The Way of Things

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Introduction

1.i I am Lying

“I am lying,” I say. Which may not be the best way to begin a work of nonfiction. But with those three words I was only saying that I was lying *with those three words*. I will of course not be lying with the other words in this work of natural philosophy. And for the following reason, I was not lying with those three either.

If I had been lying when I said “I am lying,” then I would not have been telling the truth when I said that I was lying. So it would not have been true that I was lying. Which means that I would not have been lying.

So, if I had been lying, then as well as having been lying, I would not have been lying.

And of course, I cannot have been lying and not lying—my having lied is precisely what is ruled out by my not having lied—so that was not what I was doing. I was not lying when I said “I am lying.”

Not such a bad way to begin this book, then. Except that, given that I was not lying, but telling the truth, when I said that I was lying, I must have been lying. To tell the truth is to tell it how it is. And yet, how could I have been lying? Given that I was not lying, it was not the case that I was lying.

That seems logical; and yet, logically impossible. Have I just reasoned myself into a corner? That would still not have been a bad way to begin this book, because this book is about a similar corner that some of our most logical thinkers have been reasoning themselves into. But in fact, I was reasoning myself into a narrow doorway, not a corner.

Lying is speaking falsely with the intention to deceive, and I had said “I am lying” for educational purposes, not in order to deceive you. So although I was not lying, I could perhaps have been speaking falsely. And how else could I have been speaking, when I said “I am lying”?

Does the reasoning above show that I was speaking falsely? It may look logical enough, now that it does not seem to be putting me in a corner of logical impossibility. But what if someone who was trying to deceive you had said those three words? Or what if someone said the following six words:

“These six words are not true,” says Howard Uno helpfully.

If Howard was telling us the truth, when he said those six words, then he was giving us a true description of them. And he described them as “not true.” So, if he was telling the truth, then they were not true.

But of course, if he had been telling the truth, then his words would have been true.

So, if he had been telling the truth, then something that was not true would have been true. Like he was telling the truth and pigs were flying.

Which seems to mean that Howard was not telling the truth. Except that if those six words of his were not true, then his “these words are not true” would have been a true description.

Again, something that was not true would have been true. Like he was not telling the truth and pigs were flying.

But if the previous piece of reasoning really had meant that Howard was not telling the truth, then surely that very similar piece of reasoning should mean that it was not the case that he was not telling the truth. Which would mean that he was telling the truth. Which the previous reasoning would have ruled out.

So, those two pieces of reasoning cannot both have been logical, not if logic is consistent.

And logic surely is consistent. Thinking logically takes us from truths *only ever* to truths. That is essentially the definition of *logical* thinking.

Those two pieces of reasoning did look logical, though. So in a word, what Howard said was paradoxical. The word paradoxical comes from the ancient Greek for *contrary to common sense*. And it is only common sense that by reasoning logically—as I did seem to be doing—we should end up with a clearer view of things, not with logical impossibility.

Things are logically impossible when logic rules them out. Round squares, for example, are logically impossible. A round square would have no corners because it is round, and four corners because it is square, and zero not equalling four shows that there are no round squares. But what does that paradoxical reasoning about Howard’s six-word sentence show?

It cannot show that that sentence does not exist, of course. Could it be showing that it has no meaning? But consider the following sentence:

These words are six in number.

That sentence was obviously true, with its obvious meaning, because those are six words. So if I point to that six-word sentence and say, “these six words are not true,” then what I say is obviously false, and hence meaningful. Could the same words, said of themselves by Howard, really be, with that slightly different meaning, meaningless?

Consider my “I am lying.” Those words were apparently false, not meaningless. Would they have lacked meaning if they had been spoken by someone who was trying to deceive? Surely the meanings of our spoken words do not depend so strongly upon our secret intentions.

For thousands of years, philosophers have used such questions to introduce their subject. In the twentieth century—to cut a long story short—most of our most scientific philosophers came to suspect that this liar paradox was caused by our not having evolved to be perfectly logical, by our not having evolved the ability to think in perfectly consistent ways about each and every assertion that anyone might make.

Further evidence for that being the most probable explanation was provided by other logical paradoxes, some of which are described later in this book. And that explanation seemed to be a realistic hypothesis, especially after the first world war, as astronomical observations showed that our intuitions about space—which had always seemed as irrefutable as our logical intuitions—were only approximately true.

Could failing to solve some logical puzzles really have amounted to any evidence against the consistency of logic, though? Should it not rather have been assumed that logical solutions might be found one day?

A perfectly logical explanation of the liar paradox would put that question into stark relief. And there is a perfectly natural—and fairly obvious—solution, which will be outlined in the next section of this introductory chapter. It will be elucidated at length in an appendix (at the end of this book), where you will see how logical it is, and where you will find solutions to many of those other paradoxes.

Which raises the question, what were those philosophers thinking?

My suspicion is that it was wishful thinking. As the twentieth century began, scientific philosophers were surprised by a much more paradoxical puzzle, whose logical solution seems to require—as this book will be explaining—the existence of something that has no place in most scientific worldviews. In most scientific worldviews, our not having evolved to be able to be perfectly logical is much more realistic. So, that very paradoxical puzzle could have looked like good evidence that we had not.

Those philosophers, having already rejected the logical solution, would not have thought that a better solution might be found one day. However, scientific worldviews come from looking logically at the facts, not from rejecting logic if it fails to deliver the desired results. Perhaps those philosophers worried about that. Perhaps they took the liar paradox—and other logical puzzles—to be providing supporting evidence. If so, then that would have been a kind of wishful thinking.

Anyway, I was pulling at the loosest threads in the fabrications of those philosophers—which I had become interested in via the physics of the future (see chapter 4)—when I noticed something incredible, and decided to write a book about it. What I noticed was so incredible, this book would have been unbelievable had I not kept it perfectly logical. So, this is going to be a very logical book. But thinking logically, like thinking mathematically, does get tedious. So perhaps it is time for a short break. And it is a sunny day at the seaside…

Down on the beach, a boy is looking for shells, and a few yards ahead of him something is glistening. It is off-white and roundish, and as the boy approaches it, he can see that it is rippled with ridges and furrows. The closer he gets to it, the more it looks like a large shell, worn down and polished by the sea. But when he gets there, he finds that its furrows are the imprints of fingers.

He knows that he has found something handmade—in all likelihood (a shell could have eroded like that, but it would have taken a lot of unlikely knocks and scrapes)—and in light of that, he sees quite clearly that it is a bowl in the shape of a shell.

He decides to keep his shell collection in it. And as he stoops to pick it up, he wonders if it had been lost by picnickers. Now, if he had found something much stranger than a bowl, he might still have been wondering what it was. A piece of modern art? Something from a sunken galleon, encrusted with oddly eroded barnacles? An alien artefact with a chameleonic surface that broke down soon after its spaceship crashed into the sea? Probably not that, of course; although the stranger the object, the stranger the possibilities that could reasonably be entertained, and the more it would be a matter of opinion what was reasonable. And this book is about something much stranger than a bowl on a beach.

This book is about a mathematical fact that appears nonsensical—to mathematicians. One thing, two things, three things and so on and so forth; that is a very simple pattern. But it has a property that makes the whole numbers look as though they are coming into existence endlessly. Since most of us think of numbers as existing timelessly, this property—which was discovered at the end of the nineteenth century—is paradoxical. Some mathematicians thought that it might be explained by the fact that mathematics is created by mathematicians. Although most mathematicians take themselves to be describing patterns that exist more objectively.

What I noticed was that the paradoxical property might—just possibly—be a telling impression left on whole numbers by the creator of all things. Such a possibility would not normally be of scientific interest, of course, but it is a logical possibility, as you will see in chapter 4. And the nonsensical appearance of this mathematical fact does indicate that there are no other logical explanations for its existence. Mathematicians are nothing if not logical. The logical structure of this fact is laid out very clearly for the general reader, in chapter 3, so you will be able to see for yourself just how paradoxical it is. It is so paradoxical, chapters 3 and 4 amount to a proof that there is a creator of all things—in all likelihood (other explanations for the existence of this incredible fact are possible, but none are perfectly logical).

That is an extraordinary claim, but it is not hopelessly unrealistic because this fact has already proved to be extraordinarily puzzling. This book will be solving a puzzle that has had mathematicians stumped for well over a hundred years. And it was not the difficulty of the mathematics that had them stumped. This fact is less complicated than a lot of school mathematics (as you will see in the first two sections of chapter 2). No, those mathematicians were stumped because this fact appeared to be logically impossible, as described in chapter 3.

There is a solution to the mathematical puzzle of how such a simple pattern could have such a paradoxical property (it is outlined in the second section of chapter 2). But that solution is only a logical possibility if all things are created out of nothing but the creative power of a creator whose knowledge increases endlessly, as described in chapter 4. Taking that solution seriously would have meant flying in the face of the prevailing atheism of academia, which had only just emerged from under the shadow of theology (and in the face of that theology, which described a God who was timelessly omniscient).

Academics interested in the philosophy of mathematics (such as Bertrand Russell) investigated other approaches to solving this puzzle, approaches that seemed more realistic to them. And a wide range of approaches was thoroughly explored throughout the twentieth century, because none of them panned out. None was perfectly logical, and such academics did want to be logical. They wanted to be objective. But the sheer size of the literature they produced—in pursuit of the truth (in every conceivable direction except the direction in which the truth happened to be)—meant that the academic problem came to be defined by technical difficulties. The original puzzle was effectively buried beneath increasingly complicated and convoluted formalisms. Until now.

That literature does amount to a lot of evidence that chapter 4 describes the only logical solution to the puzzle of chapter 3. But it is not an easy read. This book is much simpler, and more straightforward. The next chapter begins with the relatively simple fact that led to that literature. And then it is just a matter of seeing what follows logically from the existence of that fact. I say “just a matter” because logic does not ask you to believe anything. On the contrary, to prove a claim is to answer all reasonable doubts about it, and so the more you doubt my extraordinary claim, the more this book can do its job of proving that claim.

Still, the answering of all reasonable doubts does make looking at things logically hard, especially when it is something as puzzling as this mathematical fact. This fact is so puzzling that, for example, the pre-Socratic idea that there are not really numbers of things in the world was resurrected by twentieth-century philosophers. If there are not really numbers of things in the world—just the underlying material of the universe (which is modelled mathematically by physicists, but which is not necessarily made of the things named in their equations)—then there would be no objective facts about whole numbers; and in particular, chapter 3 would not be describing an objective fact.

Twentieth-century philosophers debated, among many other matters, whether there really are numbers of things, and the related question of the objective existence of whole numbers. Now, the literature on those questions—like most of the academic literature relating to this very puzzling mathematical fact—uses a lot of symbols, so it does look scientific. But the sciences are based on distinct observations, made with individual pieces of scientific equipment. It is simply a fact that there are numbers of things in the world. You and I are two people, for example; and those words "You and I" are clearly three in number. Much of this book involves the stating of the obvious—that is how extraordinary the evidence for my extraordinary claim is.

It will not be a walk in the park, to follow chapters 3 and 4 step by logical step, which is why I have topped and tailed those two chapters. They are topped with two introductory chapters, and followed by answers to doubts that it would not be unreasonable to entertain. Which is a lot of logic. But reading this book will be worth it. In light of the proof of this book, you will be able to see much more clearly what many other facts are actually saying about the nature of reality. By the end of this book, an incredibly puzzling fact will have become a surprisingly enlightening fact. But first things first. And first impressions of very puzzling things can be very deceptive.

1.ii The Lie of the Land

Perhaps you are not expecting much from this book. You may, for example, be thinking that there is a lot of evidence that this world is not the creation of a good God. But while there is a lot of evidence this this world is not the creation of anyone like, say, Santa Claus, that is of course not the same thing. The creator of all things out of nothing would be, in relation to creation, a bit like how novelists are to their novels. Whether they are novels about Santa Claus, or something more realistic, is a different kind of question to whether some given pages of text were produced by monkeys dancing about on typewriters, or by a writer.

The nature of that difference should be clearer by the end of chapter 4. But perhaps you are wondering how any purely mathematical fact could possibly show that God probably exists. You might think that showing that would require something more like “God Exists!” appearing, in giant letters of unearthly fire, above a city or two.

Suppose that such words did appear, though. They would most likely be a practical joke, or an advert for a product called God, something like that. And even if such possibilities could all be ruled out—say, by examining the fire and finding it to be nothing that we humans could produce—it would still be more rational to suppose that it was the work of aliens, rather than the work of God. Aliens might want us to think that they were angels, in order to control us more easily. But why would a God want to look like such aliens? Whereas, leaving a telling impression on whole numbers is not something that aliens could possibly have done.

Another reason why you may not be expecting very much from this book is that an argument that God probably exists is an argument that probably contains at least one mistake. That is because there are plenty of other arguments that have claimed to show logically that God probably exists. And most, if not all, of them have failed to show so much. For some of those arguments, their failure might be a matter of opinion, but the others do show that an argument that God probably exists is an argument that probably contains a mistake.

Nevertheless, the argument of this book is different. Those other arguments simply failed to be logical enough. But following the discovery of the mathematical fact that this book is about, some otherwise very scientific philosophers have been debating how best to modernise logic—how best to redefine the word logical—or in other words, how best to replace logic with something else—such as a deliberately inaccurate mathematical model of logic (in technical terms, a non-classical symbolic logic)—which would then be called logic. And those philosophers would hardly have entertained the rejection of logic had there not been some argument, for something unbelievable, which had not simply failed to be logical.

Logical thinking is basically just thinking of the kind that would only ever takes us to truths, if the descriptions that we were thinking about were all true. So, changes to the meaning of logical are naturally associated with changes to the meaning of truth. Twentieth-century philosophers have therefore been debating (among other things) how best to modernise the meaning of the word truth. That might seem reasonable, because a lot of the logical puzzles solved in the appendix, such as the liar paradox, are paradoxical facts about truth. But of course, to pursue the truth by redefining the word truth is not very scientific.

This being a logical, not a theological book, those other arguments that there is probably a God can simply be ignored here. Although perhaps I should briefly mention a couple of them. Like me, the British philosopher Richard Swinburne, in his *The Existence of God* (published by Oxford University Press in 1979, with a second edition in 2004), used mathematics in order to try to show that there is (very) probably a God. Unlike me, he used a scientific type of analysis to explain all of our data about the world. Now, I say "type of analysis" instead of "analysis" because, while the numbers that he plugged into his statistical analysis might seem realistic if your worldview is already close to his, they are unlikely to if it is not. However, his numbers might seem more realistic when you have finished this book. I will be establishing a relatively high logical probability of there being the kind of God that Swinburne was writing about. Furthermore, his book does show how little evidence there actually is against the existence of God.

Secondly, to illustrate what I meant by "simply failed to be logical enough," the following is a simple version of the ontological argument, which has been around for a thousand years or so, in one form or another. It is, to begin with, presumably better for beings to be very good, than for those beings not to be very good. Does it follow that the best possible being would be very good? That does seem to follow. Furthermore, it is clearly better for a very good being to exist, than for that being not to exist. Does it follow that the best possible being would exist? If so, then we would seem to have a logical argument for the existence of the best possible being. And most religious believers do think of their God as the best possible being.

But what if you believe that nothing could be better than, say, eating cake? The best possible thing might be your favourite cake, in your opinion. Would eating one of them right now be best? You might think so. But would it follow that you are now eating cake? Of course not. Making a possibility better—say, by imagining cake now rather than later—is one thing; making that possibility actual—replacing the word *imagining* with the word *having*—is quite another. That simple distinction is what that version of the ontological argument seems to have flown in the face of. Now, the ontological argument did become more complicated, as such objections to it were replied to, and new ones raised. However, this is not a theological book, so we can leave the ontological argument—and other arguments that there is probably a God—there.

