Lesson 27

Part I Vector Subspace
Part II Null Space
Part III Span
Part IV Basis
Part V Dimension
Part VI Hilbert Space: Infinite Dimensional Space
Part VII Fourier Series: Analog to Digital

You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:
MAT313S24-lesson27-last-first
and share editing of that document with me sormanic@gmail.com.   You will also include your homework and any corrections to your homework in this doc.

Parts I-V are required as they review theorems and definitions we learned before for the final and analyze how they work on vector spaces in general.   We have a different playlist for each of the parts.   Parts VI and VII are important for math majors, physics majors, and engineers as they concern Hilbert Space and Fourier Series and the conversion of analog to digital sound.   Each Part has its own playlist.

Please do the homework like classwork: Immediately when you get to it. It is much easier this way.

Lesson 27

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V}$
- Associativity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{u} + \vec{w})$
- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$
- Compatibility Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
- Distribution over Scalar Addition Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$

Defn: A Linear Map $F: V \to W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+u) = F(v) + F(u)$
- Preserves Scalar Mult: $\forall v \in V \quad \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

Lesson 27 (not 29)

Review of Concepts we learned about vectors in $\mathbb{R}^n$ and see now that they are interesting to apply to vector spaces. (function spaces)

← Analog ⟷ Digital Sound

Watch [Playlist 313F20-27-Part1](#)

Part I
Vector Subspaces

<u>Defn</u> V is a vector subspace of a vector space W if $V \subset W$ (it a subset of W) and has the same addition and scalar multiplication

V is a vector space itself.

<u>Thm</u>: We need only check
- Closed under Vector Addition

  and
- Closed under Scalar Mult.

<u>Proof</u> We must check $\boxed{\text{Add Identity} \in V}$

We must check $\boxed{\text{inverses in } V}$

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following **ten** properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V}$ ✳

- Associativity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{u} + \vec{w})$ ✓

- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$ ✓

- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$ ⟵

- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$ ⟵

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$ ✳

- Compatibility Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$ ✓

- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$ ✓

- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$ ✓

- Distribution over Scalar Addition Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$ ✓

Defn: A Linear Map $F: V \rightarrow W$
- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+w) = F(v) + F(u)$
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Part I

Vector Subspaces

Defn V is a vector subspace of a vector space W if $V \subset W$ (it a subset of W) and has the same addition and scalar multiplication V is a vector space itself.

Vector Subspace

Thm: We need only check

- Closed under Vector Addition
  and
- Closed under Scalar Mult.

Proof We must check / Add Identity $\in V$
We must check / inverses in V

Some Useful Lemmas:

are true on any vector space

Lemma: $\vec{0}$ is unique

There is only one vector $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$ $\forall \vec{v} \in V$.

Easy to see in $\mathbb{R}^2$ $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
in function spaces $0(x) = 0$ constant
Skip the proof in general.

Lemma: $0 \in \mathbb{R}$ and any $\vec{v} \in V$, $0\vec{v} = \vec{0}$.

Easy to see in $\mathbb{R}^2$ $0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 v_1 \\ 0 v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$
in function spaces $(0f)(x) = 0 \cdot f(x) = 0$ const
Skip the proof in general.

Lemma: Inverses are Unique

Easy to see in $\mathbb{R}^2$ inv of $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ was $\begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$
in function spaces took $-f(x)$.

Lemma: $-1 \in \mathbb{R}$ and $\vec{v} \in V$ then $-1\vec{v} = -\vec{v}$ the inverse of $\vec{v}$.
Easy to see in $\mathbb{R}^2$ and function spaces.

V is a vector space itself.

Vector Subspace

Thm: If V nonempty subset of W which is
- ● Closed under Vector Addition
  and
- ● Closed under Scalar Mult of W.

then V is a vector subspace of W.

Proof We must check Add Identity $\in V$

We must check inverses in V

Show Add Identity is in V

① Take $0 \in \mathbb{R}$ and any $\vec{v} \in V$ ① Lemma
$$0\vec{v} = \vec{0}$$

② $\vec{0} = 0\vec{v} \in V$ ② Given Closed under Scalar Mult

& Thus $\vec{0} \in V$

Show inverses are in V

[Hw] Use our given properties and one of our lemmas to prove inverses are in V (2 lines)

QED

Some Useful Lemmas:
are true on any vector space

Lemma: $\vec{0}$ is unique

There is only one vector $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$ $\forall \vec{v} \in V$.

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★ **Lemma:** $0 \in \mathbb{R}$ and any $\vec{v} \in V$, $0\vec{v} = \vec{0}$.

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in function spaces took $-f(x)$.

Lemma: $-1 \in \mathbb{R}$ and $\vec{v} \in V$ then $-1\vec{v} = -\vec{v}$ the inverse of $\vec{v}$.

Easy to see in $\mathbb{R}^2$ and function spaces.

Vector Subspace Thm:

If V is a nonempty subset of W which is

✳ Closed Under Addition

✳ Closed Under Scalar Mult.

then V is a vector subspace of W

Proof:
Must check all properties of a vector space hold.

✳ Closed Under vector addition (given)

✳ Closed Under scalar mult (given)

Check off 6 properties using that they are true for vectors in W and all vectors in V are in W //////

→ Additive Id Property (proved using a lemma)

→ Additive Inverses Prop (HW! also proven with a lemma)

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V}$ ✳
- Associativity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{u}) + \vec{u} = \vec{v} + (\vec{u} + \vec{w})$ ✓
- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$ ✓
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$ ←
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$ ←

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$ ✳
- Compatibility Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$ ✓
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$ ✓
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$ ✓
- Distribution over Scalar Addition Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$ ✓

QED

Vector Subspace Thm:

If V is a nonempty subset of W which is

* Closed Under Addition
* Closed Under Scalar Mult.

then V is a vector subspace of W

Proof: Must check all properties of a vector space hold.

