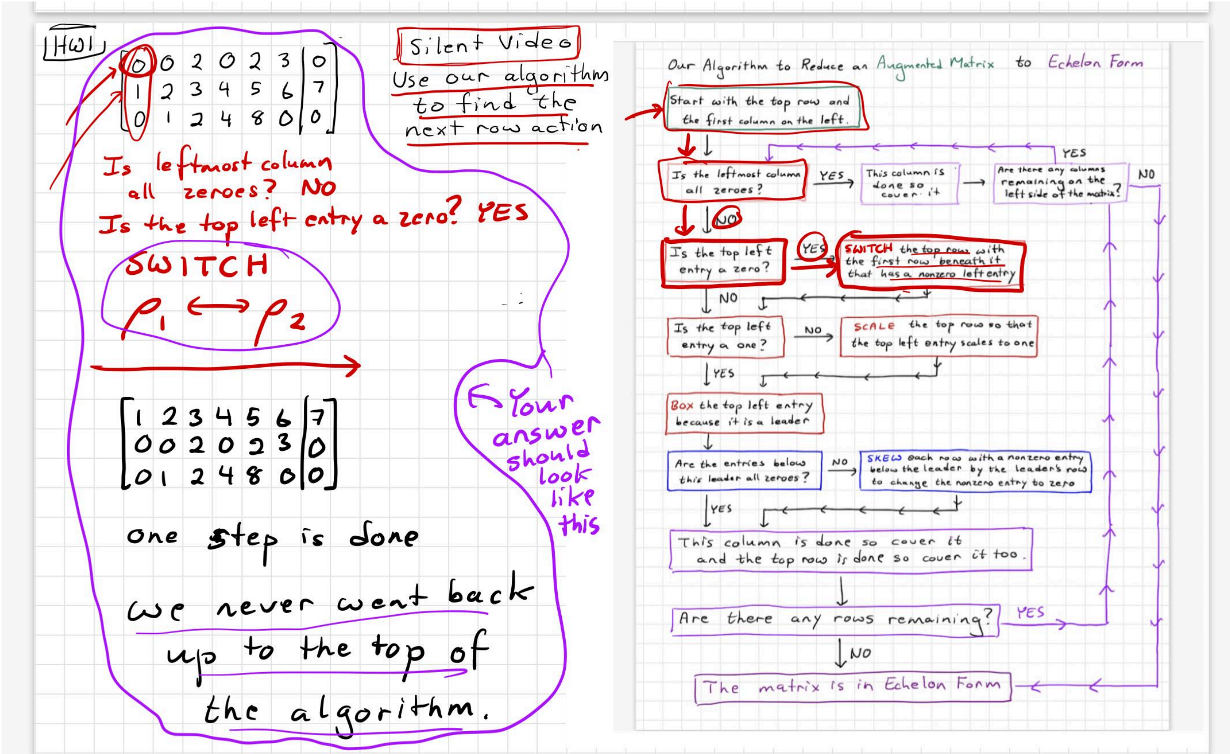
$$\begin{aligned} x+2y+z&=0\\ 0x+y+z&=2\\ x+4y+2z&=3 \end{aligned}$$

