The Mandelbrot Set in Pre Calculus

Outline: Total lesson is about 3 hours

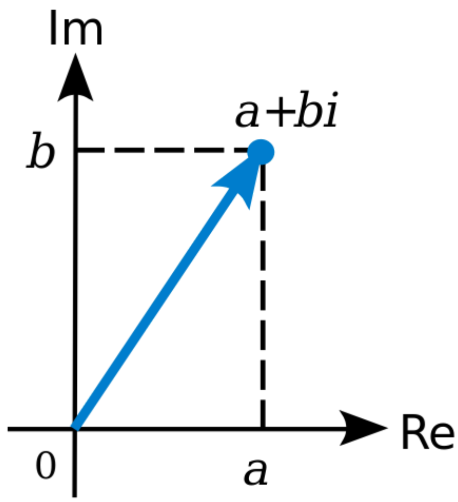
Helpful links:

- Mandelbrot Interactives from Dan Anderson: `http://recursiveprocess.com/mandelbrot/`
- The Mandelbrot Set from Numberphile: `http://youtu.be/NGMRB4O922I`
- Mandelbrot Set: how it is generated: `https://www.youtube.com/watch?v=8ma6cV6fw24`
- Music video for Jonathan Coulton's song Mandelbrot Set `https://www.youtube.com/watch?v=ES-yKOYaXq0`
- Experimental Variants of Mandelbrot Fractals: `http://mathematica.tumblr.com/post/99748587388/nanozen-mandelbrot-fractals`
- How to Fold a Julia Fractal `http://acko.net/blog/how-to-fold-a-julia-fractal/`

Introduction to the Complex Plane

- What are complex numbers?
- How does the complex number plane work?
- Size of complex numbers? What does that mean?
- Arithmetic with complex numbers?
  - Addition, subtraction, multiplication, division?

Reminders:



$$|z| = |a+bi| = \sqrt{a^2+b^2}$$

Diagram from Wikipedia.

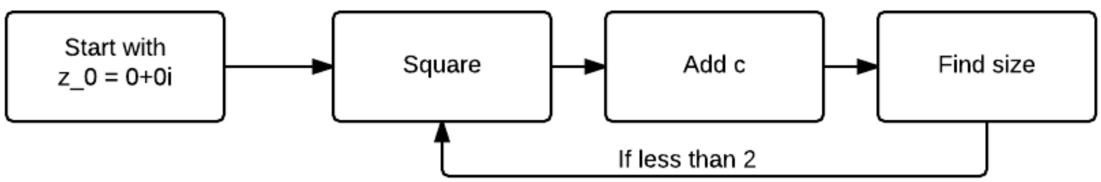
Introduction to the Mandelbrot Set

- The Mandelbrot Set exists on the complex plane. It is defined as the set of all points where:

  $$z_0 = 0 + 0i$$

  $$z_{n+1} = z_n + c$$

  or:

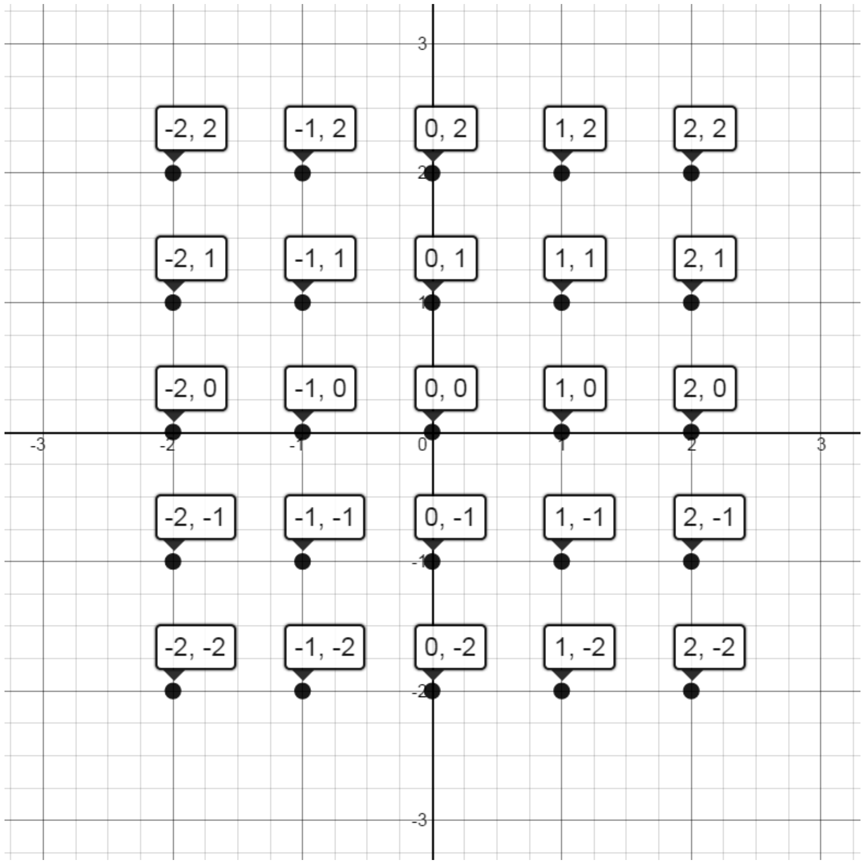
  is bounded. In other words, where the iteration doesn't blow up in size. We're going to call anything that grows past a magnitude of **2** blowing up.

- Since the Mandelbrot set is checking every single coordinate on the complex plane, we need to start with an approximation.
  - Do we need to check the coordinate, $5 + 12i$? Why or why not?

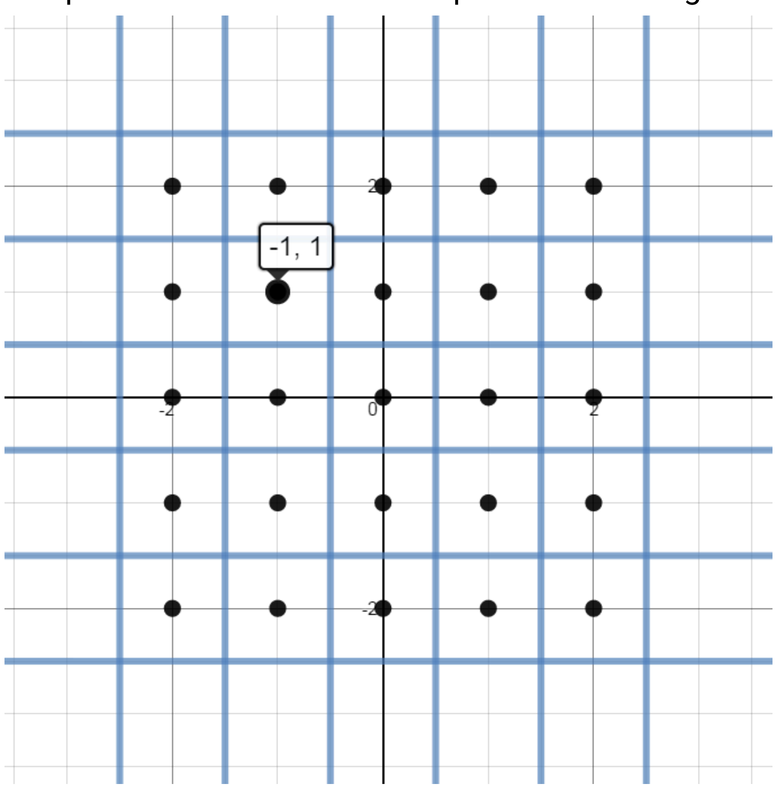
    $$z_0 = 0 + 0i,\ size\ of\ 0$$

    $$z_1 = (0 + 0i)^2 + (5 + 12i) = 5 + 12i,\ size\ of\ \sqrt{5^2 + 12^2} =$$

- Let's consider the following points and we'll check them by hand to see if they are in the Mandelbrot set.



Interpret the coordinates as a square surrounding each coordinate.



- Clicker ([link found here]())
  - Assign each student a number, if less than 25 students, then feel free to assign students at the corners of the plane with more than one box). If more than 25 students, try using the birthday labels, where the students keep track of their birthday box, students born between the 25th and 31st of the month get to choose their favorite number. The position determines the constant, ***c***, that the student is assigned to.