* Closed Under vector addition (given)
* Closed Under scalar mult (given)

Check off 6 properties using that they are true for vectors in W and all vectors in V are in W //////

→ Additive ID Property (proved using a lemma)

→ Additive Inverses Prop (HW) also proven with a lemma

QED

HW2 $\left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\} \Big| 2v_1 - 4v_2 = 0 \right\}$

is a vector subspace of $\mathbb{R}^3$

HW3 P_3 is a vector subspace $C([0,1])$

Left page:

$\boxed{HW2}$ $\left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \middle| \, 2v_1 - 4v_2 = 0 \right\} = V$

is a vector subspace of $\mathbb{R}^3$

• Closed Under Vector Addition

① Given $\vec{v}, \vec{w} \in V$ ① defn of V

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ with $2v_1 - 4v_2 = 0$

$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ with $2w_1 - 4w_2 = 0$

② $\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$ ② defn of $+$ on V

③ $2(v_1 + w_1) - 4(v_2 + w_2) =$ ③ by algebra and step 1

$= 2v_1 + 2w_1 - 4v_2 - 4w_2$

$= 2v_1 - 4v_2 + 2w_1 - 4w_2$

$= \underbrace{\quad}_{0} + \underbrace{\quad}_{0} = 0$

④ $\vec{v} + \vec{w} \in V$ ④ by defn of V

Also check closed under scalar

$\boxed{HW3}$ P_3 is a vector subspace $C([0,1])$

• $P_3 \subseteq C([0,1])$ because polynomials are continuous

Right page:

$\boxed{HW3}$ P_3 is a vector subspace $C([0,1])$

• $P_3 \subseteq C([0,1])$ because polynomials are continuous.

• Closed under addition

① Given $p, q \in P_3$ ① by defn of P_3

$P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

$q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3$

② $p(x) + q(x) = (p_0 + q_0) +$ ② by add in P_3

$+ (p_1 + q_1) x + (p_2 + q_2) x^2$

$+ (p_3 + q_3) x^3$

③ $p(x) + q(x) \in P_3$ ③ defn of P_3

• Closed under scalar mult

$\boxed{\text{You must do}}$
for HW 3.

Watch [Playlist 313F20-27-Part2](#)

Part II Null Spaces

Defn: Given a linear map

$$F: V \to W \text{ we have}$$

$$Null(F) = \{ \vec{v} \in V \mid F(\vec{v}) = \vec{0} \}$$

Recall a linear Map preserves vector addition and scalar multiplication.

HW4 Find Null(F) where

$$F: \mathbb{R}^3 \to \mathbb{R}^2 \quad F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_3 \end{pmatrix}$$

HW5 Find Null(F) where

$$F: P_3 \to P_2 \quad F(p) = p'(x).$$

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- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

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- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$.
- Compatibility Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
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- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
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Defn: A Linear Map $F: V \to W$
- Preserves Addition: $\forall \vec{v}, \vec{w} \in V \quad F(v+w) = F(v) + F(w)$
- Preserves Scalar Mult: $\forall \vec{v} \in V \quad \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

Part II Null Spaces

Defn: Given a linear map

$F: \mathbf{V} \to \mathbf{W}$ we have

$$\text{Null}(F) = \{ \vec{v} \in \mathbf{V} \mid F(\vec{v}) = \vec{0} \}$$

Recall a linear Map
preserves vector addition
and scalar multiplication.

[HW4] Find Null(F) where

$F: \mathbb{R}^3 \to \mathbb{R}^2 \quad F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_3 \end{pmatrix}$

[HW5] Find Null(F) where

$F: P_3 \to P_2 \quad F(p) = p'(x).$

[HW6] Find Null(F) where

$F: P_2 \to P_2 \quad F(p) = 5p'(x) + 4 p(x)$

[HW4] $F: \mathbb{R}^3 \to \mathbb{R}^2$

$$\left\{ \vec{v} \in \mathbb{R}^3 \mid F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \mid \begin{pmatrix} v_1 - v_2 \\ v_1 + v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

solving the homogeneous system

$$\boxed{\begin{aligned} v_1 - v_2 &= 0 \\ v_1 + v_3 &= 0 \end{aligned}} \qquad \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_1 + v_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{pmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{pmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix}$$

$\begin{aligned} x_1 + x_3 &= 0 & x_1 &= -x_3 \\ x_2 + x_3 &= 0 & x_2 &= -x_3 \\ x_3 &= x_3 \text{ (free)} \end{aligned}$

$F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-1 \\ 0+0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \mid x_3 \in \mathbb{R} \right\} = \text{Null}(F).$$

HW6 Find Null(F) where

$F : P_2 \to P_2$ $F(p) = 5p'(x) + 4p(x)$

$Null(F) = \{ p_2 x^2 + p_1 x + p_0 \mid F(p) = 0 \}$

$= \{ p_2 x^2 + p_1 x + p_0 \mid \underbrace{5(2p_2 x + p_1)}_{} + 4(p_2 x^2 + p_1 x + p_0) = 0 \}$

$= \{ p_2 x^2 + p_1 x + p_0 \mid 4p_2 x^2 + (10p_2 + 4p_1)x + \underset{\text{const}}{(5p_1 + 4p_0)} = 0 \}$
$\overset{\text{const}}{}$