This is a logical book, about one particular proof that there is probably a God. But perhaps you think that any logical evidence that this book might contain would clearly be outweighed by all the biological and paleontological evidence for our evolution by natural selection? The God sketched in chapter 4 is not the God of Creationism, though. The evidence for our evolution by natural selection, plus the proof of this book, means that our creator has (probably) created a world in which we—or at least our bodies—evolved by natural selection. There are plenty of logical possibilities for why our creator might have done that, and one of them is sketched in the fourth section of chapter 2. In short, the God hypothesis of chapter 4 is at least as consistent with all the scientific facts as atheism is. I say “at least” because there is one fact in particular that atheism has been unable to find a perfectly logical explanation for, in over a hundred years of looking for one.

The fact in question was discovered by a German mathematician, Georg Cantor (1845–1918), at the end of the nineteenth century. At first, the other mathematicians could not believe that it was a fact. It appeared to be a proof of something impossible. So they naturally assumed that Cantor had made a mistake. But mathematicians are naturally curious, especially about mathematical matters, so they tried to find Cantor’s mistake. They interrogated his mathematics—increasingly thoroughly, because they could not find any mistakes—and eventually they had to accept that it was indeed a fact. Although that did not make it any less paradoxical.

There seemed to be, in some sense, too many whole numbers—as sketched in the first section of chapter 2—but how could there be? There are all the whole numbers that there are. Mathematicians tried to make sense of Cantor’s paradoxical discovery, and philosophers tried to help them, but it soon became clear that the mathematicians would need a way of working around Cantor’s paradox, while those interested in the philosophy of mathematics carried on trying to make sense it.

A hundred years ago, everything in mathematics was given a new definition, within a specially designed language—also sketched in the first section of chapter 2. The question “How many (of the new) whole numbers are there?” has no answer in the new mathematics, so it has no paradoxical answer. For details, see Ivor Grattan-Guinness’s *The Search for Mathematical Roots, 1870–1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel* (published by Princeton University Press in 2000).

That sea change to academic mathematics eventually trickled down to school mathematics, in the form of the new math—which you may well have heard of, as it was quite controversial fifty years ago. Doing mathematics within the new “language” is like doing geometry without pictures, and such abstraction has always put people off. For such reasons, Cantor’s paradox had little impact beyond academic mathematics and philosophy. Those who did notice Cantor’s paradox tended to assume either that the new definitions had solved the problem—although they were only ever a way of avoiding the need for a mathematical explanation—or that the paradoxical fact was inexplicable.

Does every fact need to be explained? Probably not. But the version of Cantor's paradox that is chapter 3 is an apparently logical proof of something that is logically impossible. It therefore appears to be showing logic to be inconsistent. And while we can—and should—assume that thinking logically about truths will only ever result in truths, and hence that this paradox must have some logical explanation, would it not be hard to argue that we must have evolved to be perfectly logical? Especially when Cantor's paradox appears to be showing that we did not so evolve.

We should assume that logic is consistent, if only because anything else would tend to undermine any part of our scientific knowledge that was already in dispute, say, for political or economic reasons. Fortunately, there is a perfectly logical explanation of chapter 3, which will be introduced in the second section of chapter 2, and described more logically in chapter 4.

In the third section of chapter 2 you will find a logical paradox that is easier to describe than Cantor's, but almost as difficult to solve. You will find its logical solution at the end of this book—following a surprisingly obvious solution to the liar paradox, which I shall shortly outline. When faced with an especially tricky puzzle, it usually helps to look at similar puzzles.

The liar paradox, as we have seen, is an apparently logical proof of something that is logically impossible. In any logical paradox, reasoning that appears to be perfectly logical takes us from assumptions that would ordinarily be unquestionable to some obvious untruth, such as a contradiction. Those assumptions might be so unremarkable, there do not seem to be any. There did not seem to be any in the liar paradox. But if there was nothing wrong with our reasoning, then we must have been assuming something that was not the case. Thinking logically about truths will only ever take us to truths, so if it takes us to an untruth then we must have been thinking about an untruth.

"These words are not true," Howard had said.

When someone says something to us, we naturally assume, at first, that it might be true. For all we know, without having thought about it, it might be true. But certainly, if it is not the case that it is true, then it is not true. So it seems safe to assume that anything that anyone says is either true or else not true.

Dr Humdinger—a fictional logician, and Howard's friend—thinks it a law of logic that each and every description is either true or else not true. (That "law" of logic is usually called the law of excluded middle.) But consider Humdinger's cat, who likes jumping in and out of boxes, and who is currently lying half in and half out of an overturned box.

The description “Humdinger’s cat is in the box” is not a very good description, because the cat is half out of the box. Nor is it a very bad description, because the cat is half in the box. Could it be about as true as not? (This is the middle possibility that logical thinking ordinarily begins by excluding from consideration.) If it was, then it would not so much not be the case that it was true, no more than it would be the case that it was true.

If descriptions can, in general, be not only true and not true, but may sometimes be about as true as not, then what Howard said might have been about as true as not. What he said was certainly not a very good description of itself. “These words are not true,” he had said, so if it was a true description—if what it described was as described—then it was not true. But nor was what he said so bad we could call it false unproblematically. If it was false, then what he said—that it was not true—would seem to have been correct.

Either way, his six words would have been true and not true—which they could of course not have been. Given that they had to be either true or else not true, we had logical impossibility. But if Howard’s words were about as true as not, so that from their meaning it followed that it was about as true as not that they were not true, we would have had consistency.

Since logic is presumably consistent, it follows that the law of excluded middle is a false dichotomy (a dichotomy is just a division of a whole into two parts), in such cases as Howard’s six words. This is the narrow doorway that we must go through, of logical necessity, in such cases. At least there is no corner of logical impossibility that we must be occupying, as imperfectly logical primates. And as you will see in the first section of the appendix, some of the other logical paradoxes known to the ancient Greek philosophers show this to be a commonplace doorway, for all its narrowness.

When we are thinking logically about truths, and nothing but truths, our thoughts are bound to be true. That is basically what the word “logically” means: Logic takes truths to truths. However, logic can take an untruth like a false dichotomy to an untruth like a logical impossibility. So it is certainly a logical possibility that the previous section’s reasoning about Howard’s words was paradoxical because it was fallacious—because it was based on a false dichotomy—in other words, that Howard was not so much lying, or telling the truth, as telling some kind of half-truth.

Dr Humdinger thinks that his cat is neither in the box, nor out of it. He thinks that his assertion of “my cat is in that box” would not be true. And there is a lot to be said for his view. After all, if you dropped a box of something, and it spilled out, then you might not think that it was still in the box, even if some of it was. And similarly, you might not be wrong to think that Humdinger’s cat was not in the box. Nevertheless, perhaps there is no fact of this matter. Humdinger’s cat does seem to be enjoying the possession of the box. Would you be wrong if you thought that it was in the box? And since the cat is half in and half out of the box, perhaps you would not be wrong if you thought that it was in the box about as much as not.

There are many other ways of seeing that it is at least conceivable that descriptions are not always either true or else not true, some of which were certainly known to the ancient Greek philosophers who were thinking about versions of the liar paradox. For example, if you plucked hairs one by one from my head, then I would eventually be bald. And if I did not go from not being bald to being bald with the plucking of a single hair—and it is more plausible that I would go bald more gradually—then my assertion of "I am bald" would have to be something in between untrue and true, as I gradually went bald.

You will find many more examples of descriptions that are plausibly about as true as not in the appendix to this book. And I noticed another example just this morning—a pullover that was a reddish brown—or was it a brownish red? At first it seemed to be a reddish (or perhaps a purplish) brown, but the very next time I looked at it, thinking "it is brown," it looked much redder. "It must be a brownish red," I thought. But when I looked at it thinking of it as red, it looked brown, so I concluded that its red-brown colour was about as brown as not, and about as red as not.

Presumably my "it is brown" was about as true as not, although it seemed not to be true; and my "it is red" was presumably about as true as not too, although it also seemed not to be true. Someone else might have concluded that the red-brown pullover in question was neither brown nor red. But the existence of colours that are about as red as not—or about as blue as not—can be demonstrated in a purely logical way, as follows.

Imagine an object slowly changing its colour from blue, through green to yellow, when it is certainly not blue. Imagine someone filming that, and then separating out the individual frames according to the colours of the object: Into one case goes each and every frame in which the object is blue; all the frames in which it is green go into another case; and a third case ends up containing all the pictures of the yellow object.

Suppose there was a moment before which this object was blue, and after which it was not blue. The two colours that it had, just before and just after that moment, would look the same to us, because this object is changing its colour very slowly. But one of them is blue, and the other is not blue.

Because colours are essentially how they seem to us, the alternative—of there being no such moment—may well seem more realistic; and if it does seem more realistic, then it is certainly conceivable. And in this case, as the colours changed more gradually from blue colours into colours that were not blue, there would have been times when the colours were about as blue as not. And as the description “it is blue,” said of this object, changed gradually from true to untrue, it would at such times have been about as true as not.

Perhaps it would be green, in which case it would go into the green case. But perhaps it would be about as green as not, and then it would not go into any of our three cases. Anyway, let us now return to Howard’s six words.

Howard had been trying to say that his words were not describing themselves well enough for them to count as simply true. He knew that his words were paradoxical, so he knew that they were not describing themselves very well. But it seemed to him—as he tried to tell that truth in what seemed to him to be a rather poetic way—that they were describing themselves quite well. Just not well enough for them to count as simply true. So, they had seemed to him to be about as true as not.

Howard was not lying, he was just being a bit poetic. But he had said that what he was saying was not true, and because that was what he was saying, he had, in effect, said that it was not true that what he was saying was not true. So, he had implicitly said that what he was saying was true, when he explicitly said that it was not. Now, almost everything we say comes with an implicit assertion of its own truth. However, those two apparently contradictory sides to what Howard said do seem very odd. It is hard to think of a worse kind of ambiguity.

It would make more sense—or would make sense more obviously—to say two explicitly contradictory things. To say, for example, “this is round” and “it is square” when describing the same shape. At least one of those two descriptions would presumably not have been true. The shape might, for example, have been pentagonal (having five roughly straight sides, of roughly equal length), in which case neither “this is round” nor “it is square” would have been true. It is not nearly so easy to make sense of what Howard said.

Although I suppose that a pentagon does look a bit like a circle, and only a little less like a square. A poet might describe it as round and as square, for some reason, without talking nonsense, and without actually lying. Could the two sides to the meaning of what Howard said be two equally poor descriptions of something that was such a poor description it was effectively its own contrary, but not so poor it was actually false?

A relatively simple example of a poor description of a poor description is “a very bad description,” while another is “not a bad description.” Those two descriptions are basically contraries—if one is true, the other is not true—but the poor description that they are poor descriptions of could conceivably be as true as not.

This possible solution to the liar paradox is not as simple as school mathematics. But the logical solution must be a little obscure because why else would this puzzle have seemed so paradoxical, to so many philosophers, for so long? Just how logical this possibility is should be clearer when we return to it later, in the third section of the appendix. By then you will, in the first section of that chapter, have seen a lot of other descriptions that are about as true as not.

Cantor's paradox is rather different to this logical puzzle. The descriptions in chapter 3 derive logically from mathematical descriptions, so the proof of chapters 3 and 4 is able to use the law of excluded middle. Which is just as well. Very little logical thinking can be done with descriptions that are about as true as not. Enough can be done for us to know that these paradoxes are caused by false dichotomies. But most logical thinking is based on valid dichotomies.

Another difference between Cantor's paradox and the liar paradox is that the solution to the former, described in chapter 4, is even more obscure than the solution to the latter described in the appendix. Which is only to be expected. Cantor's paradox was so paradoxical—so logical in its derivation of an untruth—that it changed the nature of academic mathematics. The solution described in chapter 4 is so otherworldly, it will be impossible to describe it in any detail. But chapter 4 does describe it well enough for it to be quite clear that it is, at the very least, a logical possibility. And that is all that the proof of chapters 3 and 4 needs. Chapter 3 is so logical—so paradoxically logical—that there is quite clearly no other logical possibility.

A logical paradox is an apparently logical argument for something that cannot possibly be true. It might be called a "proof," of an untruth. It looks like a perfectly logical proof, but it cannot be a proof, because proofs are always of truths. Logical thinking is thinking that will, if we begin by thinking about truths, only result in truths. To an untruth, logical thinking must therefore have been taking an untruth. So, if we can identify all the assumptions that a "proof" began with, then the more confident we are about all but one of those assumptions, and the less confident we are about that one, the more likely it will be that that particular assumption is not true. An ordinarily unquestionable assumption might, in that way, be shown to be highly unlikely. And a surprisingly logical paradox could, just possibly, amount to a surprisingly enlightening proof.

This kind of proof is called, in general, reductio ad absurdum, which is Latin for "reduction to absurdity." Despite the fancy name, this kind of reasoning is essentially just common sense. Suppose, for example, that we are talking about keeping pigs as pets. One of us says that pigs are not pigs. This person is of course not telling us an absurdly obvious untruth. This person is clearly saying that pigs do not make particularly messy pets (and we might have assumed that they would, because they are not common pets, so we might have pictured them on a farm, in a pen or a field). We usually assume that repeated words have the same meaning, but that assumption would make this person's assertion absurd. So we naturally deduce—probably without even thinking about it—that the repeated words mean different things in this case.

More generally, if we think that something is a certain way, and we want to prove that it is, we might begin by hypothesising (or by assuming, for the sake of argument) that it is not that way. If reasoning (or arguing) in a purely logical way about that hypothesis results in an obvious untruth—if, in other words, that hypothesis (or assumption) is reduced logically to something that it would be absurd to assert—then that hypothesis will have been shown to be untrue. Logical thinking is thinking that takes truths only ever to truths, so to an untruth it must have been taking an untruth.

We have already seen some reductios. When we moved from something like Howard telling the truth and pigs flying, to Howard not having told the truth, that was a reductio ad absurdum. For an even simpler example, let's prove something that is obvious:

To prove that there are no round squares, we begin by hypothesising that there is a round square. Because it is round, it has no corners. And because it is square it has four corners. But 0 is not equal to 4—it would be absurd to say that it was—and so the hypothesis that there is a round square has been taken, in a perfectly logical way, to an untruth. Since logic only takes untruths to untruths, it follows that it is not true that there is a round square.

Sometimes, the hypothesis that something is not a certain way will only have been shown by such a reductio to be in all probability untrue. When a proof is deductive—when it gives back to us no more than we put into it—we should not need to talk of probability, but putting into a deduction everything that a perfectly logical deduction needs can be surprisingly difficult.

There is, for example, a sense in which there are round squares: A hundred is a round square; it is a square because it is ten squared (it is the area of a square with sides of length ten), and the symbol for a hundred—100—shows why it is a round number. Of course, we were implicitly talking about round square shapes, in that proof that there are no round squares, not round square numbers. But that was not explicitly stated, and proofs are nothing if not explicit.

That sense in which there are round squares ruined that proof, but not in a serious way. It would be easy to correct the proof, by being more explicit. But quite generally, in any piece of reasoning, it is possible that something serious has been overlooked, or some other serious mistake made. Nevertheless, if we practice being careful in our reasoning, we can learn to estimate such errors well enough. This book will provide you will plenty of practice, and by the end of this book you should find that the proof of this book comes with only a very small possibility of error.

This book describes a proof that there is (very) probably a creator of all things, based on the logical version of Cantor's paradox that is chapter 3. The word very is in parentheses because it is bound to be a matter of opinion, to some extent, just how likely it is. Your reading of chapters 3 and 4 will result in such a proof, insofar as reasonable doubts are raised and answered. Chapter 5 addresses the most significant doubts, but it is to some extent a matter of opinion what is reasonable. Chapters 3 and 4 are just written words, they are akin to the symbols in a mathematical proof—it is only when we give them meanings that we understand, that they amount to a proof (of a logical probability). Written words are essentially pictures. We say that they are true, or not; but more precisely, their truth is a matter of the realism of meanings that derive primarily from words being spoken, in a context with some significant cultural dimension.

