Biology Assignment: Differential Equations and Population Models

Student Learning Outcomes

4.1.2	Explain what is meant by a solution to a differential equation
4.1.5	Identify whether a given function is a solution to a differential equation or an initial-value problem
4.2.1	Draw the direction field for a given first-order differential equation
4.2.2	Use a direction field to draw a solution curve of a first-order differential equation
4.2.3	Use Euler's Method to approximate the solution to a first-order differential equation
4.3.1	Use separation of variables to solve a differential equation
4.3.2	Solve applications using separation of variables
4.4.1	Describe the concept of environmental carrying capacity in the logistic model of population growth
4.4.2	Draw a direction field for a logistic equation and interpret the solution curves
4.4.3	Solve a logistic equation and interpret the results

Assignment Overview

There are a wide variety of growth models in biology to describe growth, either of a population of individuals or a group of cells in the body. Growth can be modeled in several ways, depending on the availability of resources, the carrying capacity of the environment, and how quickly individuals or cells are able to reproduce. A common feature of all of these growth models is that the rate of population growth depends in some way on the current population. Since the derivative can be used to describe the instantaneous rate of change, these growth models are described in terms of differential equations. This assignment reviews two models previously covered, exponential growth and logistic growth, and introduces a third, the Gompertz Equation.

Part One: Exponential Growth

The simplest model of population growth is that the rate of population growth is in direct variation with the current population. This yields the equation: $P'(t) = rP(t)$, where $P(t)$ is the population at some time t and $r > 0$ is a constant.

Exercise 1: Suppose that $P(t)$ represents the population of rabbits on an island, in thousands of individuals, t years after introduction, and growth is modeled by $P'(t) = rP(t)$

a. What would it mean if $P(2) = 4$?
b. What would it mean if $P'(2) = 2$?
c. Suppose that $r = 0.5$ and $P(2) = 6$. What is $P'(2)$, using the growth model above?
d. Use the value for $P'(2)$ calculated above to estimate the value of $P(3)$, assuming a linear rate of growth. Is this value likely to be an underestimate or an overestimate, and why?

Exercise 2: Explain why, for $P'(t) = rP(t)$, $P''(t)$ is always positive for any $r > 0$, $P(t) > 0$. What does that mean about the *concavity* of the solution to this differential equation $P(t)$?

Exercise 3: Use separation of variables to show that the general solution of the differential equation $P'(t) = rP(t)$ is given by $P(t) = Ce^{rt}$, where C is a constant.

Exercise 4: Verify that $P(t) = Ce^{rt}$ is a general solution to the differential equation $P'(t) = rP(t)$ by finding $P'(t)$ and evaluating the equation.

Exercise 5: Find the solution to the initial value problem $P'(t) = 0.5P(t)$, $P(2) = 6$. Then use your solution to find the value of $P(3)$.

Part Two: The Logistic Equation

The exponential model above assumes that growth resources are limitless, and as a result the solution it produces predicts limitless population growth. While the model can be useful to describe initial phases of growth, eventually the rate of growth must slow due to a maximum number of individuals that can be sustained in the environment. This value is called the carrying capacity. One way to model this rate is to include a constant $K > 0$ to represent carrying capacity and reflect the fact that, as the population of individuals increases, competition for resources limits population growth. The logistic model builds this in by multiplying the growth rate in the exponential model by a a factor of $(1 - P/K)$. This yields the logistic equation:

$$P'(t) = rP(t)(1 - P(t)/K).$$

Exercise 6: Suppose that $P(t)$ represents the population of deer in a forest t years after the start of observation and is modeled by $P'(t) = rP(t)(1 - P(t)/K)$.

a. If $r = 0.2$, $P(0) = 1$, and $K = 8$, use the logistic equation to find $P'(0)$. Interpret what this means in a sentence.
b. What is the value of $P'(0)$ if the value of $P(0) = 12$ instead? What does this mean about how the population is changing?
c. What is the value of $P'(0)$ if $P(0) = 8$? What does this mean about how the population is changing?

Exercise 7: Use Euler's Method with step size $h = 4$ to predict the value of $P(4)$, if $r = 0.2$, $P(0) = 1$, and $K = 8$. Then repeat this process with $h = 1$, and round values at intermediate steps to three decimals. Interpret these answers in the context of #6 above. Which of these is likely to yield a better approximation?

Exercise 8: Verify that $P(t) = \frac{Ce^{rt}}{1+\frac{C}{K}e^{rt}}$ is the general solution to the logistic equation $P'(t) = rP(t)(1 - P(t)/K)$.

Exercise 9: Find the solution to the initial value problem where $r = 0.2$, $P(0) = 1$, and $K = 8$. Use it to find the value of $P(4)$ and compare this to your estimates in #7. Interpret your answer in the context of #6 above.