$4p_2 = 0 \quad 10p_2 + 4p_1 = 0 \quad 5p_1 + 4p_0 = 0$

Solve this linear system

$\begin{array}{ccc} p_2 & p_1 & p_0 \end{array}$

$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array} \right) \longrightarrow$ row reduction

$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array} \right) \xrightarrow{P_1 \to \frac{1}{4}P_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array} \right)$

$\xrightarrow{P_2 \to P_2 - 10 P_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array} \right) \xrightarrow{P_2 \to \frac{1}{4}P_2}$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array} \right) \xrightarrow{P_3 \to P_3 - 5P_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right)$

$\xrightarrow{P_3 \to \frac{1}{4}P_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \qquad \begin{array}{l} p_2 = 0 \\ p_1 = 0 \\ p_0 = 0 \end{array}$

$Null(F) = \{ 0x^2 + 0x + 0 \}$

Null space is trivial

"Kernal is trivial"

If $F : V \to W$ has $Null(F) = 0$
then F is one to one.

Thm: If $F: V \to W$ is a linear map between vector spaces V and W and if $Null(F) = \{\vec{0}\}$ then F is one-to-one.

Proof:

① $F(\vec{v}) = F(\vec{w})$
then $F(\vec{v}) - F(\vec{w}) = 0$

② $F(\vec{v} - \vec{w}) = 0$

③ $\vec{v} - \vec{w} \in Null(F)$

④ $\vec{v} - \vec{w} = \vec{0}$

⑤ $\vec{v} = \vec{w}$

Thus F is one to one

① W is a vector space

② F is a linear map lemma

③ Def$_n$ Null (F)

④ Given Null $(F) = \{\vec{0}\}$

⑤ adding $\vec{w}$
V is a vector space
QED

Def$_n$ F is one to one means
$$F(\vec{v}) = F(\vec{w}) \Rightarrow \vec{v} = \vec{w}$$

Lemma If $F: V \to W$ is a linear map then $F(\vec{a} - \vec{b}) = F(\vec{a}) - F(\vec{b})$

Pf:

① $F(\vec{a} - \vec{b}) = F(\vec{a} + (-\vec{b}))$ @ def$_n$ of $-$

② $= F(\vec{a}) + F(-\vec{b})$ ② F pres add

③ $= F(\vec{a}) - F(\vec{b})$ ③ F pres scalar mult

QED

$5p'(x) + 4p(x) = g(x)$ has at most one solution because Null space is trivial in H6

HW7) Extra Credit

Prove:

Theorem: If $F: V \to W$ is a linear map between vector spaces V and W then $Null(F)$ is a vector subspace of V.

Hint: you only need to proup
- Closed under addition
- Closed under scalar mult

HW8) Let $F: C^3([0,1]) \to C([0,1])$ be defined $F(h(x)) = h'''(x)$

Prove $Null(F) = P_2$

(this uses calculus)

Hint: $C^3([0,1])$ = functions whose third derivatives are continuous on $[0,1]$.

$C([0,1])$ = functions that are continuous on $[0,1]$

$P_2 = \{p_2 x^2 + p_1 x + p_0 \mid p_i \in \mathbb{R}\}$ polynomials of degree ≤ 2.

Facts from Calc:
If $f'(x) = 0$ function then $f(x) = k$
If $f'(x) = k$ then $f(x) = kx + C$
If $f''(x) = ax + b$ then $f(x) = \frac{a}{2}x^2 + bx + C$

Watch [Playlist 313F20-27-Part3](#)

Part III Spans

<u>Defn</u>: Given $\vec{v}_1 \dots \vec{v}_k \in V$
the <u>span</u> $\langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle =$
$= \{ t_1 \vec{v}_1 + \dots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \}$
$= \{ \sum_{i=1}^{k} t_i \vec{v}_i \mid t_i \in \mathbb{R} \}$

$\boxed{HW9}$ Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \langle \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle$?

$\boxed{HW10}$ Is $f \in \langle g, h \rangle$ where
$f(x) = x^2$ $g(x) = (x+2)^2$ $h(x) = x+1$
is the vector space $C([0,1])$?

$\boxed{HW9}$ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = t_1 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

Solve this system?

$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{array} \right) \xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array} \right)$

row of zeroes ending in 2
So there is <u>no</u> solution
thus $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is not in the span.

If there was a solution
then you say it is in
the span.

Is $\begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix} \in \langle \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle$?

$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right)$ has a solution!

Yes $\begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$ is in the span.

Part III Spans

<u>Defn</u>: Given $\vec{v}_1 \ldots \vec{v}_k \in V$

the <u>span</u> $\langle \vec{v}_1, \vec{v}_2 \ldots \vec{v}_k \rangle =$

$= \left\{ t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \right\}$

$= \left\{ \sum_{i=1}^{k} t_i \vec{v}_i \mid t_i \in \mathbb{R} \right\}$

$\boxed{HW9}$ Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \left\langle \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$?

$\boxed{HW10}$ Is $f \in \langle g, h \rangle$ where

$f(x) = \underset{\rightarrow}{x^2} \quad g(x) = (\underset{\rightarrow}{x+2})^2 \quad h(x) = \underset{\rightarrow}{x+1}$

is the vector space $C([0,1])$?