So, standing in the way of such an understanding is, as the proof of this book begins, the fact that the God hypothesis is not a typical scientific hypothesis. We naturally associate words like God and paradox with ineffable mystery, and so they do seem to be poles apart from science and proof. And in particular, the description "creator of all things" certainly does not look very scientific. Nevertheless, thinking logically about any evidence that there might be, for the world having been created out of nothing, could of course help to answer the question of whether there is such a being; and science is all about thinking logically about the facts. And the description "creator of all things" can be given sufficient sense for the purpose of this book by the use of analogies. Analogies are useful when we are trying to thinking logically about the fundamental nature of reality, because our languages developed as they did primarily to help us to communicate about more mundane matters. Even physics had to begin with analogies. In our everyday lives, we pick things up and carry them, we pull them and push them, and sometimes we have to use more force to make them move; and so the earliest physicists were naturally thinking about motion in terms of motive forces. As they made scientific observations, they developed increasingly accurate mathematical descriptions of forcefields. But it all began with that simple analogy.

And a conveniently simple analogy for the creation of all things, out of nothing but the creative power of a creator, might be the "creation" of everything in, or implied by the words of, a story, out of the writing of that story and the reading of it. The creation of all things out of nothing should certainly not be thought of as being like the creation of a bowl out of clay, but without the clay. A magician might make bowls like that, in a fairy tale. But those bowls, that magician and everything else in that fairy tale would be "created" by that story's author and its readers.

Analogies should not be taken too literally; and in particular, this one does not imply that the world would be at all fictional, were it created out of nothing. We are, of course, not unreal. Still, the world’s creator would be, in some sense, more real than anything else. And the idea of anything being more real than the world around us might sound crazy. However, all the alternatives to there being such a God make similar assertions. Materialism, for example, says that reality is physical. It does not say that our perceptions, thoughts and feelings are not real. How could it? Their reality is one of the most obvious things in the world. But it does imply that our brains are, in some sense, more real than our minds. Although it is has proved to be very difficult to spell out precisely what that sense is. Daniel Stoljar’s “Physicalism,” in Edward Zalta’s The Stanford Encyclopedia of Philosophy—which is online at https://plato.stanford.edu/archives/win2017/entries/physicalism/—is a handy place to begin looking at the details of those difficulties.

But modern philosophy is convoluted and complicated. This book is relatively simple and straightforward. This book is about one very puzzling fact, and what it says about the nature of reality. Very little philosophy is associated with this book’s proof that there is, in all likelihood, a creator of all things, who is able to change. A creator of all things out of nothing would be above and beyond the meanings of most of our words, which get those meanings via those things. And such a creation would simply be howsoever its creator wanted it to be (much as the best fictional worlds are essentially how their writers wanted their readers to imagine them).

There is some philosophy in chapter 4, but not as much as there would be if materialism was true, and minds were produced physically by brains. Various philosophers have hoped to contribute to a future science of that mysterious process (much as natural philosophers like Newton studied alchemy, in the days before chemistry). Whereas, if there is a God, then there might simply be minds as well as brains, at the penultimately fundamental level of creation (the most fundamental part of reality would be God), with the relationship between minds and brains depending upon the purpose of creation. And that would be a theological question; whereas, this is a logical, not a theological book.

It is in a relatively straightforward way that this book proves that there is (very) probably a creator of all things. Cantor’s paradox is shown to be the mathematical face of a reductio ad absurdum of there not being a creator of all things. The question of whether there is a creator of all things is a metaphysical question which obviously has a religious dimension. But that side to things does not have to get in the way of answering that question in a more scientific way. Religious believers may well regard any proof that there is probably a God as their business, but we do not have to agree with them.

The vaguely analogous question of whether there is intelligent life on other planets may shed some light on that relatively messy facet. That question could of course be answered by considering the empirical evidence logically, for all that other considerations would arise within a subculture of UFO sightings and alien abductions. I suppose that the mere existence of those subcultures might get in the way of a scientific answer to that question; for example, talk of aliens existing could conceivably become something that scientists would want to avoid. But if aliens really did arrive from outer space—so that there was scientific evidence for their existence, in the region of reported sightings of them—then ignoring reports of alien abductions would clearly not be the best scientific practice.

That is a big "if," of course. And first impressions of any God hypothesis are bound to be coloured by the strong association that the word God has with religion; and for atheists, those would tend to be bad associations. But just as none of the things said and done by social Darwinists in the twentieth century should count against the likelihood of the theory of evolution by natural selection, so nothing that religious people have ever said or done should count against the likelihood of there being the God described in chapter 4. The God hypothesis of this book is a scientific hypothesis, because its justification is logical, and because that justification is based on a mathematical fact.

Mathematics is an exact science, with little use for a word like likelihood. There are exceptions, but they are controversial. Goldbach's conjecture, for example, which asserts something about all the even numbers, has not been proved to be true. But none of the even numbers tested—about two million trillion of them, as I write—have shown it to be untrue. So, some people conclude that it is probably true. Whether they are right to use the word probably like that, or not, my use of the word probably in "there is (very) probably a creator" is quite different.

The reason why mathematicians redefined mathematics, following Cantor's discovery—the reason why Cantor's paradox is so paradoxical—is that the only logical solution to such logical puzzles as chapter 3 is the academically unpopular hypothesis of chapter 4. Any of us might make a mistake when reasoning, but the chance of all the people who have ever looked into this paradox having made the same mistake must be phenomenally low—unless we had, with Cantor's paradox, gone beyond some natural limit to our ability to reason logically. And so the main reason for that "(very) probably" is the academic plausibility of such a natural limit to our ability to reason logically.

Most academic scientists—and hence most of those who defer to such experts when it comes to matters affecting their worldviews—believe that our minds are produced by our brains, which evolved by natural selection in a purely physical world. Why would primates—even highly evolved primates like ourselves—have evolved to have the capacity to reason in a perfectly logical way about absolutely anything? Is it not more likely that we would have evolved in such a way that we might not be unable to understand at least some of the most general and abstract facts? For such reasons, many people will regard chapter 3, not as evidence for the God hypothesis of chapter 4, but as evidence for there being such a natural limit to our ability to reason logically.

However, if there really was such a limit, then there would be much less evidence for our having evolved by natural selection in a purely physical world. That is because, although there is a lot of that evidence, it all comes from thinking logically about an even larger mass of data. And of course, if there really is such a God, then it would be unscientific to ignore logical evidence that there is. This reason to doubt the proof of this book therefore begs the question, to some extent. Science is primarily a logical pursuit of truth: Observations are thought about logically; it is that which results in the justification of the best scientific theories. To use those theories without making sufficient allowance for where they came from would be too much like this:

Measurements of lengths are usually made to the nearest whole number of the smallest divisions on the scale of the measuring device. A length exactly equal to two and a third of those divisions would be measured as 2, for example. And so a measurement of two of those lengths, lined up together—which is exactly equal to four and two thirds of those divisions—would be measured as 5. Were one to ignore the details of the measurement process, and consider only the results of the measurements, those measurements might seem to show that 2 + 2 = 5 (especially if one had some Orwellian reason for wanting to believe that 2 + 2 = 5). That would not really be an empirical proof that 2 + 2 = 5, of course (however much one wanted it to be). Empirically, the measurement was of 2 + 2 being approximately equal to 5, plus or minus 1. And the fact that 2 + 2 = 4 is a mathematical fact. It is based on the logical possibility of things, which is itself justified by there being numbers of things.

2

From Paradox to Proof

2.i Paradox

As the twentieth century began, the numbers were not adding up. Mathematicians were deducing contradictions from the assumption that there are all of the whole numbers. Which is not much of an assumption. There are, of course, all the whole numbers that there are. And whole numbers are not especially complicated, mathematically. Whole numbers are answers to “how many?” questions. How many words are there in this question? We answer that question by counting those words. It is as simple as 1, 2, 3.

Although mathematicians have never found it easy to give a definitive answer to the question of how many counting numbers there are. There are infinitely many counting numbers. But does that mean that there is an infinite whole number—the number of all the counting numbers—or does the sequence of all the counting numbers just keep getting longer and longer? It is certainly a lot of things to think about. Repeatedly adding 1, starting with 0, produces first 0 + 1 = 1, then 1 + 1 = 2, then 2 + 1 = 3, then 3 + 1 = 4, and so on. If we imagine that process going on and on, without ever stopping, then we are imagining the totality of all the counting numbers.

As you think of that process going on and on, without ever stopping, it can seem as though there never would be all of the counting numbers. And if, for whatever reason, there never are all of the whole numbers, then that could explain why mathematicians were deriving contradictions from the assumption that they did all exist. And there are lots of reasons why it can seem to make sense for there to never be all of the counting numbers. Paradoxical facts about infinity have been known about since ancient times; because of them, the ancient Greek philosopher Aristotle regarded the totality of the counting numbers as not actually, but only potentially infinite, in some sense. And Aristotle’s view became the standard view of infinity, until Cantor.

And then there was Cantor’s paradox. And following Cantor, many more paradoxes of infinity were found. They are not our concern in this book, which is only concerned with Cantor’s paradox. But for a very simple example, consider how the first ten counting numbers go all the way up to ten—1, 2, 3, 4, 5, 6, 7, 8, 9, 10—and how the first hundred go all the way up to a hundred—similarly—and so on, and on. Whereas, every single counting number is infinitely far from infinitely big, because they are all finite. So, there are infinitely many of them, but they seem not to go all the way up to the infinitely big. Do they fail to go all the way because their number is always increasing?

It is one thing to think about the entirety of all the counting numbers by thinking of the process of repeatedly adding one going on and on endlessly, quite another to think that such a process might be how they come into existence. Although things might pop into existence—presumably that is what the universe as a whole did, in the Big Bang—most people find the idea of whole numbers popping into existence unrealistic. So, what does it mean, to say that a whole number exists? Well, presumably it means that it is possible, one way or another, for there to be that many things, of some sort or another. Otherwise, there might be a whole number without it being at all possible for there to be that many things. And if there could never be anything that it could be a number of, then in what sense would it be a number?

For such reasons, it seems safe to assume that if some whole number is going to exist, then that many things are going to be possible. And given that if some whole number is going to exist then that many things are going to be possible, it follows that it was never logically impossible for there to be that many things (had it ever been logically impossible, it could never be possible). And since it was never logically impossible for there to be that many things, hence it was always logically possible for there to be that many things. And that certainly seems to mean that that whole number always existed. So, any whole number that will ever exist always did exist.

So, there should already be all the whole numbers that there will ever be—which is what most twentieth-century mathematicians believed. A few did not, and they usually cited paradoxes of infinity in defence of their views. But all the arguments against there being an infinite number of all the counting numbers turned out to have gaps in their logic. On the other hand, there is no gapless argument for the existence of such an infinite number. And in particular, there is a very subtle gap in the logic of the argument above, that there should already be all the whole numbers that there will ever be. That gap will be described in the first section of chapter 2, and it means that there is at least one logically possible solution to the puzzle of Cantor's paradox. Basically, there are not all the whole numbers that there will ever be.

Most mathematicians think that there are all of the counting numbers, and hence that there is also the infinite number that is the number of all of them. But the endlessness of the sequence of all of them is certainly counter-intuitive. And in particular, the following paradox was discussed by the Italian astronomer Galileo Galilei (1564–1642). Firstly, it is only every other counting number that is an even number. So, there are clearly fewer even numbers than counting numbers. But secondly, we can get the even numbers by doubling the counting numbers. In that way, the even numbers can all be paired up with all of the counting numbers—which means that there are just as many even numbers as counting numbers.

Galileo's paradox is that, although there are clearly fewer even numbers than counting numbers, there are, equally clearly, just as many even numbers as counting numbers. From that inconsistency, Galileo concluded that relationships like "fewer than," "just as many as" and "more than" do not make sense when they are applied to infinitely big collections. The problem is that they do make sense, though. We understand them well enough to see that we have a paradox. We have two assertions, each of which is clearly justified, but which seem to contradict each other.

Nevertheless, this is no more a contradiction than "pigs are not pigs" is; the sense in which there are fewer even numbers is not the same as the sense in which there are just as many. The sense in which there are fewer even numbers than counting numbers is obvious enough. To get the even numbers, we take some of the counting numbers away. The other sense is more important, mathematically, because it applies more widely. If we have five oranges and three lemons, for example, then, although we clearly have fewer lemons than oranges, we do not get the lemons by taking some of the oranges away. So, the first sense of fewer does not apply. It is rather the case that we have some oranges left over after we have paired up as many of the oranges and lemons as possible. This is the second sense of there being fewer lemons than oranges. And that pairing gives us the second sense of as many.

It is obvious that there are two senses at work in Galileo's paradox. But it was many years before the second was defined by mathematicians. In Galileo's day, the Church took an Aristotelian view of infinity, and regarded it as an attribute of an ineffable God. But following the invention of the calculus, by the British mathematician Isaac Newton (1643–1727), it became less plausible that infinity was ineffable. And Cantor was working at the cutting edge of the calculus, when he found a need for clarity about the second sense of as many.

A collection of things is just those things, being referred to collectively; and given any two collections of things, if the things in one collection can all be paired up with all the things in the other collection, then they have the same number of things in them. Mathematicians say that they are equinumerous. Equinumerosity is essentially a logical concept, because reference is a logical concept, and the sense of the word can in "if the things in one collection can all be paired up with all the things in the other collection" is that of logical possibility. If no such pairing is possible, there will be some possible pairing up of just some of the things in one of the collections with all of the things in the other, and that one has more things in it, the other fewer. And that word possible also has the sense of logical possibility.

Once Cantor had clarified what equinumerosity was, mathematicians saw that it was the concept that they had always been working with. In particular, arithmetic results from thinking logically about equinumerosity. It is easy enough to see how the ordinary arithmetic of the counting numbers is essentially just logic: Consider how three round things and three red things might be three things, if each is round and red, or six things, if none of the round things are red, or four or five things, if some but not all of them are red. That is just logic, and the second of those possibilities is an instance of the arithmetical fact that 3 + 3 = 6.

By thinking logically about infinite whole numbers, Cantor came up with a very simple proof that, given any whole number, even an infinite whole number, there is a bigger number. Given an infinite collection, this proof shows that the collection of all of its subcollections—a subcollection of a collection is just some of those things—is an even bigger collection. You will find a sketch of that proof at the beginning of chapter 2, and more detailed versions of it in chapter 3, because this proof is what facilitates such "proofs" as that of chapter 3, and Cantor's paradox.

Cantor's paradox concerns the collection of all the other mathematical collections. That should be the biggest mathematical collection, but Cantor's proof shows that there would be a bigger mathematical collection, the collection of all of its subcollections. Mathematicians and philosophers have come up with a huge range of possible explanations of Cantor's paradox. And some of them do seem to solve that particular puzzle rather well. But just as, as the beginning of this book, the solution to the puzzle of my "I am lying" was no solution to the paradox of Howard's six words, so none of those possible explanations of Cantor's paradox solves the logical puzzle of chapter 3.

In chapter 3, a similarly paradoxical contradiction is obtained in the following kind of way (since it facilitates a neater proof). Given the collection of all the counting numbers, there is a bigger collection, the collection of all the collections of counting numbers. And the collection of all of the subcollections of that bigger collection is, similarly, an even bigger collection. And so on. So, we have an endless sequence of bigger and bigger collections. And the totality of all the things in all those collections is an even bigger collection. So we can start again with that collection. The collection of all of its subcollections is bigger still; and so on, and so forth. We can start again, and again. There is no end to this. But because all of these mathematical objects are presumably timeless, there should be a totality of all the things in all the collections that can be obtained in this way. The problem is that there cannot be such a totality. If there was such a totality, the collection of all its subcollections would contain even more things, obtained in just that way.