HW9

$$\begin{aligned} x+y+z&=4\\ 2x+0y+0z&=4 \end{aligned}$$

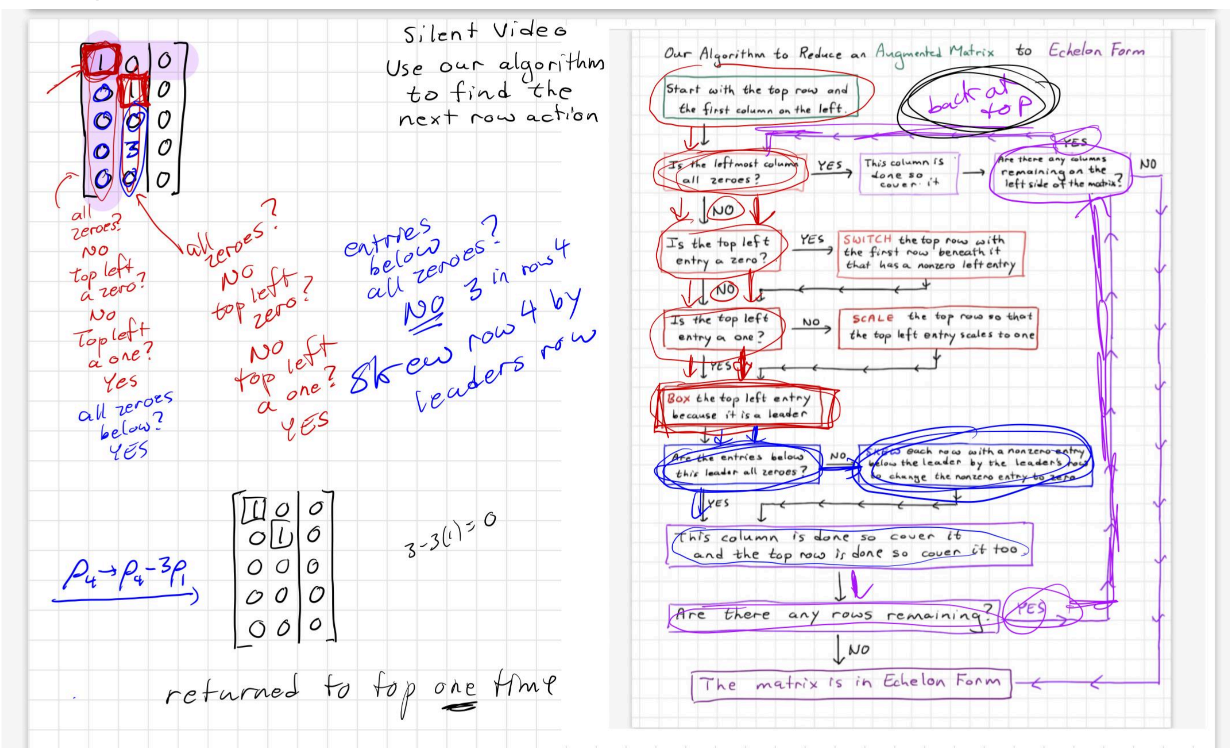
Choose one of the four times on the course webpage for your quiz 1.

Check your answer to HW1 here:



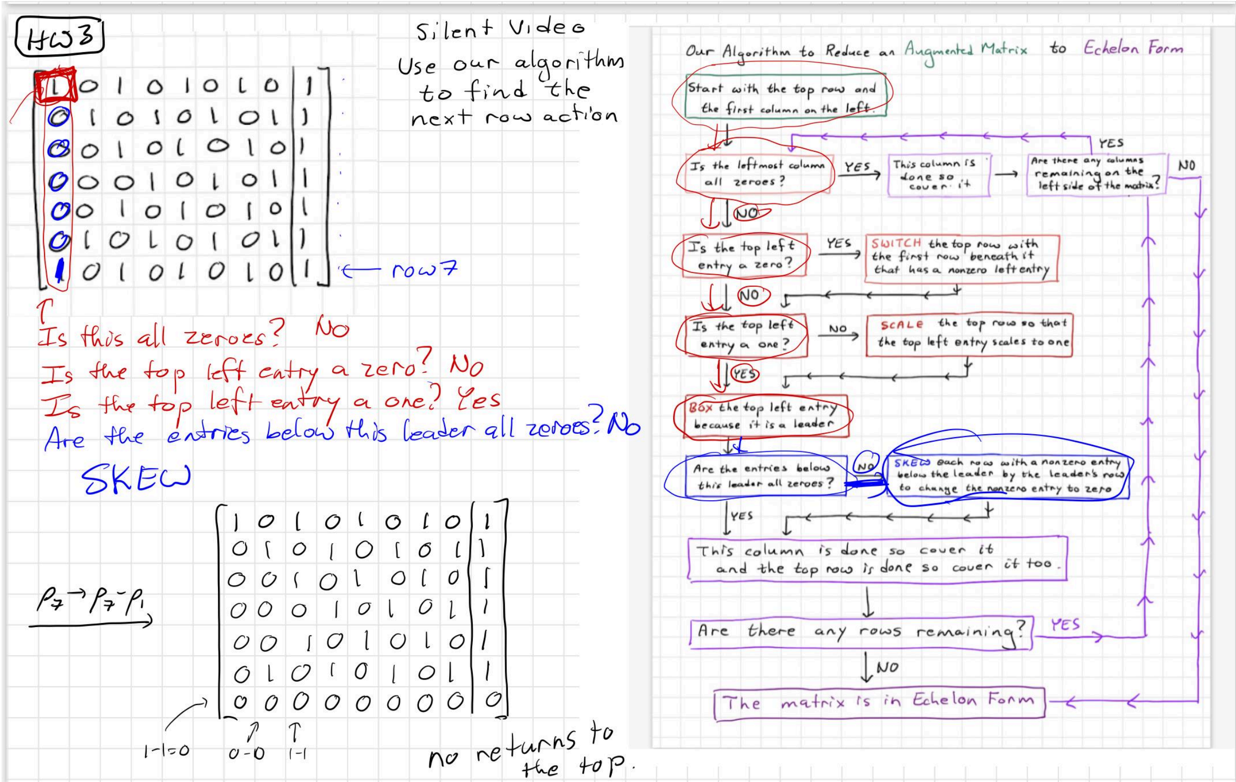
If your answer to HW1 is wrong, watch this [Video of HW1](#).