$\boxed{HW10}$ Can we find $t_1, t_2 \in \mathbb{R}$

s.t $f = t_1 g + t_2 h$

$x^2 = t_1 (x+2)^2 + t_2 (x+1)$

$x^2 = t_1 (x^2 + 4x + 4) + t_2 x + t_2$

$x^2 = t_1 x^2 + 4t_1 x + 4t_1 + t_2 x + t_2$

$x^2 + 0x + 0 = t_1 x^2 + (4t_1 + t_2)x + (4t_1 + t_2)$

Solve $1 = t_1 \quad 0 = 4t_1 + t_2 \quad 0 = 4t_1 + t_2$

$\begin{pmatrix} 1 & 0 & | & 1 \\ 4 & 1 & | & 0 \\ 4 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 0 \\ 0 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$

has a solution

Yes $f \in \langle g, h \rangle$.

Part III Spans

Defn: Given $\vec{v}_1 \ldots \vec{v}_k \in V$
the span $\langle \vec{v}_1, \vec{v}_2 \ldots \vec{v}_k \rangle =$
$$= \left\{ t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \right\}$$
$$= \left\{ \sum_{i=1}^{k} t_i \vec{v}_i \mid t_i \in \mathbb{R} \right\}$$

Thm: If $\vec{v}_1 \ldots \vec{v}_k \in W$ and
$V = \langle \vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \rangle$ then
V is a vector subspace of W.

HW11 Prove this

Defn: The Image of
a map $F: V \to W$
is $F(V) = \left\{ F(\vec{v}) \mid \vec{v} \in V \right\}$

Thm: If $V = \langle \vec{v}_1, \ldots, \vec{v}_k \rangle$
then $F(V) = \langle F(\vec{v}_1), \ldots, F(\vec{v}_k) \rangle$

HW12 Prove this.

Defn: $F: V \to W$ is onto
$\Longleftrightarrow \forall \vec{w} \in W \ \exists \vec{v} \in V$ s.t $F(\vec{v}) = \vec{w}$

Thm: Onto $\Longleftrightarrow F(V) = W$

Thm: Onto $\Longleftrightarrow \exists \vec{v}_1 \ldots \vec{v}_k \in V$
such that $W = \langle F(\vec{v}_1), \ldots, F(\vec{v}_k) \rangle$

HW13 Prove this

Part III Spans

Defn: Given $\vec{v}_1 \ldots \vec{v}_k \in V$
the span $\langle \vec{v}_1, \vec{v}_2 \ldots \vec{v}_k \rangle =$

$$= \{ t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k \mid t_i \in \mathbb{R} \}$$

$$= \left\{ \sum_{i=1}^{k} t_i \vec{v}_i \mid t_i \in \mathbb{R} \right\}$$

Thm: If $\vec{v}_1 \ldots \vec{v}_k \in W$ and
$V = \langle \vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \rangle$ then
V is a vector subspace of W.

HW11 Prove this

Defn: The Image of
a map $F: V \to W$
is $F(V) = \{ F(\vec{v}) \mid \vec{v} \in V \}$

Thm: If $V = \langle \vec{v}_1, \ldots, \vec{v}_k \rangle$
then $F(V) = \langle F(\vec{v}_1), \ldots, F(\vec{v}_k) \rangle$

HW12 Prove this.

Defn: $F: V \to W$ is **onto**
$\iff \forall \vec{w} \in W \; \exists \vec{v} \in V$ s.t. $F(\vec{v}) = \vec{w}$

Thm: Onto $\iff F(V) = W$

Thm: Onto $\iff \exists \vec{v}_1 \ldots \vec{v}_k \in V$
such that $W = \langle F(\vec{v}_1), \ldots, F(\vec{v}_k) \rangle$

HW13 Prove this

HW13 is a very short proof combining the two theorems above it with the definition of onto. You may skip it and I have also removed HW14.

Below we show the solutions to some of the more important homework HW11 and HW12 and HW15.

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{\omega} \in V \quad \vec{v} + \vec{\omega} \in V}$
- Associativity Property: $\forall \vec{v}, \vec{\omega}, \vec{u} \in V \quad (\vec{v} + \vec{\omega}) + \vec{u} = \vec{v} + (\vec{u} + \vec{\omega})$
- Commutativity Property: $\forall \vec{v}, \vec{\omega} \in V \quad \vec{v} + \vec{\omega} = \vec{\omega} + \vec{v}$
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$.
- Compatibility Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$ ⟵
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{\omega} \in V \quad t(\vec{v} + \vec{\omega}) = t\vec{v} + t\vec{\omega}$
- Distribution over Scalar Addition Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$ ⟵

Defn: A Linear Map $F: V \to W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+w) = F(v) + F(u)$
- Preserves Scalar Mult: $\forall v \in V \quad \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

HW 11

closed under addition (you do)
Closed under scalar

Given $\vec{v} \in V = \langle \vec{v}_1 \dots \vec{v}_k \rangle$
$t \in \mathbb{R}$
Show $t\vec{v} \in \langle \vec{v}_1 - - \vec{v}_k \rangle$

① $\vec{v} \in \langle \vec{v}_1 \dots \vec{v}_k \rangle$ ① by defn of span
$\vec{v} = t_1 \vec{v}_1 + \dots + t_k \vec{v}_k$

② $t\vec{v} = t(t_1 \vec{v}_1 + \dots + t_k \vec{v}_k)$ ② by fn defn scalar mult

③ $= t(t_1 \vec{v}_1) + \dots + t(t_k \vec{v}_k)$ ③ Distri

④ $= (t t_1) \vec{v}_1 + \dots + (t t_k) \vec{v}_k$ ④ Compat.

⑤ $\in \langle v_1 \dots v_k \rangle$ ⑤ $t t_i \in \mathbb{R}$ defn of span.