Cantor took his paradoxical discovery to be showing the infinity of all the whole numbers to be ineffable, and symbolic of the ineffable infinitude of his timeless God. But of course, most mathematicians thought that there should be a more logical way of thinking about such mathematical totalities. They thought about Cantor's discovery, and they thought about it, and they soon realised that they would need a way of working around it while they thought about it some more, it was so puzzling. So, a hundred years ago, mathematicians changed the meaning of whole number so that the "whole numbers" that most of them would be working with would be giving them consistent results.

Consistency could not be guaranteed, as the Austrian logician Kurt Gödel (1906–1978) proved in 1931. But the Cantorian inconsistencies would not arise in the new mathematics. And until they understood why those inconsistencies had arisen, that was the best that they could do. And at least they could be sure that the new numbers would cohere with everything else in the new mathematics. They changed the meaning of whole number by giving everything in mathematics a new definition, in a new "language" called an axiomatic set theory.

Mathematicians have since designed other "languages" which are to some extent analogous. For example, when a computer is used to do various calculations, the calculations are first translated by that computer into its machine language, so that the computer can do all the hard work. The computer code for, say, a problem in Newtonian mechanics is more complicated than the original problem, but it facilitates a much faster and more reliably accurate solution. A computer model of a mathematical model of a mechanical system is basically just a different mathematical model of that system; a better model, in various important ways. And similarly, the new definitions did not adversely affect the application of any mathematical model or method. They just allowed academic mathematicians to carry on working, effectively as normal (although actually within one new bit of mathematics), without having to worry about Cantorian inconsistencies.

Before Cantor, mathematicians had not worried about inconsistencies—and certainly not in arithmetic. But that is the measure of how paradoxical Cantor's paradox was. Before Cantor, if a mathematical innovation—like Cantor's infinite whole numbers—had led to inconsistencies, the innovation would have been rejected. But Cantor's derivation of a contradiction was so logical, mathematicians kept his innovation, and gave everything in mathematics a new definition.

Of course, arithmetic is consistent. Even the Cantorian arithmetic of infinite whole numbers is presumably consistent, it is so logical; and of course, logic is consistent. Imagine what would happen if mathematicians discovered an actual inconsistency in ordinary (finite) arithmetic. It is of course unthinkable. It would be like discovering that 2 + 2 = 5 (but with much bigger numbers). Those unrealistically hypothetical mathematicians would assume that they had made a mistake in their calculations. When they repeated their calculations and kept finding the same mistake, as they would continue to assume it was, they would conclude that some higher power (such as secret agents of a foreign government) was messing with their work. They might give up mathematics, but they would hardly conclude that arithmetic actually was inconsistent.

Axiomatic set theory was designed to save mathematicians from having to worry about some very puzzling, but fortunately very peripheral inconsistencies. So, most mathematical proofs remained essentially the same when translated into the new “language” of mathematics. Most mathematicians did not even need to know the new definitions, which could be imagined to be in a “chapter zero” of their old textbooks. Most mathematicians work within a standard set theory, but they do so implicitly, not explicitly. And you will not need to know anything about sets in order to follow the proof of this book. So do not worry if you do not understand much of this section. It is just that the invention of this new “language” was the main academic response to Cantor’s paradoxical discovery. So you ought to know a bit about it, in order to judge well the proof of this book.

Redefining their entire subject may seem like a strange thing for mathematicians to have done. But more than two thousand years earlier, the ancient Greek mathematician Euclid had written the book on replacing geometry with an axiomatic model of geometry, and that axiomatisation had worked out well enough. In the nineteenth century, mathematicians investigated geometries based on axioms slightly different to the Euclidean axioms; and a hundred years ago, axiomatic modelling was shown to be scientifically important because actual space was discovered to be, in all probability, non-Euclidean.

Euclidean space is often thought of as the natural view of space, in contrast to the non-Euclidean spacetime of modern physics. But it is, more precisely, a mathematical model of that natural conception of space. Euclidean space is a mathematical object, defined by the Euclidean axioms (axioms are mathematically precise descriptions of properties). Thinking logically about those axioms yields the whole of what was, until the nineteenth century, called geometry. For example, Euclid’s first axiom (or postulate) says that between any two points you could, in principle, draw a straight line; and his second axiom (or postulate) says that you could extend that line beyond those points, to get an infinitely long straight line. Those two axioms together mean that any two points in Euclidean space lie on such a line—which is how we naturally imagine space to be.

Sets are similarly mathematical models of collections. The axioms of axiomatic set theory say what sets are, and hence—by way of the new definitions—what everything in modern mathematics is. For example, one of the axioms of the standard set theory asserts that there is a set containing all of the new counting numbers, and the new zero. In the arithmetic that is modelled well by that set theory, there is therefore an infinite number of all the counting numbers. Mathematicians who do not believe that there are such infinite numbers do not have to use that particular axiom (it is called the axiom of infinity). But most mathematicians are implicitly using all of the standard axioms, as they build on the work of others who used them. And most mathematicians do believe that there are all the counting numbers.

Translated into the new mathematics, Cantor’s paradox became a simple proof by reductio ad absurdum that there is no new whole number of all the new whole numbers. The totality of all the new whole numbers is not a set, because the assumption that that totality is a set leads logically to Cantor’s absurd inconsistency. It does so because the new whole numbers were defined to have essentially the same properties as actual whole numbers, with this one exception (the whole point of the redefinitions being to save existing mathematics from the Cantorian inconsistencies). And since the new whole numbers are defined in terms of sets, it follows that there is no new whole number of all the new whole numbers. Cantorian inconsistencies did not arise in the new mathematics.

If you look into set theory, the first thing that you will notice is that it uses a lot of symbols. All mathematics does—although the Euclidean axioms originally used far fewer symbols. And those axioms were manipulated logically. Mathematicians usually manipulate their symbols logically. But those symbols are technically defined within a standard set theory, and the logic is technically a symbolic logic—normally classical symbolic logic, which is a very good model of ordinary logic. Symbolic logics became increasingly important in the twentieth century, originally as a result of Cantor’s paradox, but more recently because of artificial intelligence. I will have a bit more to say about them, because of their connection with Cantor’s paradox. But this is a logical book, not a symbolic-logical book.

There are very few symbols in this book. Unfamiliar symbols are strange and intimidating (like hieroglyphs and graffiti). But when we are familiar with them, they do make reading easier—which is why mathematicians use them. Reading “1931” is easier than reading “nineteen hundred and thirty-one,” for example. In themselves, mathematical symbols are no more mysterious than the use of nicknames. Indeed, the symbols for the counting numbers—1, 2, 3 and so forth—are so familiar, many of us think of them as being the numbers (although the numbers themselves are merely referred to by those symbols, just as they are by the words one, two and three). And the symbol for equality, =, is especially neat, as it is a tiny picture of two equally long lines.

The whole number zero, 0, is the number of things in something that has no things in it. Nowadays, the new “whole number” 0 is defined to be the empty set, Ø, in the standard set theory. The meaning of that symbol is given by the axioms of that theory. It has a very abstract meaning. But basically, sets “contain” mathematical objects, much as bags contain things. Sets have members, much as clubs have members, or bags have things in them.

The empty set is a set with no members, so it is a bit like an empty bag, in a story about bags. Fictional bags are howsoever their stories say they are—if a story describes a bag as bottomless, then it is bottomless, in that story—and similarly, axiomatic sets have whatever properties their axioms say they have. The existence, in standard set theory, of a set with no members is part of the axiom of infinity, for example. And another standard axiom says that different sets have different members. Those two axioms mean that there is one empty set in the standard set theory. And other axioms ensure that that set behaves just like the whole number zero, in its interactions with other sets that new “numbers” are defined to be.

The endless sequence of the counting numbers—1, 2, 3 and so forth—is naturally thought of as the product of an endless process of repeatedly adding one (having started with nothing). And the process of adding one to a “whole number” is usually defined to be the putting of the set that is that “number” into the set that is that “number.”

The whole number 1 can be defined to be 0 + 1, for example; and if you put Ø into the empty set then you get a set containing Ø. So, the “whole number” 1 is now defined to be {Ø}, in the standard set theory. It is like a bag that contains an empty bag and nothing else, in a story about bags. Those curly brackets represent sets (they are like a picture of a bag). The symbols inside the curly brackets are symbols for members of the set.

The whole number 2 is naturally thought of as being 1 + 1. And if you put {Ø} into {Ø} then you get {Ø, {Ø}}. The English word two refers to how many things there are in collections that have as many things in them as there are units in this sum: 1 + 1. But most mathematicians implicitly define 2 to be the standard set {Ø, {Ø}}. Had the symbol for the empty set been {}, the new 2 would be {{}, {{}}}, which is much harder to read. And larger numbers become very hard to read with that symbol. Whereas, symbols are all about making things easier to read.

Arithmetical facts about the new “numbers” are deduced from the axioms of the standard set theory. To prove that 1 + 2 = 3, for example, the set-theoretical adding of 1 is used in the following equations (which are laid out vertically to make them easier to see). Each equation follows from the one above it in a fairly obvious way, as listed to the right of each equation.

1 + 2 = 1 + (1 + 1)	because 2 = 1 + 1, by definition of 2
1 + 2 = (1 + 1) + 1	because 1 + (1 + 1) = (1 + 1) + 1

1 + 2 = 2 + 1 because 1 + 1 = 2, by definition of 2

1 + 2 = 3 because 2 + 1 = 3, by definition of 3

The set-theoretical axioms corresponding to each of those steps down that proof make the actual proof a bit more complicated. But even so, this set-theoretical proof is a huge improvement on the earliest twentieth-century proofs.

To prove that 1 + 1 = 2, for example, took Alfred Whitehead (1861–1947) and Bertrand Russell (1872–1970)—two British philosophers whose first degrees were in mathematics—hundreds of pages of their Principia Mathematica (published by Cambridge University Press in 1910). And of course, 1 + 1 = 2 just says that one thing and another thing make, in total, two things. Which is essentially the definition of the whole number 2.

Why did Whitehead and Russell not take 2 to be 1 + 1 by definition? Because it was just such common sense that had been leading mathematicians into inconsistency. A few years before mathematicians replaced the numbers, for academic purposes, with a slightly inaccurate model of numbers, Whitehead and Russell's 2 existed—insofar as such things can be said to exist—within the axiomatic structure of their Principia. It is no more the whole number two than any model is the thing modelled.

Axiomatic sets are mathematical models of collections. And collections are so simple, no one had seen any need to analyse them until Cantor. This is one of the big differences between the axiomatisation of geometry by Euclid—which happened long before the discovery that spacetime is non-Euclidean—and the axiomatisation of collections in the twentieth century—which followed relatively quickly upon the discovery of Cantor's paradox.

Before Cantor, properties of collections had been used in mathematical proofs, to move from one line in a proof to the next, but those uses had not been thought much about. Following Cantor, those properties were collected together. Each of those properties was self-evidently true of collections, which is why it had been used in a mathematical proof. Mathematicians described those properties using symbols because symbols facilitate generalisation, and those properties are logical properties of each and every collection. Those descriptions are the axioms of set theory.

The standard sets are defined by the standard axioms, much as a fictional character is defined by descriptions in a novel. And they give mathematicians a good model of collections, because the peripheral property discovered by Cantor is the only property not described by an axiom. Cantor's paradox is instead used to prove that the collection of all the standard sets is not itself a standard set. Standard set theory is a relatively consistent model of mathematics, but it is not an accurate model because there is a collection of all the standard sets. Indeed, mathematicians will sometimes be proving things about all the standard sets, and they are then working with that collection. The words “all the standard sets” refers to all of them collectively, and a collection of things is just all of those things being referred to collectively.

The set-theoretical proof that 1 + 2 = 3 is primarily a proof within an axiomatic set theory. It is a proof that 1 + 2 = 3 only insofar as those axiomatic sets are identified with those counting numbers, and they are only a deliberately inaccurate model of them. The real proof that 1 + 2 = 3 is much simpler than the set-theoretical proof:

1 + 2 = 1 + (1 + 1)	because 2 is the number of units in 1 + 1
1 + 2 = 1 + 1 + 1	because 1 + (1 + 1) = 1 + 1 + 1
1 + 2 = 3	because 3 is the number of units in 1 + 1 + 1

The step from the first equation to the second is justified as follows. When it comes to the question of equinumerosity—the question of whether the things in two collections can all be paired up, or not—it does not matter how the things in a collection come to be in the collection. So, it does not matter how the units are added together, in the equation 1 + (1 + 1) = 1 + 1 + 1, only that there are the same number of units on both sides of that equation.

The proof of this book is an equally real proof, because the paradoxical property discovered by Cantor is a real property of logically possible things. That is, after all, why mathematicians needed to invent set theory.

There clearly are numbers of things in the world. Collections of things are discovered in the world, not invented by mathematicians. And so their properties are discovered, not stipulated in axioms. Those properties can be theorised about, of course. And the standard set theory is a very good theory of collections of things. But while theories are perfect descriptions of theoretical entities, they are incomplete and potentially inaccurate descriptions of reality.

Analogously, fictional characters are exactly as they are described in their canonical stories, but historical figures were not necessarily as history describes them. Embarrassing facts about the biggest figures are sometimes uncovered. And at the end of the nineteenth century, there had been found to be too many logically possible collections of things—too many whole numbers—for all of our most reasonable assumptions about whole numbers to be true.

These are four words. And that word four has a clear, natural meaning. Some scientists—and many atheist fans of science—think that Cantor’s paradox shows that the modern set-theoretical meaning is more scientific. But how scientific is it, to favour the set-theoretical meaning on the grounds that the natural meaning is part of a proof of something that they do not believe? It is one thing to favour it when there seems to be no logical alternative. But as this book shows, there has always been a perfectly logical alternative. And real scientists will always have been assuming that there was a logical solution to the logical puzzle of Cantor's paradox, even if it was proving to be hard to find. So, an interesting discovery had been made, at the end of the nineteenth century. But it was far from clear what had been discovered.

Following the set-theoretical redefinition of mathematics, it was mostly left to philosophers to make what sense they could of Cantor's puzzling discovery. The paradoxical nature of infinity had been the business of philosophers since ancient times. And in the twentieth century, philosophers found several related paradoxes, beginning with Russell's paradox, which is one of many logical puzzles solved in the appendix. Following Russell, logicians investigated a wide range of revisions of logic, primarily by varying the axioms of good mathematical models of logic. With a suitable revision of logic, atheist philosophers would have been able to call "logical" any explanation of such paradoxes as Cantor's that cohered well with the best scientific theories. Since those revisions were mathematical models of logic, they certainly looked scientific. But they never did find one that they all liked. And of course, rewriting logic so that it only gives you what you find sensible was never very scientific.

By plumbing the logical depths, those philosophers had only muddied the logical water. They never did find any properly logical explanation of Cantor's paradox. And there is a perfectly logical solution to that incredible puzzle (and equally logical solutions to all the other logical paradoxes). It would make sense for there to be this puzzling fact about the sizes of logically possible collections of things—a fact that I shall shortly be describing—if the nature of things derives from their creation out of nothing but the creative power of a creator whose knowledge is able to increase.

Although there is—as we saw in the first section of chapter 2—an apparently logical argument that there should already be all the whole numbers that there will ever be, that argument contains a logical flaw if there is such a God. That is because some logical possibilities can—as you will see later in this section—become more finely differentiated over time, so that they increase in number; and the logical possibilities associated with whole numbers could like that if there is such a God, as you will see in chapter 4.

By invoking God, my explanation does use a sledgehammer to crack a nut. But Cantor's paradox has already proved itself to be extremely puzzling. It is a very large coconut of a puzzle. The details of all the failed attempts to find another logical explanation are a mass of involuted symbols. But the underlying reasons for those failures are not nearly as complex as the attempts themselves had to be, to look plausible. So, this book is able to describe in relatively plain English—but with sufficient logical detail for there to be no reasonable doubt about it—a proof that there is (very) probably a creator whose knowledge can increase.

