Summary Calculus Workshop 2019 🌤️

University of Arizona

Calculus 3 Workshop: Scalar Projection and Vector Line Integrals

Worksheet #5 📝

by Ken Yamamoto

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Introduction

The **line integral of a vector field** plays a crucial role in vector calculus. Out of the four **fundamental theorems** of vector calculus, three of them involve line integrals of vector fields. **Green's theorem** and **Stokes's theorem** relate line integrals around closed curves to double integrals or surface integrals. If you have a **conservative vector field**, you can relate the line integral over a curve to quantities just at the curve's two boundary points. So, it's worth the effort to develop a good understanding of line integrals! 😉

We won't have time to cover all of these fundamental theorems, but we will at least familiarize ourselves with **line integrals**. This way, you'll have a head start in understanding these more advanced theorems in vector calculus in your semester course. 👍

Yesterday we covered line integrals over **scalar fields**. Today, we will finally learn about line integrals over **vector fields**. 🤓

From Yesterday

In this Worksheet, we will continue to explore the same surfaces below we worked with in **Worksheet #2**, **Worksheet #3**, and **Worksheet #4**. 🎨

A. $f(x,y) = y$

B. $f(x,y) = x$

C. $f(x,y) = x + y$

D. $f(x,y) = 2x + y$

E. $f(x,y) = x + 2y$

F. $f(x,y) = x^2 + y^2$

G. $f(x,y) = 4 - x^2 - y^2$

H. $f(x,y) = x^2 - y^2$

Parametrizing a Line/Curve

Recall how we determined equations in **vector form** for lines and curves in two and three dimensions in **Worksheet #1**. Refresh your memory and practice by determining the **vector function** $\vec{r}(t) = \langle x(t), y(t) \rangle$ for the two-dimensional lines and curves C from the previous problems. Determining the vector function for a line or curve is also known as **parametrizing the line or curve** or finding a **parametrization for the line or curve**. 🤓

1. Write a **vector function** of (i.e., **parametrize**) the two-dimensional straight line C that connects… ✏️
   a. $(0,0)$ and $(10,0)$.
   b. $(0,0)$ and $(0,10)$.
   c. $(0,0)$ and $(10,10)$.

2. **Parametrize** the two-dimensional circle C centered at the origin with radius equal to 10. ✏️

Tangent Vectors to Space Curves

3. Write the **vector function** for the **line tangent** to the parametrization of the circle C centered at the origin with radius equal to 10 that you obtained in Problem #2 at three different t values. ✏️

4. How does obtaining the vector function of the tangent line to this two-dimensional curve compare to obtaining the tangent line of a function in single-variable calculus? Write at least a few sentences. Use clear and precise language! ✍️

Scalar Projection

In **Worksheet #3**, we learned about the **dot product**. Specifically, we learned the **algebraic definition** of the dot product.

📖 **Algebraic Definition of Dot Product**

If $\vec{a}, \vec{b} \in \mathbb{R}^d$, then the **dot product** of $\vec{a}$ and $\vec{b}$ is the real number $\vec{a} \cdot \vec{b}$ given by

- $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$ where $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$ for $d = 2$.
- $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ for $d = 3$.

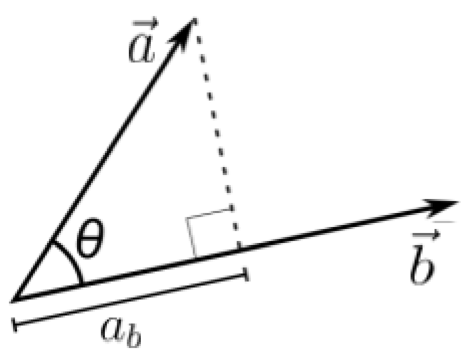
It can be shown that this is equivalent to the **geometric definition** of the dot product.

📖 **Geometric Definition of Dot Product**

If $\vec{a}, \vec{b} \in \mathbb{R}^d$, then the **dot product** of $\vec{a}$ and $\vec{b}$ is the real number $\vec{a} \cdot \vec{b}$ given by

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \, ||\vec{b}|| \cos\theta$$

where $||\cdot||$ denotes **magnitude** of the vector and θ is the **angle** between vectors $\vec{a}$ and $\vec{b}$.



5. Using trigonometry, show that the (scalar) **component** a_b of a vector $\vec{a}$ in the direction $\vec{b}$ is given by

$$a_b = ||\vec{a}|| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. ✏️

6. Using the **geometric definition** of the dot product, show that

$$a_b = \vec{a} \cdot \hat{b}$$

where

$$\hat{b} = \frac{\vec{b}}{||\vec{b}||}$$

is the unit vector in the direction of $\vec{b}$. ✏️

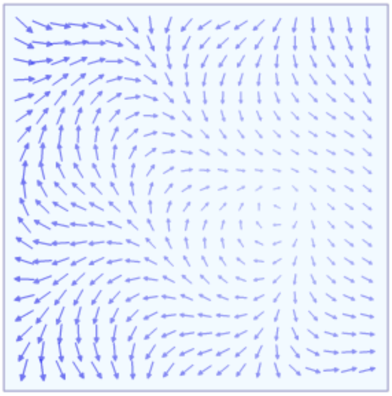
Line Integral over a Vector Field[1]