You do closed under addition. carefully name the properties

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- **Closed under Vector Addition:** $\boxed{\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V}$
- **Associativity Property:** $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{u} + \vec{w})$
- **Commutativity Property:** $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$
- **Additive Identity Property:** $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$
- **Additive Inverses Property:** $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- **Closed under Scalar Multiplication:** $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$.
- **Compatibility Property:** $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$
- **Scalar Identity Property:** $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$
- **Distribution over Vector Addition Property:** $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
- **Distribution over Scalar Addition Property:** $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$

Defn: A Linear Map $F: V \to W$
- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+w) = F(v) + F(u)$
- Preserves Scalar Mult: $\forall \vec{v} \in V \quad \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

HW 12 $V = \langle r_1, \dots, v_k \rangle$

Show $F(V) = \langle F(v_1) \dots F(v_k) \rangle$

Proof:

① $F(V) = \{ F(\vec{v}) \mid \vec{v} \in V \}$

　① by defn of Image

② $= \{ F(t_1 \vec{v}_i + \dots + t_k \vec{v}_k) \mid t_i \in \mathbb{R} \}$

　② defn of span
　$V = \langle v_i \dots v_k \rangle$

③ $= \{ F(t_1 v_i) + \dots + F(t_k \vec{v}_k) \mid t_i \in \mathbb{R} \}$

　③ F pres addition

④ $= \{ t_1 F(v_i) + \dots + t_k F(v_k) \mid t_i \in \mathbb{R} \}$

　④ F pres scalar

⑤ $= \langle F(v_i), \dots, F(v_k) \rangle$

　⑤ by defn of span
　QED

HW15 Are $(x+1)^2$, $5x$, 8
a basis for P_2?

• Do they span P_2?

Given any $p \in P_2$ $p = p_2 x^2 + p_1 x + p_0$
Find $t_1, t_2, t_3 \in \mathbb{R}$ s.t.

$p_2 x^2 + p_1 x + p_0 = t_1 (x+1)^2 + t_2 (5x) + t_3 8$

$= t_1 (x^2 + 2x + 1) + t_2 5x + t_3 8$

$= t_1 x^2 + (t_1 2 + t_2 5) x + (t_1 + t_3 8)$

$p_2 = t_1$ $p_1 = t_1 2 + t_2 5$ $p_0 = t_1 + t_3 8$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & 0 & 8 \end{pmatrix} \left| \begin{array}{c} p_2 \\ p_1 \\ p_0 \end{array} \right.$$

Solve to
t_1, t_2, t_3

solve this system + show
it has a solution.

Are they lin indep?

$t_1 (x+1)^2 + t_2 (5x) + t_3 (8) = 0$ function

↘ same work $p_i = 0$

$0 = t_1$ $0 = t_1 2 + t_2 5$ $0 = t_1 + t_3 8$

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 2 & 5 & 0 & | & 0 \\ 1 & 0 & 8 & | & 0 \end{pmatrix} \text{ solve}$$

Is the only solution
$t_1 = 0$ $t_2 = 0$ $t_3 = 0$.

Finish this.

HW15 Are $(x+1)^2$, $5x$, 8
a basis for P_2?

• Do they span P_2?

Given any $p \in P_2$ $p = p_2 x^2 + p_1 x + p_0$
Find $t_1, t_2, t_3 \in \mathbb{R}$ s.t.

$p_2 x^2 + p_1 x + p_0 = t_1 (x+1)^2 + t_2 (5x) + t_3 8$

$\quad = t_1 (x^2 + 2x + 1) + t_2 5x + t_3 8$

$\quad = t_1 x^2 + (t_1 2 + t_2 5) x + (t_1 + t_3 8)$

$p_2 = t_1 \quad p_1 = t_1 2 + t_2 5 \quad p_0 = t_1 + t_3 8$

$$\begin{pmatrix} 1 & 0 & 0 & | & p_2 \\ 2 & 5 & 0 & | & p_1 \\ 1 & 0 & 8 & | & p_0 \end{pmatrix}$$

Solve to
t_1, t_2, t_3

Solve this system + show
it has a solution.

Are they lin indep?

$t_1 (x+1)^2 + t_2 (5x) + t_3 (8) = 0_{\text{function}}$

→ same work $p_i = 0$

$0 = t_1 \quad 0 = t_1 2 + t_2 5 \quad 0 = t_1 + t_3 8$

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 2 & 5 & 0 & | & 0 \\ 1 & 0 & 8 & | & 0 \end{pmatrix} \quad \text{solve}$$

Is the only solution
$t_1 = 0 \quad t_2 = 0 \quad t_3 = 0$.

Finish this.

Part IV Basis

Watch [Playlist 313F20-27-Part4](#)

Part IV Basis

<u>Defn</u>: $\vec{v_1} \dots \vec{v_k}$ are a basis
for a vector space V
if $V = \langle \vec{v_1}, \vec{v_2}, \dots \vec{v_k} \rangle$
and $\vec{v_1}, \vec{v_2} \dots \vec{v_k}$ are
  linearly independent.

<u>Defn</u>: $\vec{v_1} \dots \vec{v_k}$ are linearly
  independent if
$$t_1 \vec{v_1} + t_2 \vec{v_2} + \dots + t_k \vec{v_k} = \vec{0}$$
$$\Downarrow$$
$$t_1 = t_2 = \dots = t_k = 0.$$

Standard Basis

easy check
lin indep

for $\mathbb{R}^2$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\mathbb{R}^2 = \left\{ t_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; t_1, t_2 \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \;\middle|\; t_1, t_2 \in \mathbb{R} \right\}$$

for $\mathbb{R}^3$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

for P_2 x^2 x 1

$$P_2 = \left\{ t_2 x^2 + t_1 x + t_0 \cdot 1 \;\middle|\; t_i \in \mathbb{R} \right\}$$
$$= \langle x^2, x, 1 \rangle$$
easy to check lin indep.