Religious believers whose God is totally unchanging need not be too troubled by the proof of this book. They can simply believe that God is as far above and beyond our logic as He is above and beyond time. Religious belief is a matter of faith, rather than evidence; it is a cultural, rather than a scientific matter. But rational atheists are also cultural creatures (all sane humans are), and so they face the following dilemma:

They can continue to be perfectly rational—to form their beliefs by thinking logically about the available evidence—or they can continue to be atheists—to believe that there is (very) probably no God—but they can no longer be both.

Science is all about thinking logically about the evidence. So you might think that there would not be much of a dilemma here. However, atheism is a big part of the scientific worldview. And that worldview can make perfectly consistent logic look unrealistic. Much of this book will therefore be about defending logic. In the next section, we will be looking at logic and logical paradoxes more generally.

2.ii Proof

Cantor’s proof that given any infinite collection, there is a bigger collection is called Cantor’s diagonal argument, because it works by way of the leading diagonal of an infinite square matrix. But we should begin with a much smaller matrix.

The following is a vertical list of six horizontal sequences, each of which has six digits, each of which is either 0 or 1. The leading diagonal is 1, 0, 0, 1, 0, 0 (those digits are underlined).

1	1	0	1	1	1
1	0	1	1	0	0
1	1	0	1	0	0
1	1	1	1	0	1

0	0	1	1	0	1
1	1	0	1	1	0

Suppose that you want a seventh sequence of six terms, each of which is either 1 or 0.

You could try to guess one, and then check it against each of those six sequences. But it is simpler to replace each 1 in the leading diagonal with 0, and each 0 with 1. That sequence—0, 1, 1, 0, 1, 1—differs from the first sequence in the first place (and in other places, although that is irrelevant), from the second sequence in the second place (at least), from the third in the third and so forth. So, it is clearly a seventh sequence, not one of those six.

And that way of finding another sequence is not just simpler. The beauty of it is that it works just as well when we have a million sequences of a million terms each. Or an infinitely long list of sequences, each of which is infinitely long.

So, let's suppose that there are an infinite number of things (such as the counting numbers). If we consider just some of those things, we are considering a subcollection of them. And it is also logically possible—whether or not it is physically, or mentally possible—to select for consideration (or for any other purpose) any one of those things. Each of the subcollections, and each of the original things, is there, as part of that collection, and so its selection is a logical possibility. So, all of these are logically possible selections from our original collection. That collection is also a logically possible selection, since all of those things are there.

Furthermore, the selection of none of them could also be called a logically possible selection. It might be more natural to say that it was not a selection, but this is mathematics. And when classifying conic sections, mathematicians regard a point as a circle with a radius of zero, and call it a degenerate conic (proper conics are curves like circles, ellipses and parabolas). They do that in order to simplify the presentation of their proofs. And I will be including this degenerate selection for the same reason. Cantor's paradox does arise for subcollections, but it is much easier to read the proof if we consider logically possible selections.

A logically possible selection, from our infinitely many things, is, then, some or one or all or none of them. There is a logically possible selector, if the God described in chapter 4 is logically possible. So, let's consider a logically possible selection from our infinitely many things.

Each of our infinitely many things will either be in that selection, or else it will not be in it. So, in order to describe the selection, we can simply give each of the things that we started with a label In, when it is in the selection, or else the label Out.

The different ways of giving each of our infinitely many things a label that is either In, or else Out, are descriptions of the logically possible selections from our infinitely many things.

To see this more clearly, it may help to consider a simpler—a finite—example of this kind of thing. Suppose we have the first six counting numbers: 1, 2, 3, 4, 5 and 6. Each of the six sequences in the matrix at the start of this chapter could then—if we read the 1 as In and the 0 as Out—be a description of a selection from those six numbers. The first sequence, for example (it was 1, 0, 0, 1, 1, 1), would represent the selection of 1, 2, 4, 5 and 6.

So, consider the collection of all the logically possible selections from our infinite number of things. If the size of that collection was the same infinite number, then we could, in theory, pair up of all of those things with all of those possible selections. And we could think of that pairing as a vertical list of all those things, next to a vertical list of those possible selections, each of them a horizontal sequence just as long as the vertical list and composed of some combination of symbols like 1 and 0 (or In and Out).

The putative pairing has been represented as an infinite square matrix of 1s and 0s. And much as we got our seventh sequence at the start of this section, we can get from that infinite square matrix a description of a possible selection that could not be in that pairing.

That means that no such pairing is possible. And that means that the size of the collection of all the logically possible selections from our infinite number of things is not that infinite number. Furthermore, since it is a bigger collection—it contains such selections as the selection of just one of those things, for each of those things—its size is a bigger infinite number.

That is basically the same as Cantor's diagonal argument.

What Cantor originally meant by his uses of the German word menge, which is usually translated into English as set, was something like an imaginary bag, containing mathematical objects (such as other sets, or numbers). Nowadays, a mathematical set is defined by the axioms of an axiomatic set theory, and that is certainly not what we are interested in here. We are interested in objective reality, and in thinking logically about it.

But like our logically possible selections from a given collection, Cantor's subsets of a given set were some or one or all or none of the things in that set. The subsets of a given set are those sets that contain nothing that is not in the given set. For example, {1, 3} is a subset of {1, 2, 3} because all the things in {1, 3} are in {1, 2, 3}.

It is called a proper subset of {1, 2, 3}. For an example of an improper subset, note that because there is nothing in the empty set, there is nothing in it that is not in {1, 2, 3}. So, the empty set is also a subset of {1, 2, 3}. The corresponding subcollection does not exist, which may well be why mathematicians call this an improper subset. (Similarly, some mathematicians say that subsets with just one thing in them are not proper subsets, and a single thing is not a collection. And another improper subset is the entire set. The corresponding collection is not a subcollection.)

The set of all the subsets of a given set is called the powerset of that set. The powerset of {1, 2, 3} is {{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}. The following paragraph explains why the powerset has that name (if you are interested).

There are clearly eight sets in the powerset of {1, 2, 3}. And there are twice as many in the powerset of {1, 2, 3, 4}, because there are those eight sets plus the eight that you get if you put a 4 in each of those eight. And there are similarly thirty-two subsets of {1, 2, 3, 4, 5}, and so on. Now, eight is three twos multiplied together—mathematicians say “two to the power of 3”—and there are three numbers in {1, 2, 3}. And sixteen is four twos multiplied together—two to the power of 4—and there are four numbers in {1, 2, 3, 4}. And so on.

But that is enough mathematics. Let’s see the paradox. Cantor’s diagonal proof had shown that for each and every set, its powerset is bigger than it is. And in particular, the powerset of the set of all the other sets is bigger than that set of all the other sets—which is inconsistent with that set containing all the other sets, because all those subsets are sets. That inconsistency is Cantor’s paradox.

The version of Cantor’s paradox in standard mathematics—in mathematics constructed within the standard set theory—is part of a proof that there is no set of all the other sets. That proof is very simple: were there such a set, there would be an inconsistency. If a mathematician did want such a set—a set of all the other standard sets—then an axiom could be added to the standard axioms, asserting that there was such a set. This would not be inconsistent, it would just give that mathematician a new set theory, with one more axiom than the standard set theory. There would be no set of all of the other sets in that new set theory; but again, we could add another axiom to that new set theory, if we wanted to.

The standard response of mathematicians to Cantor’s paradox was, in effect, to mathematically model the endless creation of unimaginably large whole numbers. The original problem had been that the existence of a whole number is essentially the logical possibility of that many things. And logical possibilities are essentially timeless. If something is ever going to be logically possible, then it was always logically possible. That problem was side-stepped by mathematicians, who effectively replaced mathematics with their mathematical model. Basically, they redefined the whole numbers. That was fine for the mathematicians, because it did not affect their proofs (which had always used axiomatic properties) or the mathematical models used by others (which are often translated into computer code nowadays, anyway). But because of the connection with logical possibilities, there was still a logical problem to solve. There is a solution to that problem, because logical possibilities can increase in number, as you will shortly see; although that did not completely solve the problem, because this solution does need there to be a God. First things first, though.

Chapter 3 begins with a collection of things: a, b and c.

The collection of all possible selections from that collection is—if we use curly brackets for collections, and Ø for the selection of nothing—{Ø, a, b, c, {a, b}, {a, c}, {b, c} and {a, b, c}. That is clearly a bigger collection. We can now consider all of the possible selections from that collection, forming an even bigger collection. And then all possible selections from that even bigger collection. And so on, endlessly.

There are all the things in all those collections, which is infinitely many things. And there are also all the possible selections from that collection, which is an even bigger infinite collection, according to Cantor's diagonal argument. And there are all possible selections from that collection—and so on, endlessly. And there are all the things in all of those collections, and so on and so forth. Basically, there is always another, bigger collection—even when you already have an endless sequence of bigger and bigger collections.

Anything that is possible—such as a possible selection, from some possible collection—was always logically possible. So, all of the things in all of those collections should already exist (in whatever way such things exist). There should be some huge totality of all the things obtainable in that way. However, there are even more possible selections from that totality. That huger collection was obtained in exactly the same way as all the others were, so it should already be part of that huge totality, not bigger than it. But Cantor's diagonal argument shows that it is bigger.

We have a contradiction, a collection that is bigger—but which cannot be bigger—than another collection. In other words, pairing up all the things in one collection with all the things in the other is—and is not—logically possible. There are very few, if any, places in that argument where a mistake could have been made, it was such a short argument. And chapter 3 is not a long chapter. Most of it is taken up with describing Cantor's diagonal argument. So, this contradiction really does seem to cast doubt on the reliability of logic. After all, Cantor's paradox did lead to a sea change in mathematics and logic in the twentieth century.

By the end of this book, you will have seen that the only reasonable doubt is whether those possible selections were always distinct possibilities. Perhaps they become more numerous as they become individually distinct. It is relatively easy to see how that kind of thing could happen with other sorts of possibilities. Imagine, for example, a creator of things out of nothing wanting to create you, in an imaginary world in which you do not yet exist.

She begins with a perfectly detailed conception of you, and then she actualises that possibility. Basically, she makes you. Presumably she could then actualise that possibility again, making an identical duplicate of you (perhaps in an identical universe). So, there was always the possibility of you, and there was always the possibility of your duplicate—a different possibility, because you are not your duplicate.

This logically possible creator could keep on creating identical duplicates of you indefinitely. For each one that she creates, there was always the possibility of that particular instance of someone just like you. Now, if there were originally all of these distinct possibilities—one for each duplicate that she could create—then, given that there is no limit to how many she could create (there being no limit to how many parallel universes she could create), there would be an impossibly large number of them. It would be like the number of all the sets in a set theory. Could there have been some limit to her creative power? But once she had reached her limit, the duplicates that she could not create would still be logically possible creations of a different logically possible creator.

Alternatively, there were not originally all of those distinct possibilities. This logically possible creator began with the power to create someone like you, so there was originally the possibility of someone exactly like you. Then she created you, at which point she could refer to you directly, as distinct from any identical duplicate that might also create (perhaps in an identical universe). Just by creating you, there would seem to be this new possibility—the possibility of you in particular, as distinct from an identical duplicate—independently of the creation of any actual duplicates. And what would make the possibilities of those duplicates different, in their turn, would be those duplicates actually existing. They were always possible, but their possibilities become distinct from each other—and from the original, more general possibility of any such thing—when the duplicates become distinct.

At the very least, the latter option does seem to be one of the realistic options, for such perfect duplications. And as you will see in chapter 4, it enables us to have a logical solution to the logical puzzle of chapter 3. Indeed, it seems to be the only logical solution. So, we will have that reason to think that this option is not only one of the realistic options, for such perfect duplications, but is actually how such logical possibilities would be. Another reason is how unrealistic the former option—of an impossibly large number of original possibilities—seems to be; although that reason is insufficient in itself. Only logical impossibility could rule out any of these options.

The latter option was not ruled out as logically impossible by the fact that there was always the possibility of you, and there was always the possibility of your duplicate. Those two possibilities were always there, but they were originally indistinct aspects of the general possibility of someone exactly the same as you. There was originally only that possibility. There were two distinct possibilities when there was you and your duplicate. And then, once that distinctness exists, it is naturally read into the original possibility.

Consider a slightly more abstract example of the same kind of thing. Suppose that this creator wants to create a golden apple. She begins with a perfectly detailed idea of a golden apple, and then creates that apple. She could then create another, identical apple. There was always the possibility of the first apple, and the possibility of the second apple. Now, try to imagine her not creating the first apple, but only the second. That may seem straightforward. We begin by imagining two identical apples, a first one and a second one, and then we are asked to imagine just the second. So, she begins with a perfectly detailed idea of some particular apple, and then creates that apple. But that is exactly the same as the creation of the first apple. The only thing that made the second apple distinct from that one was the existence of that one. So, it seems to have been the second apple that we have imagined her not creating. It seems that these particular possibilities do only become distinct if, and when, there are distinct duplicates.

Would the apple that she made in the latter case—in the second of these logically possible worlds—be the same apple as the first one that she made (in the first of them)? To see what that question means, it may help to go back to your imaginary creation, by this fictional God. If she created a copy of you in an identical parallel universe, then that person would of course not be you. What if she had only created that universe, not this one? Would that identical person be you, or not? If we think of that person as being in a different universe, then presumably not; but if we think of that person as being in the only actual universe, then presumably that person would be you.

That is a very abstract puzzle, though. And it is not as though we need to know its answer. God knows, would be my response to such a philosophical question. Still, that question may shed a bit of light on a suggestion made by some philosophers. Note that there was always the possibility of the first golden apple that she made, and the possibility of the second one, and the third and so forth. So, there were always infinitely many possibilities—impossibly many of them, apparently; although this “impossibility” is not a problem for us, because these possibilities are defined in terms of numbers. They are defined only as well as numbers are. Anyway, the suggestion is that you would be the first person just like you—assuming that there are not actually any other duplicates of you in existence—and that your particular possibility was always the possibility of the first person just like you. That would answer the previous “philosophical” question. Although only by stipulating a convenient answer, not by discovering a true solution. And in any case, this suggestion would not work with the following, slightly more complicated example of the same kind of thing.

It is logically possible for a space to have nothing in it but three identical balls, equally spaced at the corners of an equilateral triangle; or, similarly, four identical objects at the corners of a square, or five in a pentagon, and so on. Because such objects are identical, even down to their relationships with everything else, there is nothing to tell them apart before they are created. And none will be the first to be created. When they are created, each can be individually referred to. Once they are created, there was always, for each object, the possibility of that particular object; but those possibilities would originally have been indistinct parts of their collective possibility.

So, it seems that possibilities can become more numerous, even though something that will be possible was always possible. That does not immediately solve Cantor’s paradox, because the logical possibilities that whole numbers describe—the logical possibilities of such numbers of things—are not obviously the same kind of logical possibility. Quite the contrary. How could whole numbers be coming into being?

Some mathematicians, following Cantor’s discoveries, did think of whole numbers as coming into being. As those mathematicians did their mathematics, they took themselves to be creating all the things in their mathematics. But while that is a reasonable way to think about theories—such as axiomatic set theories—it hardly does justice to the objectivity of the whole numbers. These really are five words. The counting number five exists in an objective kind of way. And there are similarly all the counting numbers that there are.

Nevertheless, the logical possibility of all things being created out of nothing does mean that there is the logical possibility of whole numbers coming into being, as you will see in chapter 4. As you will see in chapter 3, logically possible selections from collections seem, as a matter of logical necessity, to become more and more numerous. Chapter 3 is a perfectly logical “proof,” of a contradiction, from the reasonable assumption that logical possibilities are timeless. And you will see in chapter 4 that logically possible selections from collections could become more and more numerous, if they were being individuated as they were thought of by such a creator.

The Cantorian problem with whole numbers is essentially the same as that problem with logically possible selections from collections. So, we will then have a logically possible solution to the logical puzzle of Cantor's paradox. And if there are no other logically possible solutions—to the puzzle of chapter 3 (and similarly, all other versions and variants of Cantor's paradox)—then we have an argument for the existence of a God of the kind described in chapter 4.