Check your answer to HW2 here:



If your answer to HW2 is wrong, watch the [Silent Video of HW2](#).

Check your answer to HW3 here:



If your answer to HW3 is wrong, watch the Silent Video of HW3.

STUDENT QUESTIONS:

QUESTION: If I follow the flow chart, and I reach the box that says SCALE, how do I know which number to scale by?

SCALE explanations: (to change a nonzero leader to 1)

Suppose your leader is a 5 and you are changing it to 1,
then you scale the leader's row by (1/5)
because (1/5)5=1 will appear in the leader's place.
Note any zeroes in that row will go to (⅕)0=0
and all other numbers in the row will be multiplied by ⅕ as well.

If the leader is -2 and you are changing it to 1,
then you scale the leader's row by (-1/2)
because (-1/2)(-2)=1 will appear in the leader's place.
Note any zeroes in that row will go to (-1/2)0=0
and all other numbers in the row will be multiplied by (-½) as well.

If the leader is 1/8 and you are changing it to 1,
then you scale the leader's row by 8
because 8(1/8)=1 will appear in the leader's place.
Note any zeroes in that row will go to (8)0=0
and all other numbers in the row will be multiplied by (8) as well.

In general if the leader is R and you are changing it to 1,
then you scale the leader's row by (1/R) as long as R is not 0
because (1/R)R=1 will appear in the leader's place.
Note any zeroes in that row will go to (1/R)0=0
and all other numbers in the row will be multiplied by (1/R) as well.

QUESTION: If I follow the flow chart, and I reach the box that says SKEW, how do I know which skew to choose?

SKEW explanations: (to change a nonzero entry below a leader to 0)

Suppose you have an entry 5 in row4 below a leader in row1,
then you skew by the leaders' row,
row4 to row4-5row1,
because 5-5(1)=0 will appear under the leader.

Suppose you have an entry -8 in row5 below a leader in row2,
then you skew by the leaders' row,
row5 to row5+8row2,
because -8+8(1)=0 will appear under the leader.

Suppose you have an entry 3/7 in row4 below a leader in row2,
then you skew by the leaders' row,
row4 to row4-(3/7)row2,
because 3/7-(3/7)1=0 will appear under the leader.

In general if you have an entry k in rowi below a leader in rowj,
then you skew by the leaders' row,
rowi to rowi -k rowj,
because k-k(1) =0 will appear under the leader.

QUESTION: Where did we learn SWITCH, SCALE, and SKEW?

We first learned SWITCH, SCALE, and SKEW in Lesson 2 Part II:

Part II: How to Solve Any Linear System

Watch [Playlist 313F21-2-8to12]

"How to solve a Linear System" is in video 313F21-2-8 and the Example 1 rewritten using this technique is in video 313F21-2-8 and 313F21-2-9:

How to Solve a Linear System:

Step 1: Row Reductions to Echelon Form

Row Actions which are reversable

- scale • $\rho_i \to k\rho_i$ where $k \neq 0$
- skew • $\rho_i \to \rho_i + k\rho_j$
- switch • $\rho_i \longleftrightarrow \rho_j$

later we will set up an algorithm for this step.

We've continued to practice using the exact same actions in our classwork today and will continue to use them all semester.