A **vector field** $\mathbf{F}(x, y)$ associates a vector to every point in a domain (rather than a **scalar** as in a **scalar field**). For example, the **gradient** of the surfaces A-H defines a **two-dimensional vector field** (more precisely, a ***gradient*** **vector field**) that associates a **gradient vector**

$$\mathbf{F}(x, y) = \nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

for every point $(x, y) \in \mathbb{R}^2$. 🤓

Vector fields are often used to model, for example, the **speed and direction of a moving fluid** throughout space, or the **strength and direction of some force**, such as the **magnetic or gravitational force**, from one point to another point.



📐 Geometric Intuition of Vector Line Integral

The line integral over a two-dimensional vector field $\mathbf{F}(x, y)$ along a curve C is the integral of the **component of the vector field that is tangent to the curve** with respect to the arclength element ds.

Intuitively, a two-dimensional **line integral over a vector field** captures how much a two-dimensional curve C aligns with the vector field $\mathbf{F}(x, y)$. (See the animation.)

📖 Line Integral over a Vector Field

The line integral over a two-dimensional vector field $\mathbf{F}(x, y)$ along a curve C that starts at $t = a$ and ends at $t = b$ is

$$\int_C \mathbf{F}(x, y) \cdot \mathbf{T}(x, y)\, ds = \int_a^b \mathbf{F}(x(t), y(t)) \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$

where the **unit tangent vector** $\mathbf{T}(x, y)$ to the curve C at the point (x, y) is

[1] The animation in this section for the line integral over a scalar field is taken from the [Wikipedia article on line integrals](https://en.wikipedia.org/wiki/Line_integral).

$$\mathbf{T}(x,y) = \frac{\langle x'(t), y'(t)\rangle}{\sqrt{(x'(t))^2 + (y'(t))^2}} = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$$

and $\mathbf{r}(t) = \vec{r}(t) = \langle x(t), y(t)\rangle$ is the parametrization for the curve C.

(Note that vectors are usually written in bold r or $\vec{r}$. In fact, the **vector field** F could have just as well been written as $\vec{F}$.)

Typically, we write

$$\int_a^b \mathbf{F}(x(t), y(t)) \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \, d\mathbf{r}$$

where we use the shorthands

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t)) = \mathbf{F}$$

and

$$d\mathbf{r} = \langle dx, dy\rangle.$$

<u>**Don't Take the Formula for Granted**</u> 🤔

7. Show that the left- and right-hand sides of the formula for the line integral over a vector field are equal... ✏️

$$\int_C \mathbf{F}(x,y) \cdot \mathbf{T}(x,y)\, ds = \int_a^b \mathbf{F}(x(t), y(t)) \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \, dt$$

   by using the fact that

$$\mathbf{T}(x,y) = \frac{\langle x'(t), y'(t)\rangle}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

   and

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

8. Show how

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \, d\mathbf{r}$$

in the above formula for the line integral over a vector field. ✏️

9. Make sense of the above definition for the line integral over a vector field. How does this **line integral over a two-dimensional vector field** capture how much a two-dimensional curve C aligns with the vector field $\mathbf{F}(x, y)$? 🤔

Practice ✏️

10. Compute the **line integral** over the (gradient) **vector field** described by the **gradient** of the surfaces C, F, and H along the two-dimensional straight line C that connects… ✏️

    a. $(0,0)$ and $(10,0)$.
    b. $(0,0)$ and $(0,10)$.
    c. $(0,0)$ and $(10,10)$.

11. How does the value of the line integral over the vector fields compare? How does this make sense with the fact that the **line integral over a two-dimensional vector field** captures how much the two-dimensional curve C aligns with the vector field $\mathbf{F}(x, y)$? 🤔

12. Redo Problem #10 but with C in the opposite **orientation**, e.g., going from $(10,0)$ to $(0,0)$ instead. What do you notice? ✏️

13. Compute the line integral over the (gradient) vector field described by the gradient of the surface F over the **two-dimensional circle** C centered at the origin with radius equal to 10. ✏️

14. Come up with curves C that will result in a **zero line integral** over the gradient fields corresponding to surfaces C, F, and H. Show that you indeed obtain a zero line integral. What do you notice about these curves C? ✏️

Be Creative 🎨

15. Create your own surface $z = f(x, y)$ and compute its **gradient field**. Also create your own curve C. Compute the **line integral over this vector field** along the curve you chose. ✏️

Fundamental Theorem of Calculus for Line Integrals 📚

16. Compute $f(x, y)$ at the **start-point** (i.e., $t = a$) and **end-point** (i.e., $t = b$) for surfaces C, F, and H for the paths C described in Problem #10. Then compute… ✏️

$$f(x, y)|_{t=b} - f(x, y)|_{t=a}$$

What do you notice? 🤔