Of course, there is always the possibility of something having been overlooked. But insofar as it is unlikely that there are any other logically possible solutions, it is likely that such a God exists. Doubts about the explanation described in chapter 4 being the only logical explanation are addressed in chapter 5, as are doubts about the reliability of logic. Many atheists will find it more plausible that Cantor's paradox shows us a natural limitation to human logic. So, the appendix also answers some doubts about the reliability of logic, by solving some relatively simple logical puzzles that could similarly be said to be showing logic to be inconsistent.

My explanation of Cantor's paradox needs only the most unimaginably huge whole numbers to be endlessly coming into being. Some mathematicians regard the counting numbers as temporal, but my explanation works either way. That is not a bad thing. It is only natural for us to defer to mathematicians, when it comes to mathematical questions. And the mathematicians are, to some extent, in two minds about the counting numbers.

Furthermore, my explanation does not necessarily need the whole numbers to be coming into being in time; they just have to be coming into being in something like our temporal dimension. As you will see in chapter 4, my explanation is surprisingly able to respect the intuition that whole numbers are timeless. The key feature of my explanation is the existence of a creator of all things.

Now, it is only natural for us to defer to mathematicians, when it comes to mathematical questions. And most academic mathematicians do assume that numbers are axiomatic sets. However, they do so for the purpose of proving things (theorems) about mathematical models (or theories), by deducing models of those things from the standard axioms. When they count things—such as the lines in one of their proofs—they use the same numbers that we all use when counting things. They are the same numbers as Newton used. Should we not assume that Newton knew what numbers are? Or have mathematicians discovered that Newton was wrong about numbers, much as physicists have discovered that he was wrong about space?

One problem with thinking that twentieth-century mathematicians had discovered that earlier mathematicians had misunderstood the whole numbers, much as twentieth-century physicists discovered that earlier physicists had been mistaken about the nature of space and time, is that while physicists can rationally doubt their intuitions about space by thinking logically about the empirical evidence, mathematicians cannot rationally doubt their intuitions about arithmetic by thinking logically about the evidence, because arithmetic is essentially the logic of there being numbers of things. It would be too much like rationally doubting reason itself.

Although, whole numbers are basically properties of possible collections, much as colours are properties of possible surfaces. And colours are not really on those surfaces—out there in the world—but are really in our minds, created by our brains as part of our perception of the world. Colours appear to be out there, in the world, but they are not really out there. So, this analogy might make you wonder whether there really are numbers of things. Could Cantor's paradox be showing that they are not?

But you and I really are two people. There really are people—and houses, cars, countries, islands, trees, birds, stars, atoms, words, colours, shapes. There are lots of things, in the real world. Some of those things are less objectively things than others (you might see three coloured objects, for example, where someone who was colour-blind saw only one). But it does not matter to the logic of this proof, how such things exist, it only matters that there are numbers of them, in the sense of equinumerosity. Which is a very logical sense. Still, a question mark was thrown over logic by Cantor's "proof" of inconsistencies in the logical properties of the largest conceivable collections of mathematical objects. And Russell, thinking about Cantor's paradox, brought a range of other logical paradoxes into play (as you will see in the second section of the appendix).

Russell was originally a big fan of logic. He had, for example, cited the universality of logic when arguing against nationalist philosophies. And he argued for atheism logically. But as mathematics turned into set theory, he came to favour replacing logic with symbolic logic.

Much as mathematicians replaced numbers with mathematical models of numbers—via mathematical models of collections, called axiomatic sets—so logicians have been replacing logic with mathematical models of logic, called symbolic logics. And classical symbolic logic is a very good mathematical model of ordinary logical thinking. But although we could discover that space was non-Euclidean by thinking logically about the empirical evidence, how could we discover that logic was non-classical, by thinking logically? We could find a logical puzzle that none of us could solve logically. But how could we know that no one would ever solve it logically?

Being mathematical, symbolic logic does look very scientific. But science is all about verifiable evidence, and its logical explanation. And if the logical pursuit of truth led scientists to an inconvenient truth, would they change logic until it gave them what they wanted? Or would they revise their beliefs, in light of what logic was showing them? Such questions are complicated by the fact that we are all cultural creatures. And the culture of academic science is essentially atheist.

Russell called the mathematical modelling of logic “mathematical logic,” naturally enough. Nowadays, symbolic logic is usually called “formal logic,” as though logical arguments in actual—or as scientific philosophers tend to say, natural—languages are bound to be informal. Whereas, it is only by using our languages that we can give meanings to symbols. The only way to get a rigorously logical argument is to put enough work into an argument in an actual language, such as English.

Logic was always about the form—the shape, or structure—of logical arguments. And mathematics can be regarded as the science of abstract structure, and so as logic became more mathematical, it was easy to think that it was becoming more logical. Symbolic logic does suit the logic of deductions, which is the logic of mathematics.

The logic of Cantor’s theorem—which is what Cantor’s paradox becomes within set theory—is deductive. In that theorem, it is deduced—using a reductio ad absurdum argument—that the collection of all the sets is not itself a set. But the logic of the proof of this book is the logic of explanations. This book is about understanding where Cantor’s paradoxical reasoning went wrong, and why it has taken this long to notice.

2.iii Deduction

There is a very puzzling paradox, called Curry’s paradox, which was originally a symbolic-logical paradox devised by an American logician, Haskell Curry (1900–1982). It was designed to be a problem for some of the earliest symbolic logics, and it remains problematic for some recent symbolic logics (such as those that allow true contradictions). The problem is to get symbolic-logical axioms from which no inconsistencies can be deduced, but which do enable all known proofs to be mathematically modelled.

More generally, logicians investigate what it is that makes logical arguments logical. “What do all—and only—logical arguments have in common?” they wonder. But the question of which arguments are logical is one that has to be decided before we can know what all the logical ones have in common. And to make an argument formal, in the sense of modern logic, is to make a mathematical model of it. And to get an actual argument from a formal argument—which is a lot of symbols—we have to give meanings to those symbols. And that meaning has to be something that can be described in an actual language, because that is how we think. Logic is actually about thoughts, which are naturally expressed linguistically, much as arithmetic is actually about numbers, of things in general.

In general, logic just gives back to us some of what we have already got. That is why logical thinking is reliable thinking. If we begin with truths, and we think logically, then whatever we end up with must be right. For example, from the fact that we are all animals, it clearly follows that any particular one of us is an animal. The logic of that becomes clearer when we lay that reasoning out as three facts, the third of which is deduced from—in other words, it follows logically from—the first two:

All humans are animals.

And the Queen is a human.

Therefore, the Queen is an animal.

And yet, calling the Queen an animal seems wrong. Does that give us any reason to doubt the reliability of logic? Of course not. We did begin with a couple of facts, and there was then a perfectly logical deduction. But what came next was just a joke. Still, you will soon be seeing a deduction that might make anyone doubt the reliability of logic. Even logicians have argued that such deductions—logical paradoxes, more of which are in the appendix—show logic to be unreliable.

Such logicians would say that it was only our natural logic that had been shown to be unreliable, much as our natural intuitions about space have been shown to be unreliable. But what such logicians call "natural logic" is what most scientists call logic. And physicists did take a logical look at the physical evidence against Euclid's fifth postulate, when they found spacetime to be non-Euclidean. And you will see in the appendix that logical paradoxes do not really show logic to be unreliable—no more than did that joke.

Which raises the question, why would any logicians have such a low opinion of logic? This book is a logical argument that there is (very) probably a God, based on some paradoxical mathematics. And the school of philosophy to which most of our philosophers belong—mainstream philosophy—began life with Russell's approach to that paradoxical mathematics. As a rule, mainstream philosophers are big fans of science. And the sciences do come from looking logically at a lot of evidence, in pursuit of reliable truths. But many scientists are atheists, and so many mainstream philosophers are too.

In the eyes of atheists, my claim that this book contains a logical argument that there is (very) probably a God must look like this: there is a logical argument that there is (very) probably a round square. If you came across an argument that there is (very) probably a round square, and you could see nothing wrong with its logic, you would certainly not believe that there were (very) likely to be round squares. You would probably assume that you had missed something. And it is similarly reasonable for atheists to at least begin by assuming that my argument is flawed. They may well not want to waste their time looking for its flaw. Even those that are reading this book will be likely, either to imagine that they have found a flaw that is not really there—lots of logical fallacies are very easily committed—or else to conclude that it was very well hidden.

Religious people are similarly likely to dismiss my argument for a God that could easily appear not to be theirs. Although there are many other starting positions, where real people are, and it is those at which this book is aimed. Now, if the best mainstream philosophers find nothing wrong with the logic of my argument, then they would certainly not want to conclude that it had beaten them. But they would be able to observe that, while we presumably evolved to have a reasonably reliable way of reasoning (or while we were presumably created in God's image), our reasoning is unlikely to be perfectly reliable, even at best. Of course, if academics doubted logic only when it told them something that they did not like, then they would not look very scientific. However, there are other logical paradoxes, superficially related to Cantor's paradox, but whose logical solutions have nothing to do with God. Those puzzles have been taken by such atheists as Russell to be evidence that our natural logic is slightly inconsistent when it comes to things like Cantor's paradox. You will be able to judge for yourselves, because the logical solutions to those puzzles are relatively simple, and are described in the appendix.

As you will see, the obvious logical solution to chapter 3's logical version of Cantor's paradox is basically that the logical possibilities that whole numbers describe—logically possible collections—become more and more finely differentiated or individuated, endlessly. Possibilities can be like that, as we saw at the beginning of this chapter. And logically possible collections could be individuated as they were thought of by a God who is able to change. Given that all actual collections of things would be created out of nothing but God's creative powers, were there a God, there is nothing very mysterious about the individuation of the logically possible collections by such a God (how things could be created out of nothing is more mysterious). Whereas, it is inconceivable that logically possible collections could become ever more individuated by themselves. That is why this can amount to a proof that there is probably a God.

Philosophers at the beginning of the twentieth century naturally looked elsewhere for a solution to Cantor's paradox. And there are arguments that could make you doubt the reliability of logic, such as the following versions of Curry's paradox. First, though, I ought to run through how that joke about the Queen worked. It is not so much that there might not have been a joke; rather, the way that joke worked is related to the ways in which those arguments fail to be perfectly logical. Those ways all involve linguistic subtleties.

At the start of this chapter, there were two facts, but there was also the language in which they were expressed. We naturally read the words “all humans are animals” as saying that we are all members of the animal kingdom. That is because it is with that reading that those four words are most clearly true. When we read words, we subconsciously give them meanings in such a way as to make the most sense we can of them.

We do the same kind of thing when we hear words (which is how we were able to learn our first languages). And we do something similar when we recognise familiar faces. Now, many of us find it harder to recognise familiar faces when we see them out of their usual contexts. When it comes to recognition, context is important. So, we naturally read the second sentence—the Queen is a human—as another biological fact.

Given those two facts, it follows logically that, in the same biological sense, the Queen is an animal. And the reason why “the Queen is an animal” seems to be to saying that the Queen is beastly is that that is the normal meaning of the words “is an animal” when that words is applied to a person. Even though the Queen is not, in this suddenly salient sense, an animal—even though the very idea is absurd—normal meanings of words and wordss are, of course, the biggest factor in how we subconsciously make sense of words. At the same time, the two facts being biological facts, and their context being a logical deduction, we have the appearance of a proof. So, we have a “proof” of an absurdity—a logical joke—as well as a logical deduction.

There are several “proofs” of untruths in this book. They are the natural setting for Cantor’s paradox, which is a seemingly logical derivation of an impossibility from some simple arithmetical facts. This book began with versions of the liar paradox—seemingly logical derivations of impossibilities from mere words—and some very similar words support another kind of paradoxical reasoning, an English language version of Curry’s paradox. Let’s begin with an obvious fact about truth.

If someone says that snow is blue, and I say “that is true,” then I am clearly agreeing that snow is indeed blue. That is the meaning of “is true,” in such contexts. Given any claim, these two are effectively the same: Firstly, the claim being true; and secondly, whatever it was that the claim was claiming being the case. Now, consider the following claim:

If this claim is true, then there is a God.

What does it mean, for that claim to be true? It means that what that claim is claiming is the case. And what that claim claims is that if that claim is true, then there is a God. So, if that claim is true, then it is the case that if that claim is true, then there is a God:

If that claim is true, then if that claim is true—which it would then be—there is a God.

So, if that claim is true, then there is a God.

So, what that claim claims has been found to be the case. In other words, the claim is true.

So, we have just found the claim to be true, having earlier found that if it is true then there is a God. And from those two findings, it follows that there is a God.

Does that prove that there is a God? Of course not. And it is easy to see why it did not. We need only consider claims like these:

If this claim is true, then pigs fly.

If this claim is true, there is a round square.

They are essentially the same as “This claim is not true,” but they are also very similar to “If this claim is true, then there is a God.” So we can reason about them in the same sort of way.

If “If this claim is true, there is a round square” is true, then what it says is the case:

If the claim is true—which it would then be—there is a round square.

So, if the claim is true, there is a round square. (Let us call this One.)

And because that is what the claim claimed, the claim is true. (Let us call this Two.)

So, the claim is true (from Two), and if the claim is true, there is a round square (from One), which means that there is a round square.

It being impossible for there to be a round square—and similarly, it being a fact that pigs do not fly—this “proof” that there is a round square must have contained a mistake. Whatever that mistake was, a similar mistake will presumably have been made in the earlier “proof” that God exists, because those two arguments clearly had the same logical structure.

Now, that might seem to be a problem for me. This book as a whole describes a proof that there is (very) probably a God. And I hope that every bit of that argument will seem perfectly logical. I would like you to conclude that there is (very) probably a God. But have I not just given you a reason not to conclude that? Those two “proofs” seemed perfectly logical, did they not? And yet a round square is not even possible.

But in fact, the proof of this book is based on there being a solution (in chapter 4) to the puzzle of a similar “proof” of a contradiction (in chapter 3). And you will find a solution to the puzzle of those two “proofs” in the appendix. Curry’s paradox is not really a problem for me, because I am not trying to pull the wool over your eyes. I want you to see how different the proof of this book is, to “proofs” that there is (probably) a God.

What should already be surprisingly clear is that you do not have to be crazy to wonder about the reliability of logic. Those two “proofs” did look perfectly logical. And you might think that that must amount to some evidence that they were logical. However, we should be very reluctant to see those “proofs” as providing any evidence against the reliability of logic. If logic was unreliable, then there would be no reliable way in which to think about that possibility. Still, you now know (if you did not already know) that we can be told untruths by arguments that look perfectly logical. It will therefore be harder for you to trust any argument—even good ones—until you solve this puzzle.

Mathematicians must have felt like that about Cantor’s paradox, at the beginning of the twentieth century. They needed to know why the puzzling inconsistencies were arising, so that they could be sure of the mathematics that they were doing. They have now been working within a standard set theory for a hundred years, presumably because they have had good reasons for thinking that standard set theory is a very good model of mathematics. And the solution of this book, in chapter 4, is well modelled by the standard set theory. All that is to come, though. Let’s try to resolve the puzzling “proofs” of this section.

If logic is reliable, and if all the steps that followed the claim “if this claim is true, there is a round square” were logical steps, then there must have been something wrong with that claim. And it is like saying “this claim is not true.” Since there is a solution to the liar paradox at the end of this book, let’s put this English version of Curry’s paradox on hold until then. Until then, at least, we should be assuming that logic is reliable, and that we can think logically enough if we try hard enough.

To get from the facts of any matter, to the truth of that matter, we need to think logically. Logical thinking is reliable because it gives back to us some of what we already had. But it does more than tell us what we already know. What it gives back to us can help us to focus on where our thinking can be improved, and so it can help us to improve our explanations of things. In the next few chapters, you should be able to see how the probable existence of God follows logically from the existence of these words. And that will shed new light on old problems in physics.