If our functions how
to check lin. indep?

$\boxed{\cos(x), \sin(x)}$

are thes lin indep?

$t_1 \cos(x) + t_2 \sin(x) = 0$ function

so this sum is 0 for any
x we plug in.

plug in x=0

$t_1 \cos(0) + t_2 \sin(0) = 0$

$t_1 \cdot 1 + t_2 \cdot 0 = 0$

$\boxed{t_1 = 0}$

plug in $x = \frac{\pi}{2}$

$t_1 \cos(\frac{\pi}{2}) + t_2 \sin(\frac{\pi}{2}) = 0$

$t_1 \cdot 0 + t_2 (1) = 0$

$\boxed{t_2 = 0}$

Yes $\cos(x)$ and $\sin(x)$ are
lin indep.

$\boxed{\text{HW16}}$ Are

$\sin(x), \sin(2x), \sin(4x)$

linearly independent?

Try this
using the
same method
plugging in values
of x where
different functions
are 0.

Try $x = \frac{\pi}{2}$ $x = \frac{\pi}{4}$ and another x.

Thm If $\vec{v}_1 \ldots \vec{v}_k$ form a basis for V then every $\vec{v} \in V$ has exactly one set $t_1, \ldots t_k \in \mathbb{R}$ such that $\vec{v} = t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k$.

(that is, there exists a unique solution)

. Usually Difficult to find the t_i

Recall:

Orthonormal Basis

$\vec{v}_1 \ldots \vec{v}_k \in \mathbb{R}^n$

s.t $\vec{v}_i \cdot \vec{v}_i = 1$

$\vec{v}_i \cdot \vec{v}_j = 0$ if $i \neq j$

Easy to find t_i for any $\vec{v}$

$$t_i = \vec{v} \cdot \vec{v}_i$$

Check $\vec{v} = t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k$

HW21 Check this works.

We will do HW17-20 in Part 5.

Part V Dimension

Watch [Playlist 313F20-27-Part5](#)

V. Dimension

Thm: If $\vec{v}_1, \ldots, \vec{v}_k$ are a basis for V and also $\vec{w}_1, \ldots, \vec{w}_m$ are a basis for V then $m = k$

Defn: $\uparrow$ Dimension of V

[HW17] Prove the theorem
  Extra Credit: hint find our old proof in $\mathbb{R}^n$ and imitate.

[HW18] Find the dimension of $P_3 = \{p_3 x^3 + p_2 x^2 + p_1 x + p_0 \mid p_i \in \mathbb{R}\}$

Defn: V is infinite dimensional if it has no finite basis.

Thm: If there is a sequence $\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots \in V$ such that for every $k \in \mathbb{N}$ $v_1, v_2 \ldots v_k$ are lin. indep then V is infinite dim'l.

[HW19] Prove this theorem

[HW20] Use this theorem to show $C([0,1])$ is infinite dim'l.

$\boxed{HW19}$

Proof by Contradiction to show V has no finite basis

① Assume on the contrary it does have a finite basis $w_1 \dots w_m$ ① Indirect Hypothesis Unjustified.

② $V = \langle w_1, \dots w_m \rangle$ ② defn of basis. and $w_1 \dots w_m$ are lin indep

③ Take $k = m+1$ ③ by given info for our thm. $v_1, v_2 \dots v_{m+1}$ are linearly indep.

④ Each $v_i \in V = \langle w_1 \dots w_m \rangle$ So $\exists\ t_{i1}\ t_{i2} \dots t_{im} \in \mathbb{R}$ s.t. $\boxed{v_i = t_{i1} w_1 + t_{i2} w_2 + \dots + t_{im} w_m}$ ④ by defn of spam

⑤ $s_1 v_1 + s_2 v_2 + \dots + s_{m+1} v_{m+1} = 0$

$\Rightarrow s_1 = s_2 = \dots = s_{m+1} = 0$

⑤ by defn of lin indep

⑥ $s_1 (t_{11} w_1 + t_{12} w_2 + \dots + t_{1m} w_m)$

$+ s_2 (t_{21} w_1 + t_{22} w_2 + \dots + t_{2m} w_m)$

$+ \quad \dots$

$+ s_m (t_{m1} w_1 + \dots + t_{mm} w_m)$

$+ s_{m+1} (t_{m+1\,1} w_1 + \dots + t_{m+1\,m} w_m) = 0$

$\Rightarrow s_1 = s_2 = \dots = s_{m+1} = 0$

⑥ Sub step 4 into step 5

⑦ $\left(s_1 t_{11} + s_2 t_{21} + \cdots + S_m t_{m\,1} + S_{m+1} t_{m+1} \right) \omega_1$