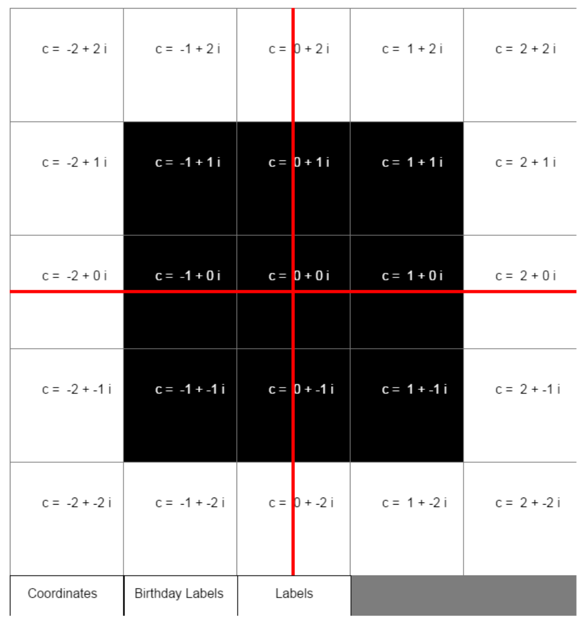
  - With a real number (try 1+0i and reassign these students), run the iteration through as a class:

    $z_0 = 0 + 0i$, by definition

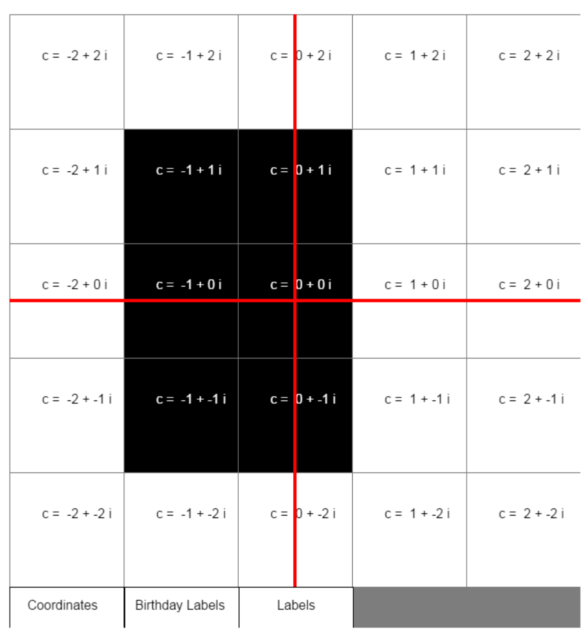
    $z_1 = z_0^2 + c = (0 + 0i)^2 + (1 + 0i) = 1 + 0i$

    Find size of $z_1 = \sqrt{1^2 + 0^2} = 1$. Since this is < 2, continue to z_2.

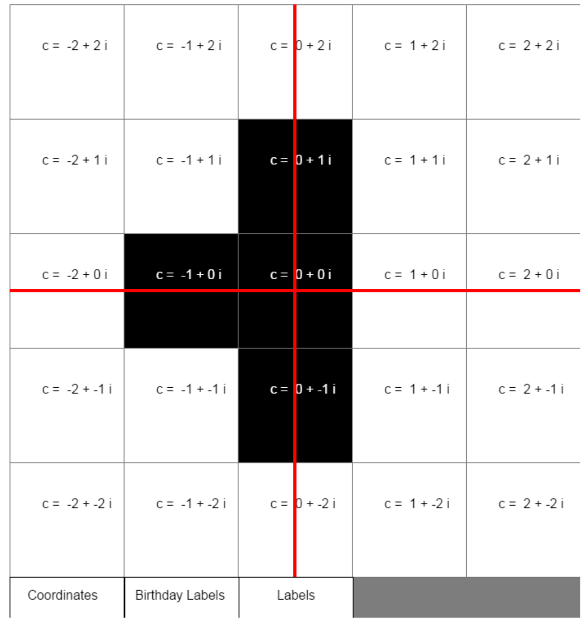
  - Have everyone in class run one step of the process. As they finish, record their answers by clicking on the appropriate coordinate. If they have a z_1 that is less than 2, then color their square black, otherwise white. The clicker should look like this after step 1:



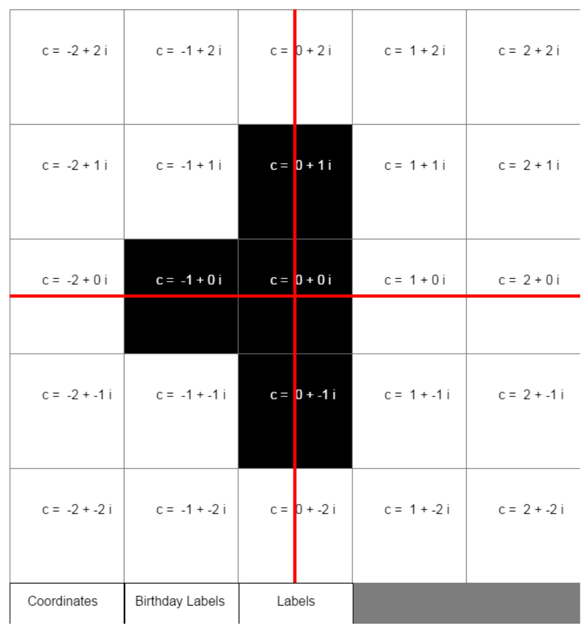
  - Have all the remaining black squares run a second step. Anyone whose square has blown up is reassigned to error checking. As they finish, record the results of this step again. The clicker should look like this after step 2:



- After step 3 it should look like this:



- After step 4 it should look like this:



- Uh oh, it's not progressing. How can we improve our picture?
  MORE PEOPLE! (what does this mean? breaking up the boxes into more pieces)
  MORE STEPS!
  MAKE IT AUTOMATIC!

- How do you make it automatic? Have the class square (a+bi) and tell you the real parts and the imaginary parts. Spoiler alert, $(a + bi)^2 = (a^2 - b^2) + (2ab)i$
  If we can give the computer the instruction on how to square a complex number, then the computer can do this boring work error free and quickly.

- Mandelblocks (link found here)
    - Show the code for Mandelblocks (specifically the calculate_mandelbrot part where it is using the fact from above that $(a + bi)^2 = (a^2 - b^2) + (2ab)i$
    - Click on the **resolution +1** button to replicate what the class did by hand.
    - Ok, now increase the people, i.e. increase the **grid resolution**.
    - Reset all and click once on the coloring mode. Start increasing the **resolution** and **grid resolution**. The color of the block will now depend on how quickly it took to blow up. If it blows up after 1 iteration then it's red.

    - *I want a better picture! Let's treat each pixel as a block.*

- Mandelbrot (link found here)
    - Click through the **resolution +1**. Ask the class to think about the first step. Does it make sense that it's a circle? Why?

    - *What's next? Zoom!*

- Mandelbrot Zoom (link found here)
    - Click on the fractal to zoom. Right click to zoom out.

    - Note the width of the window and how far you can zoom in until it "breaks". This is because the data structures can't store these small numbers accurately enough.

    - What about powers of z_n other than two??