You should not be too surprised to find the probable existence of God following logically from the existence of such things as the words in this sentence, precisely because logical thinking just gives back to us some of what we already had. From the existence of such words, it follows logically that there is probably a language, and hence a sophisticated social structure of language-users—and hence, in all likelihood, whatever it is that makes the existence of such people most likely. And so it is not too odd that the probable existence of God should follow logically from the existence of such things as words.

Most of us nowadays would expect logic to say that there is probably no God. That is because people tend, insofar as they are logical, to think that there is no God. For example, although four in five Americans believe in something like a God, it is the other way round for scientists at the best American universities. About four in five of them believe that there is probably no God.

When scientists use a word like probably, they tend to have in mind a rather narrow range of fairly high probabilities. Scientific witnesses in a trial, for example, can usually put figures to their small chances of being wrong, because they will have been working with just such figures. But the use of the word probably in “there is probably no God” is more like when a juror sees that a defendant is probably guilty.

Both scientific work and jury duty are a matter of trying to see what the evidence itself is showing. With or without figures, they are a matter of logic, not opinion (for all that we all begin with all sorts of prejudices). If the evidence in a trial shows that the accused is, not just probably guilty, but clearly guilty—a probability greater than 90% is sometimes suggested—then the jury should find the accused guilty. Because that is a serious matter, the jury should deliberate as logically as possible. They should pursue the truth of the matter.

The same is true of scientific research—and also philosophical investigations. Scientists are obviously a bit like detectives. Philosophers can seem more like lawyers; but to read this book is to pursue the truth of the matter of the existence of God, by examining one particularly compelling piece of evidence, the version of Cantor’s paradox that is chapter 3. The question of the existence of God—in the sense of a creator of things in general—is primarily a question about the nature of reality. It is more like a scientific question, than a religious question.

The question of whether there is (very) probably a God, or not, is a question like this: Was each thing made to be the way it is deliberately, or are most things the way they are unintentionally? That is a question about the nature of things in general—a metaphysical question. The see what metaphysics is, it may help to consider the following list, which is in order of increasing generality of subject matter: Criminal profiling, psychology, biology, physics, metaphysics. Physicists try to discover the laws of physics, and philosophers doing metaphysics might try to clarify the meaning of “laws” in that context. Or they might try to say, as clearly as possible, what theories of space and time are theories of. Or what the meaning of “theories” is, in that context. And so on.

A lot of questions are asked in metaphysics, much as they are in mathematics, which is almost as general. An object of any kind, together with any other thing, are two things. That is just what the word two means. But any sort of pattern can be studied in mathematics. And logic is even more general. Logic is the study of how we should think, about anything. Logic usually means deductive logic, the study of deductions from facts. But it can also include the study of explanations of facts, such as the scientific method. Scientific explanations aspire to be as accurate as deductions, if only in their estimates of their chances of being wrong. But that is only an aspiration. Explanations are inevitably a matter of imprecise likelihoods.

That is just the way of scientific explanations. Consider, for example, how natural the Euclidean view of space is. It was regarded as self-evidently true, for thousands of years. And by the end of the nineteenth century there was, moreover, so much scientific evidence for the truth of the Newtonian view of the solar system—in which space is Euclidean—that it seemed to be a scientific fact. Nevertheless, in 1919 some observations of mercury showed that the physics of the Swiss physicist Albert Einstein (1879–1955)—in which spacetime is non-Euclidean—was correct. So, space had only ever been probably Euclidean, even if a lot of thinkers had taken its being Euclidean to be a matter of logical necessity. And so now, we could only (at best) know that spacetime is probably non-Euclidean. The huge amount of evidence that now exists for Einsteinian physics may well make it quite mysterious how the Einsteinian view of spacetime could possibly go the way of the Newtonian view of space. But it was no less mysterious at the end of the nineteenth century how space could possibly fail to be Euclidean. And there are some intriguing mysteries in physics nowadays, just as there were at the end of the nineteenth century.

2.iv Explanation

For a much simpler example of an explanation, imagine that you are in a disused warehouse, looking at half a brick. It is surrounded by shattered glass, underneath a broken window. The hypothesis that it broke the window would explain your observations—up to a point. You have no idea why it would have been thrown at the window. Still, you have no reason to doubt that that is what happened. So, you naturally assume that the half brick broke the window.

Consequently, you have no reason to examine the warehouse. But if you did look around, you might find other halves of bricks surrounded by shattered glass, and none of them underneath broken windows. Our half brick could be part of a work of modern art. Or, an examination of the glass might show it to have been shattered by a bullet, indicating that the half brick was put there later, to conceal a shooting. There are lots of theoretical possibilities. But if you did look around, you would probably find little more than dirty rubble and a few beer cans. The window was probably broken by the half brick. Had you examined the warehouse, and found nothing of any significance, you could take yourself to be knowing that the half brick broke the window. We could say that it very probably broke the window. But for you to have wanted to examine the warehouse, you would have had to have been investigating something a bit more serious than a broken window.

This book presents some damning evidence against atheism. Atheists are unlikely to see it that way, of course. Many of them will think that because there might be other explanations of chapter 3 (a logical version of Cantor's paradox) than chapter 4 (a God hypothesis), nothing has been shown either way. But atheists have been looking for atheist explanations of Cantor's paradox for over a hundred years, and their failure to find a logical explanation has to mean something. The reason why chapters 3, 4 and 5 (which surveys the range of alternative explanations) have to mean something can be seen most clearly by way of analogous scenarios.

Imagine, for example, a scientist assessing the evidence for evolution, and concluding that evolution is a fact, that it very probably occurred by means of natural selection, and that there is probably no God. I point out that there might be other theories of how evolution occurred. There is a lot more evidence out there, I say, and some of it would paint a different picture. My words are already covered by the word "probably" in what the scientist said, though. So, they carry no weight. Similarly, a defence lawyer who, following the presentation of some damning evidence by a prosecuting counsel, simply observed that there is probably a lot of other evidence out there, and that some of it would paint a different picture, would hardly convince a jury to ignore that evidence.

You had no reason to look around the warehouse, in that fictional scenario. But having noticed one broken window, you notice that the next window along is also broken. Beneath it, surrounded by shattered glass, is the other half of the brick. Continuing on, you see when you get outside that one of the window frames has been painted red, the other blue. And over them an oversized spectacles frame has been fixed to the wall. A sign underneath says "The Spectacle of the End of Society." Searching that words on your phone, you find this to be a work of situationist art by Howard Uno, who had used the half bricks to break the windows. Your hypothesis was true (unless what you read online was untrue), although not in the way that you had imagined.

There are lots of possibilities, some of them even more unlikely. Aliens, for example, are not impossible. The windows might, just possibly, have been broken when an alien fired a ray-gun at another alien. Covering up signs of their presence with “art” might be their standard procedure (as exemplified by crop circles). And there are so many other, highly unlikely possibilities that they cannot all be thought of, let alone ruled out. Indeed, how do you know that aliens are highly unlikely? Any evidence you have for its unlikelihood could, in theory, have been produced by these powerful aliens, as part of their cover up.

On the other hand, even if an alien ray-gun had broken the windows, and you had made a thorough examination of the glass and seen how strangely it had been broken, you would not have acquired any real evidence for aliens. You would only have got some evidence that human scientists had been using the deserted warehouse to test a secret device. UFOlogists may well disagree, but it would be perfectly rational, in the absence of any real evidence for aliens, for us to simply ignore that possibility.

Whether a claim is extraordinary or not is to some extent a cultural (and subcultural) matter. But it is of course not the case that anything goes. Juries, for example, are quite capable of pursuing the truth, despite their varied memberships. And a jury will naturally ignore the logical possibility of aliens. The unquantifiable possibility that some secretive and very competent power has been interfering with evidence, for their own unfathomable reasons, without leaving any sign of their presence behind, is just not a reasonable doubt for a jury to entertain, whether that power is alien, angelic, demonic, foreign, political or commercial.

When scientists are looking for a scientific explanation of their observations, they can usually ignore the logical possibility of God. The metaphysical hypothesis that all things were deliberately created is certainly not an ordinary scientific hypothesis. But as described in chapter 5, the hypothesis described in chapter 4 is the only logical resolution of the version of Cantor’s paradox that is chapter 3. That makes it unreasonable to ignore the logical possibility of God—much as a spaceship arriving from outer space, in clear view of all our telescopes, would make it unreasonable to ignore the logical possibility of aliens.

The idea that physical things might come into being out of nothing physical is certainly extraordinary. Normally, mass and energy just change from one form to another. Engines turn the energy of their fuel into the energy of motion, for example. The sum total of all the energy (including the energy equivalent of mass) in the universe remains constant. This is one of the conservation laws of physics. Nevertheless, there is a lot of evidence that the universe has been expanding, for billions of years, out from an original concentration called the Big Bang. Before then, there may well have been none of that mass or energy. That would not violate the law of the conservation of energy, it would just mean that that law did not apply beyond this universe.

Or, there may originally have been some sort of potential for the Big Bang. What we do know is that there are physical objects and people, which gives us two basic possibilities for that original potential: an impersonal one and a personal one. The hypothesis that reality is fundamentally impersonal is commonly called materialism, while theism—the God hypothesis—takes reality to be fundamentally personal.

Atheists would certainly agree that it would be wrong to rule out materialism just because it is odd. And it is odd to think of people as being nothing but chemicals. We do not naturally think of each other, let alone ourselves, as being biochemical robots. Nevertheless, materialism is a reasonable metaphysical hypothesis, at least to begin with (we may find other reasons to reject it). And it is admittedly no less odd to think of the universe being created by an extraordinary person, out of nothing but that person's ability to create. Ordinary people cannot create mass/energy out of nothing. Nevertheless, just as materialism is a reasonable metaphysical hypothesis, so is the God hypothesis.

Most atheists think that there is (very) probably no God. So, they do think that God is a logical possibility. Some have argued that God is not a logical possibility, but their arguments have never shown that the kind of God described in chapter 4 is logically impossible. Most were aimed at the classical conception of God. You will get the general idea from a typical example, and the following argument against the logical possibility of a God has been around for about a thousand years. Can God make a stone that is too heavy for anyone—including God—to lift? If not, then there is something that God cannot do: make such a stone. But even if God could make such a stone, there would be something that God could not do: lift that stone. Either way, there is something that God cannot do. Whereas, most religious believers would say that there is nothing that God cannot do.

This book describes a reason why the world was (very) probably created out of nothing but the creative power of its creator, who is able to change. Such a God can make an entire universe, obeying any laws desired. There is, in that sense, nothing that this God cannot do. And in particular, no stone would be too heavy for God to make it rise up, if God wanted it to. Not being able to make a stone that was too heavy for God to lift would seem to be a reflection of God's power, not a limitation of it. Although perhaps such a God could also create someone, in the universe, who God identified with, in some essential way. And a stone could perhaps be too heavy for that person (who was essentially God) to lift. Either is compatible with the existence of the God that this book shows (very) probably exists.

The main thing, so far as this book is concerned, is that only a God could individuate the paradoxical possibilities of chapter 3; whether such a God could also do other things is hardly our concern here. What does matter to the (very) probable existence of such a God is that none of the scientific data is less likely under that God hypothesis than it is under such alternative hypotheses as materialism. Now, I can hardly go through all of that data here; but one fact does stand out: Is the existence of the God described in chapter 4 not made less likely by all the evidence for our evolution by some process that certainly looks a lot like natural selection? Natural selection is a very bloody business.

Surprisingly, the obvious nastiness of the world is actually some evidence that we will probably survive death, within a wider world that is probably a lot better than it looks. That there is such light at the ends of the tunnels of our inevitable deaths will certainly surprise those who take the obvious nastiness of the world to be showing the world to have no responsible creator: Why would a good God create people who would choose to do evil things? Surely a good God would create good people. And why would a good God give those good people poor lives? But a lot of people do live poor lives, not least because people do evil things.

That is called the problem of evil. But while it can be a problem for religious belief, because religious believers say that God is good, it is not a problem for the proof of this book. This book shows logically that the world is (very) probably created out of nothing but the creative power of a creator who is able to change, and the logic of that proof is unaffected by whether that creator is good or not. Although there are reasons to think that any creator of this world out of nothing would be good, in some objective sense. If we knew all about creation and we thought it evil, but its creator thought it good, why would it not be us that was getting that wrong?

I should add that if we knew more about creation, then it might not seem so evil anyway. Solutions to the problem of evil often take the form of stories that show ways in which the creation of a good God could be like the world as we know it. Analogously, primitive humans naturally regarded the world as flat. They could see that it was. But we now know that it is roughly spherical, as a matter of physical fact. The evidence for that fact trumps our everyday experiences, and that evidence was always logically possible. And there are lots of ways in which the creation of a good God could be like the world as we know it. If you think that the world being much as it is described in some such story would indeed solve the problem of evil, and if you find no other solutions satisfactory, then you might rationally think that the horrors of life on earth amount to some evidence for the truth of something like that story, given the proof of this book.

Whether it would amount to much evidence is a matter of opinion. There are a lot of different stories, just as there are a lot of different ideas about what is wrong with the world. But for example, the following is one possibility. There is, to begin with, no problem of evil standing in the way of a good God creating good people in a good world, of course. So, let's suppose that that is what God originally did. There will then have been lots of people in a spiritual, rather than a physical world. And only good things will have been happening to them there.

Nevertheless, some of them might have wanted to spend a very limited amount of time in a less heavenly world. God would presumably have facilitated that, had it been a good idea; and for such reasons as the following, it could, just possibly, have been a good idea.

God transcends creation—even the heavenly parts of it—much as authors transcends their stories, and so the relationships that created people could possibly have with their creator would be quite limited. Nevertheless, those important relationships might be improved if these people spent some time in a world that was even further removed from their creator. Now, we cannot possibly know much about such things (those of us who are not saints). However, this certainly seems to be a logical possibility.

And this world of ours certainly seems to fit that bill. Biology is almost entirely reducible to physics, and physics seems to have no need for the God hypothesis. Furthermore, the relationships that these people have with each other might also benefit from their being briefly in such a physical world. From their heavenly perspective, it might be like going camping, or going on an adventure holiday.

For one reason or another, some of these good people might have chosen, in a well-informed way, to be incarnated for a relatively short time in a much less heavenly world. Now, it could be that things would not go well for all of these people, especially the more adventurous ones. Some of them might even end up becoming evil. However, a God could have guaranteed that all of these people would end up at least as well off as they started. Appropriately therapeutic worlds could have been included in creation; indeed, this world of ours might be therapeutic for some, an adventure holiday for others.

The main thing, so far as this logical possibility is concerned, is that we might all—in previous lives—have chosen wisely to be here, or to be somewhere that had led to us being here for equally good reasons. The fact that we do not remember choosing to be born is no evidence against this hypothesis, because few of us can even recall being babies. Indeed, all the known facts are consistent with this hypothesis. Whereas, the logical fact described in chapter 3 is probably inconsistent with atheism.

Mainstream philosophers, following in Russell's footsteps, seek to explain the scientific view of the world. And our best scientists do tend to think that there is probably no God. But for many atheists, atheism is a cultural belief, a matter of social identity rather than scientific evidence. If you were to ask them what would be good evidence for the existence of God, they might suggest something like a walking and talking statue of Jesus telling us which is the correct denomination of Christianity. The nonexistence of anything like that might account for the confidence that most of them feel about there being no God. But as we saw in chapter 1, even such an extraordinary spectacle as the words "God exists," written in giant letters of unearthly fire, would be no more evidence for the existence of God than things not being where we expected them to be would be evidence for the existence of mischievous pixies.