$+ \left(s_1 b_{12} + \cdots + S_{m+1} t_{m+1\,2} \right) \omega_2$

$+ \cdots + \left(s_1 t_{1m} + \cdots + S_{m+1} t_{m+1,m} \right) \omega_m$

$= 0$

$\Rightarrow \boxed{s_1 = s_2 = \cdots = 0.}$

⑧ Since ω_1 to ω_m span V
and are lin indep.
So if $= 0$ then

$s_1 t_{11} + s_2 t_{21} + \cdots + S_{m+1} t_{m+1,1} = 0$

$\cdots$

$s_1 t_{1m} + \cdots + S_{m+1} t_{m+1,m} = 0$

$$\begin{pmatrix} t_{11} & t_{21} & \cdots & t_{m+1,1} \\ & \cdots & & \\ t_{1m} & t_{2m} & \cdots & t_{m+1m} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$m+1$ columns and m rows
So there is at least
one free varriable.

$$\left\{ \begin{pmatrix} s_1 \\ \vdots \\ s_{m+1} \end{pmatrix} = \text{Something} \atop \text{not just } 0 \right\}$$

Then we do **not** get

$$\boxed{S_1 = S_2 = \cdots = S_{m+1} = 0}$$

which means
$V_1, V_2 \cdots V_{m+1}$ are **not**
linearly independent.

⊗ In thm they are lin. indep
So Indirect Hyp. is false
in step 1
So V does **not** have a finite
basis QED

Thm

If $\vec{v}_1 \ldots \vec{v}_k$ form a basis for V then every $\vec{v} \in V$ has exactly one set $t_1, \ldots t_k \in \mathbb{R}$ such that $\vec{v} = t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k$.

(that is, there exists a unique solution)

Proven using properties of vector spaces.

Proof:

① $\vec{v}_1 \ldots \vec{v}_k$ are a basis for V — ① given

② $V = \langle \vec{v}_1 \ldots \vec{v}_k \rangle$ — ② by def^n of basis

③ $\vec{v} = t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k$ has a solution $t_1 \ldots t_k$ — ③ by def^n of span

④ Is there only one solution?
Check $\vec{v} = s_1 \vec{v}_1 + \ldots + s_k \vec{v}_k$ — ④ step 3

$t_1 \vec{v}_1 + \ldots + t_k \vec{v}_k = s_1 \vec{v}_1 + \ldots + s_k \vec{v}_k$

⑤ $t_1 v_1 - s_1 v_1 + \ldots + t_k v_k - s_k v_k = \vec{0}$ — ⑤ by add inverses

⑥ $(t_1 - s_1) v_1 + \ldots + (t_k - s_k) v_k = \vec{0}$ — ⑥ by dist prop

⑦ $t_1 - s_1 = t_2 - s_2 = \ldots = t_k - s_k = 0$ — ⑦ v_k are lin indep

⑧ $t_1 = s_1 \quad t_2 = s_2 \ldots t_k = s_k$ — ⑨ add s_i

QED

Part VI Hilbert Space

Watch Playlist 313F20-27-Part6

Part VI
Hilbert Space

$$\ell_2 = \left\{ (v_1, v_2, v_3, \ldots) \mid \sum_{i=1}^{\infty} v_i^2 < \infty \right\}$$

$\underbrace{(v_1, v_2, v_3, \ldots)}_{\text{infinite sequence of real numbers}}$

So $(1, 1, 1, 1, 1 \ldots) \notin \ell_2$

$1^2 + 1^2 + 1^2 + 1^2 + \ldots = \infty$

But $(1, 1, 1, 1, 0, 0, 0 \ldots) \in \ell_2$

$1^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + \ldots$

$= 4 < \infty$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \leq 1$

$v_1 = \sqrt{\frac{1}{2}} \quad v_2 = \sqrt{\frac{1}{4}} \quad v_3 = \sqrt{\frac{1}{8}} \ldots$

this is in ℓ_2

Part VI
Hilbert Space

$$l_2 = \left\{ (v_1, v_2, v_3, \ldots) \;\middle|\; \sum_{i=1}^{\infty} v_i^2 < \infty \right\}$$

$\underbrace{}$ infinite sequence of real numbers

So $(1,1,1,1,1\cdots) \notin l_2$

$1^2 + 1^2 + 1^2 + 1^2 + \ldots = \infty$

But $(1,1,1,1,0,0,0\ldots) \in l_2$

$1^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + \ldots$

$\qquad = 4 < \infty$

Hilbert Space is a Vector Space

addition:

$(v_1, v_2, \ldots) + (w_1, w_2, \ldots)$

$\qquad = \underbrace{(v_1 + w_1, \; v_2 + w_2, \; \cdots)}$

• Check is in l_2
   to see we are closed
      under addition

show:
$(v_1 + w_1)^2 + (v_2 + w_2)^2 + \ldots$
   is finite

use:
$v_1^2 + v_2^2 + \ldots$ is finite
$w_1^2 + w_2^2 + \ldots$ is finite.

Hilbert Space is a Vector Space

ADDITION

$$(v_1, v_2, \ldots) + (\omega_1, \omega_2, \ldots)$$

$$= (v_1 + \omega_1, v_2 + \omega_2, \ldots)$$

$$\underbrace{\qquad\qquad\qquad}_{\text{must check this is in } \ell_2}$$

- Closed Under Addition

  show:
  $$(v_1 + \omega_1)^2 + (v_2 + \omega_2)^2 + \ldots$$
  $$\text{is finite}$$

  use:
  $$v_1^2 + v_2^2 + \ldots \text{ is finite}$$
  $$\omega_1^2 + \omega_2^2 + \ldots \text{ is finite}$$ $\Big\}$ difficult

- Additive Identity
  $$(0,0,0 \ldots) \text{ (easy)}$$
- Additive Inverse
  $$(-v_1, -v_2, -v_3 \ldots) \in \ell_2 \text{ (easy)}$$
- Associative (easy)
- Commutative (easy)