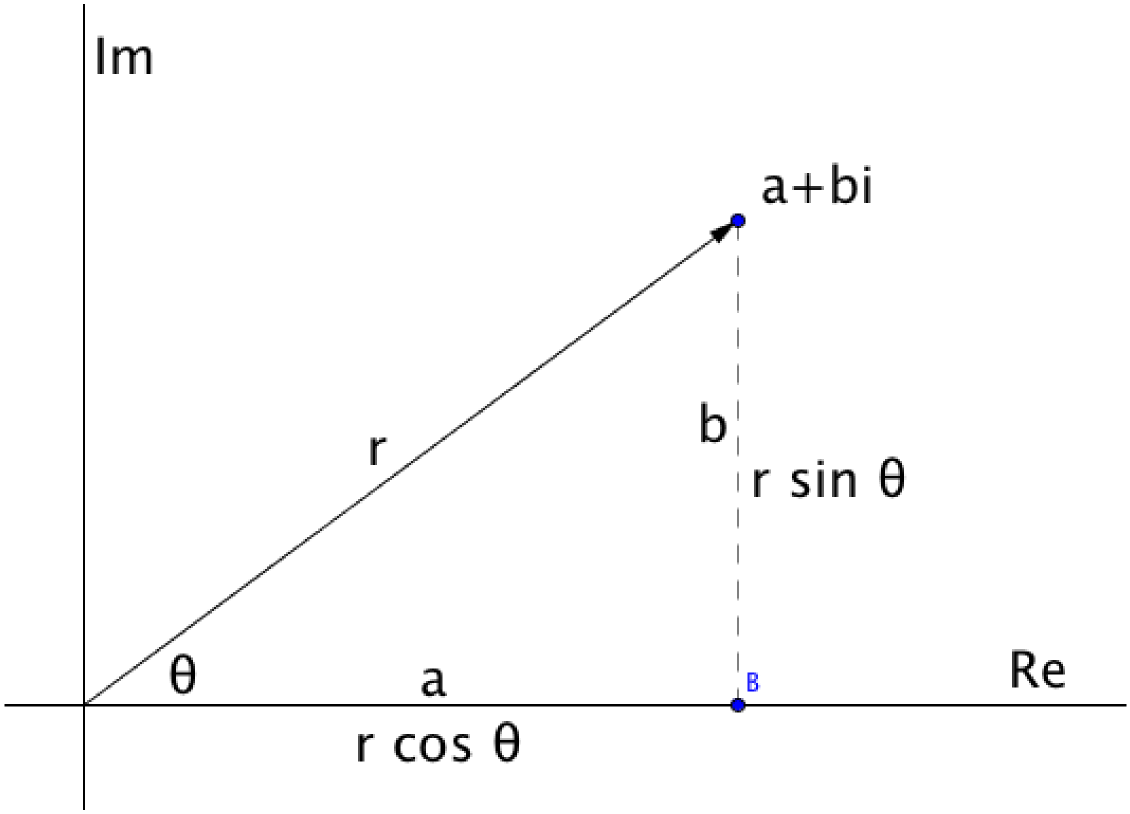
    - Ok, let's expand $(a + bi)^3$ so that the program can find the third degree Mandelbrot

    - What does the 4th degree Mandelbrot look like?

    - 10th degree?

    - 1.5th degree?

- Mandelbrot Family (link found here)
  - How can we make this process general? How can we tell the computer to graph $(a + bi)^n$ when we don't know what *n* is yet?
  - Polar form



$$r = \sqrt{a^2+b^2}$$

$$\theta = tan^{-1}\left(\frac{b}{a}\right)^*$$

$$so \ ...$$

$$a+bi = r(cos\theta) + r(sin\theta)i = r(cis\theta)$$

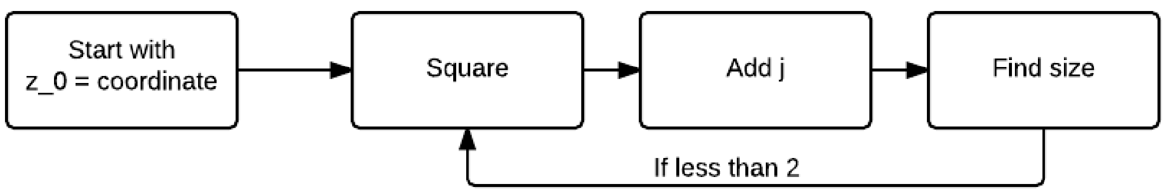
- Where does polar form get us?
  deMoivre's Theorem!

$$(a+bi)^n=(r\,cis\theta)^n=r^n cis(n\theta)$$

- The math is so much easier, operations with real numbers instead of binomial expansion. And n doesn't have to be an integer!
  So:
  - Convert from rectangular to polar
  - Use deMoivre's Theorem
  - Convert back to rectangular.

- Julia Set ([link found here]())

  Let's consider the following rule (Julia Set):
  - Come up with a constant *j*



- Mandelbrot Experimental ([link found here]())

  Let's consider the following rule