SCALAR MULTIPLICATION

$$\underset{\text{in } \mathbb{R}}{t} \, \underset{\text{in } \ell_2}{(v_1, v_2, v_3 \ldots)} = \underbrace{(tv_1, tv_2, tv_3 \ldots)}_{\text{show in } \ell_2}$$
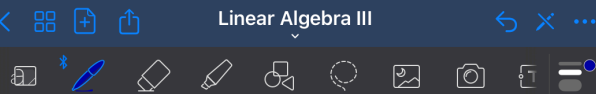
- Closed under scalar mult
  (difficult)
- Compatibility
  (easy) $(st)v = s(tv)$
- Scalar Identity $1v = v$
  (easy)
- Distribution $t(v+w) = tv + tw$
  (easy)
- Distribution $(s+t)v = sv + tv$
  (easy)

Extra Credit
for Completing
the proof.

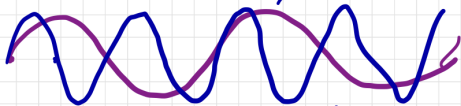**Part VII Fourier Series Analog to Digital**

Watch [Playlist 313F20-27-Part7](#) which has one video made by me and then a few others with professional sound and graphics.

VII Fourier Series.

Consider $\{\sin(j\pi x) \mid j = 0, 1, 2\ldots\}$

$\sin(\pi x)$ $\sin(2\pi x)$
higher and higher
frequencies

Consider linear combinations
of these sine waves

$f \in \langle \sin(\pi x), \sin(2\pi x), \sin(3\pi x)\rangle$
then f is periodic

Solve for $t_1, t_2, t_3 \in \mathbb{R}$
s.t

$$f = t_1 \sin(\pi x) + t_2 \sin(2\pi x)$$
$$+ t_3 \sin(3\pi x)$$

Trick $\int_0^{2\pi} \sin(m\pi x)\sin(n\pi x) = 0$
        if $m \neq n$

$$\boxed{k} = \int_0^{2\pi} \sin(m\pi x)\sin(m\pi x)\, dx$$

Find this

We can call them L_2 orthonormal

$$\frac{1}{k}\int_0^{2\pi} f(x)g(x)\, dx = \begin{cases} 0 & f \neq g \\ 1 & f = g \end{cases}$$
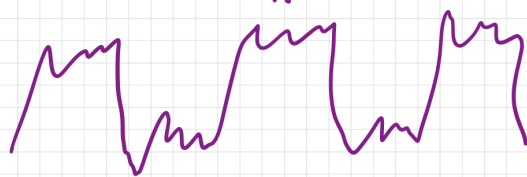
"dot product" for function

Consider

$\sin(k\pi x)$ to be
a note played

$t\,\sin(k\pi x)$ ← frequency

↖ amplitude — loud

Play a few notes at
once

$t_1 \sin(\pi x) + t_2 \sin(2\pi x) +$
$\qquad \cdots + t_k \sin(2\pi k x)$
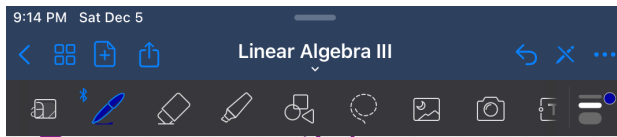
Start with a "nice"
function $f \in L_2$

$$\int_0^{2\pi} f^2(x)\,dx \quad \text{Lebesgue}$$
$$\text{finite.}$$

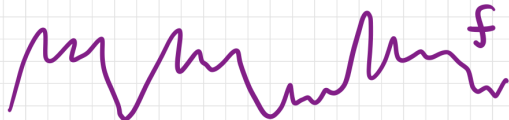$$t_j = \frac{1}{k} \int_0^{2\pi} f(x)\sin(j\pi x)\,dx$$

$$f(x) = t_1 \sin(\pi x) + t_2 \sin(2\pi x) +$$
$$\qquad + \cdots + t_n \sin(\pi n x)$$

Fourier Coefficients of f

Fourier
series $f(x) = \displaystyle\sum_{j=1}^{\infty} t_j \sin(j\pi x)$

Start with a sound

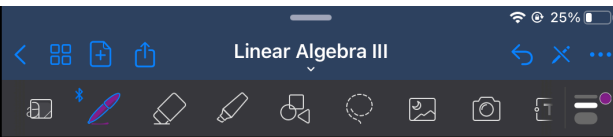Sound wave

 f

Compute the Fourier
coefficients

$\{t_1, t_2, t_3, t_4 \ldots t_N\}$

$f \longmapsto (t_1 \cdots t_N)$

analog
sound   digital
               sound

Stop N when the frequency
is high enough a human
cannot hear.

$f \xrightarrow{F} (t_1, t_2, t_3 \cdots)$

$f \in L_2 = \left\{ f \mid \int_0^{2\pi} f^2(x)\,dx < \infty \right\}$

$(t_1, t_2 \cdots) \in \ell_2 = \left\{ (t_i \cdots) \mid \sum_{i=1}^{\infty} t_i^2 < \infty \right\}$

Hilbert Spaces
  F is a linear
     isomorphism
  one to one
     and
     onto

Consider $\mathbb{C}$ valued functions
$f(s) = a(s) + ib(s)$
$t_i \in \mathbb{C}$ $\mathbb{C}$ Fourier
          series.

Direct links to the professional videos:

Sounds https://youtu.be/3lAMpH4xF9Q

saw wave https://youtu.be/YUBe-ro89I4

3blue1brown   https://youtu.be/r6sGWTCMz2k