To see the logic of this more clearly, suppose that we have some monkeys in a cage with some typewriters, and that they type a play that could have been written by Shakespeare. It would be natural for us to think that the pages that they had actually typed had been swapped for the pages of that play. If we could rule that out, we might wonder if the monkeys had been trained to type the play. And if we could rule that out, then perhaps some nanotechnology introduced into the monkeys' brains and/or arms and fingers was to blame. It would of course be impossible to rule out alien nanotechnology that was so advanced, it could even infect our detectors.

In olden times, such monkeys would have been presumed to be possessed by demons. How will scientists react to a perfectly logical proof that there is probably a God? They may well consider such a proof to be the business of mainstream philosophers. However, because of the underlying paradox, mainstream philosophers have been rejecting logic, in favour of symbolic logics. They do it in defence of the scientific worldview. But scientists form their worldviews logically not symbolic-logically.

Scientists might already have chosen logic over atheism, had they not been deferring to mainstream philosophers in such matters. And then, mainstream philosophers would presumably have chosen logic over atheism too. Those who have a scientific view of the world might therefore think of this book as correcting a philosophical mistake that may well have arisen for sociological, rather than logical reasons. As you will see, in chapters 3 and 4, Cantor's paradox had presented twentieth-century academics with pretty good evidence for there being a God. It was a bit like the way in which aliens arriving in spaceships from outer space would present us with good evidence for there being aliens. Naturally, cultural factors complicated that presentation.

At the end of the nineteenth century, classical theism—a collective name for theologies that take God to be absolutely infinite and unchanging—was losing ground to atheism, in the cultural background of academic science. So, although classical theists—such as Cantor—could explain Cantor's paradox in terms of God's ineffability, the paradoxical fact was mostly a problem for atheists. And they could explain it in terms of our being highly evolved primates. Why would natural selection have resulted in our being able to understand such abstract mathematics? During the twentieth century, the mainstream philosophy of Cantor's paradox got very complicated. But this book is not about any of that. Nor is this a religious book. This book is about the relatively simple fact that Cantor's paradox does indeed present us with a choice between logic and atheism, because there is a genuinely logical explanation of Cantor's paradox, for all that it invokes the logical possibility of a God of a relatively logical kind.

The logic of explanation is much simpler than the history of ideas, although not as simple as deductive logic. One of the complications is nicely illustrated by the possibility of aliens having already arrived and hidden themselves. Is that unlikely, because nothing in the universe can travel faster than light, and even the nearest galaxies are millions of light-years away? But such secretive and powerful entities could conceivably have misled us about all sorts of things. It is logically possible that, in order to make their presence here seem less likely to us, they fabricated the evidence that nothing in the universe can travel faster than light (which is basically just a few observations that are difficult to make, plus a lot of very complicated mathematics). For such reasons, it is impossible to know just how unlikely such secretive aliens are. We simply assume that they are not here.

How can it be rational for us to treat unknown likelihoods as though they were zero probabilities? Now, that is a problem for mainstream philosophers to solve. Presumably it is rational to so treat them, and that is all that we need to know here. None of the scientific evidence for that universal speed limit is actually thrown into question by this possibility. It is not a reasonable doubt for scientists to entertain. The philosophical problem is known as the problem of Cartesian doubt, or Cartesian scepticism. It is named after the French mathematician and philosopher René Descartes (1596–1650), as are Cartesian coordinates, which you may be familiar with from school mathematics, incidentally. The problem of Cartesian doubt is made worse by there being a lot of other possibilities that seem, in similar ways, to give us cause to doubt our knowledge. That makes it worse because adding up a lot of even very low probabilities might result in a probability that is not low. So, even if logicians could show each one to be very unlikely—and that does not seem to be possible—they would be no better off anyway.

In practice, we simply ignore each and every one of these possibilities, when we are trying to explain things, which does add up to our ignoring them all. Cartesian doubts are not reasonable doubts, and so when it comes to explaining things logically, we can simply ignore a Cartesian doubt, on the grounds that it is Cartesian. That is relevant to the proof of this book, because the possibility that we evolved to have an imperfectly consistent natural logic does seem to be a Cartesian doubt. It would certainly not ordinarily be a reasonable doubt. Imagine a defence lawyer suggesting to a jury that they ignore their logical deliberations about the evidence—evidence that clearly indicates the guilt of the accused (when it is considered logically)—on the grounds that, because the accused should be presumed to be innocent, they ought to assume that logic is simply revealing one of its natural imperfections. Such a lawyer would be laughed out of court. Still, it might be different for an extraordinary proof. So, this matter is considered further in chapter 5.

One of Descartes' examples was that you might, for all you know, be having a very vivid dream right now. Perhaps you are in a coma—people do fall into comas. Or do you think that dreams are never like your current experiences, so that you can know that you are not now dreaming? But if this was a very vivid dream, then your recollection of what dreams are like would be a recollection in a dream, and would therefore be suspect. Now, if you were in a coma, then the objects that you see around you would not be real objects, and so a lot of your beliefs about them would not be true. That would not be your face in the mirror, for example. Whereas, you naturally assume that it is.

What if you were having an extraordinary experience? Would it then make sense for you to wonder if you were dreaming? But why assume that real life cannot occasionally be strange? We would hardly be able to learn much about the world if we did. It is, moreover, precisely in such testing times that things should be taken seriously. It never makes sense to wonder if you are dreaming.

A similar example of a Cartesian doubt is that you could, just possibly, have been hypnotised, by a hypnotist who has made you forget that people can be that good at hypnotism. Since any of our common-sense beliefs could, in theory, have been put into our heads by such a hypnotist, how could we hope to justify the common sense that this has not happened? That is an academic problem, for logicians to address, not a reasonable doubt to have about any of your common-sense beliefs.

Another one of Descartes' examples was that an evil demon might have bewitched you. A modern version can be found in the 1999 movie The Matrix. In that movie, humans lived simulated lives, while their bodies were kept alive in vats. A vast computer fed data directly into their brains. Since they shared the same simulation, this is different to the Cartesian sceptical scenario in which you are dreaming. So, how do you know that you are not in such a simulation? Your data on how unlikely that is would be generated by an unknown computer, were this the case, so it would not be reliable data. We naturally assume that that is not the case—indeed, you could argue that the word reality refers to the world around you, whatever the nature of its underlying substance(s)—but some people have argued, as follows, that they are probably computers.

If purely material objects like brains can have feelings—as many scientists think they can—then purely material objects like the very complex supercomputers of the future probably can too. The universe being so very big, there are likely to be a lot of artificial intelligences. And many of them may well be living simulated lives, as there would be little else for them to do. Since some of those lives might be just like these lives of ours, and since there are likely to be a lot of them—the universe being so very big—hence, this argument concludes, we are unlikely to be the real thing. But the problem with that argument is that it undermines itself. Its probabilities are based on data that would be suspect, were we living in a simulated environment.

For a more unusual Cartesian sceptical scenario, consider how there might, for all we know, be parallel universes that are exactly the same as this one. This universe of ours is certainly physically possible. And an extra dimension that we have not noticed from within this universe—there being no motion in that dimension, within this universe—is certainly a logical possibility. So, there might be other universes that are exactly the same as ours, off in that extra dimension. And it is also logically possible for objects to be instantaneously swapped between these universes. There would be no physical consequences of such a swap, so we have no evidence that such swaps do occur. But nor do we have any evidence that they do not. We naturally assume that they do not happen—so we are, in effect, assuming that they are very unlikely—but there is no evidence either way.

To see that we do assume that they do not happen, consider what would happen if I was swapped for one of my duplicates. There would be no physical difference, of course. And my duplicate would not notice that he was suddenly in this universe. But if my mother saw my duplicate waiting at a bus-stop, she would not know that I was waiting for a bus, because it would not have been me.

The problem for logicians is to explain how, when it is me who she sees waiting for a bus, she can know that I am waiting for a bus. She is not just assuming that it is me, when she can see that it is. But it does make sense that, in order for us to be able to gain any knowledge, about anything, we need to be able to recognise things as the same things in different situations. How else could we acquire knowledge about them? Perhaps that is why we naturally assume—unless we actually have evidence to the contrary (and perhaps even then, since it never makes sense to wonder if you are dreaming)—that things remain the same things.

And similarly, it is likely that we have to assume that logic is perfectly consistent, in order for us to be able to think properly about anything. If we start with the facts, then thinking logically about those facts will only ever take us to truths. Logic enables us to avoid problems because it helps us to keep our thoughts realistic. Logic is a way of being very careful, and so it can get rather tedious, especially if we are thinking about the bigger picture. So, there is a well-known shortcut.

We simply give a fair amount of credence to the views of the professionals at thinking logically, about the bigger picture. Now, most academic biologists and physicists are atheists – and for good reason. There is little or no physical evidence of anything supernatural; whereas, the biological and paleontological evidence supports very strongly the theory of evolution by natural selection. To cut a long story short, the culture of academic science is essentially atheist. Which is a pretty good reason why you should be an atheist.

However, all that scientific evidence is consistent with there being some sort of God. It is just that there has not been—until now—any genuinely scientific need to postulate the existence of any kind of God, to explain any verifiable data. Data itself says little or nothing about the stuff whose properties the data describes. And the academic scientists themselves tend to say that there is (very) probably no God, not simply that there is no God. And that fine distinction turns out to be crucial here. It is not that there is any actual evidence for there not being a God. Whereas, assumptions that are ordinarily unquestionable—and which are almost certainly true if atheism is true—can be taken logically to conclusions that are not consistent with each other, conclusions that cannot all be true. Since logical thoughts always take truths to truths, it follows that atheism is (very) probably not true. Which is a very logical reason why you should not be an atheist.

People do have reasons other than logical ones for their beliefs. For example, they might be very loyal to the culture within which they grew up. But academic scientists should certainly prioritise thinking logically, over any particular belief. As a result of the following chapters, the culture of academic science will therefore have to move away from atheism. And the question of what should replace it is logically a metaphysical question (rather than a religious, a cultural, a political or even an economic question). So, the following chapters should lead to greater academic interest in metaphysics.

3

Extraordinary Evidence

3.i Logical Possibilities

Howard reenters the picture, juggling an apple, a banana and a carrot. He is juggling them well enough, but eventually he will drop something. This time the question is what will he be juggling after a minute? There are eight possibilities:

Firstly, he might still be juggling all three of them.

Secondly, it is only the apple and the banana.

Thirdly, it is the apple and the carrot.

Fourthly, the banana and the carrot.

Fifthly, just the apple.

Sixthly, the banana.

Seventhly, the carrot.

And finally nothing, if he dropped all three.

Those possibilities are logical possibilities. For all we know for sure, the world might be deterministic; it might be like a machine that always works like clockwork, and only one of those eight logical possibilities is physically possible. Or perhaps you know just how good at juggling Howard is, and you are confident that only the first of them is a real possibility.

But in any case, we might be on firmer logical ground if we begin with the logical possibility of something that actually is the case, and Howard is a fictional artist. And there really are three pieces of food in a bowl in front of me. There is a red apple, a yellow banana and an orange carrot, in an off-white bowl that is shaped like a shell. The world as a whole is a bit like that bowl.

Still, I guess that you only have my word for that. And a juggler is obviously as possible as a bowl of fruit. But we are headed for a very paradoxical contradiction, so we will soon be looking for anything that might, just possibly, not be true.

This chapter is a description of a very logical derivation of a contradiction. We shall be assuming nothing but apparent truths. Not even the truth of atheism will be explicitly assumed, so that we will seem to be finding a true contradiction. But we can of course rule that possibility out. When a certain kind of God is described in chapter 4, it will become clear where the contradiction comes from. It comes from assuming the non-existence of such a God. But to begin with, the more that the conclusion of this chapter seems like a true contradiction, the more justifiably confident you can be that there was nowhere else for that contradiction to have come from.

We begin with some things. They can be any things at all. About those arbitrary things we shall be assuming only what we would never ordinarily think to question. And we will be assuming some obvious properties of collections of things in general (such as would never have been questioned had it not been for Cantor's paradox). And yet, with relatively few, very logical steps we will have reached a contradiction. We will therefore be motivated to uncover and thoroughly question those assumptions. But it would be very tedious to begin with such doubts. This being so paradoxically, most of them will seem like Cartesian doubts (are there things, do possibilities have to be logical, can humans be logical and so forth).

We might begin with three words. They most clearly do exist as distinct things: each word is distinct enough from every other word for you to be able to read these words. Even more simply, we could begin with the letters A, B and C. Written English requires that such letters are clearly distinct from each other—just like words are—and the following argument will use only such facts about A, B and C as that they are distinct from each other.

But we are heading for a paradoxical contradiction. And given such a logical derivation of a contradiction, we might wonder if it is showing us that the things that we began with do not really exist. And it is a fact that words, letters and, similarly, numbers (such as those that Cantor began with) do not exist in the way that, for example, rocks do. We could begin with a single grain of sand, and logic would give us a fact about infinity that was so paradoxical it was incredibly meaningful. But it will be simpler if I let A, B and C be names for our three items of food.

Let A be a red apple, B a yellow banana, and C an orange carrot. A, B and C are in a bowl that contains nothing else. And they are clearly distinct enough for us to be in no reasonable doubt that there are three of them. It does not matter that they are in other ways a little indistinct. Some of their molecules may well, to some slight extent, be mixing together in the air above the bowl. Nevertheless, the apple can be picked up, leaving the banana and the carrot in the bowl. Or they could all be juggled with. A few molecules here or there are completely irrelevant. A, B and C are distinct enough from each other for one of them to be selected from the others. It does not matter that in other circumstances, such as putting them in a blender, these three items of food might become indistinct from each other.

We have three items of food, A, B and C, in a bowl. With such wordss as "those three items of food" we are referring to them collectively. And their collection remains the same, whether they are in the bowl or not. That collection can be written in a symbolic way like this:

A, B & C

It is just those three items of food, being referred to as though they are one thing, but not being treated as though they were anything other than three things.

Collections of things are all of those things referred to collectively. Whereas those things are many things, their collection is one collection. (A similar move from plurality to singularity occurs with the words “a lot of things,” where a lot is singular, and the things are plural.)

Those three items of food are not one thing, they are three things—or are they? Scientists might say that they are really a lot of molecules; but let’s leave such doubts until we are desperate. We could, after all, have started with three molecules (or three letters). And we already have a more serious doubt here—three things are not one thing: three is not one. This doubt is that collections may not really exist, over and above the things that they are collections of. After all, a collection of things is just those things, being referred to collectively. There is, in that sense, no such thing as a collection. There are just those things.

But if we are only talking about collections as a way of talking about the things in them, there is nothing wrong with talking about collections. And we will not be getting the contradiction because we have mistaken three things for one thing. The contradiction we are heading towards will have arisen because of an impossibly huge number of things. But really, there will only ever have been these three items of food. So, where did all those things come from? Well, you will shortly see. But to begin with, the idea that given any two or more things, we also have yet another thing: their collection—that is not what we want to begin with.

What is it that takes any two things, say a and b, and gives us another thing, the collection a and b? If we wrote that collection as a set, {a, b}, it could seem quite mysterious what “{” and “}” were denoting. Why would Cantor’s contradiction not show that it does not really exist—not as we have been conceiving of it? Why not describe such a mysterious and paradoxical, but also mathematically useful thing with axioms, for mathematical and scientific purposes? Well, there is no reason why not. But the collections of this chapter are not axiomatic sets. This is not a mathematical chapter. This is a logical book, and each and every collection referred to in it is simply all the things in that collection.

When we have a dozen things, we do not have this mysterious mathematical object, a dozen (of something); we have one thing and another thing, and another, …, and a twelfth thing. It is just much easier to say that they are twelve things. Cantor’s paradox was so paradoxical—it was unbelievably paradoxical for the atheists of the twentieth century—and so clearly mathematical, it made mainstream philosophers shine a very focused light on the philosophy of mathematics. “What are numbers?” they wondered. “Where are they? If they are nowhere, do they really exist?” And “if they do not really exist, are they like fictional characters?” Such intense scrutiny seemed to vaporise them. But there is nothing very mysterious about the counting numbers; or rather, there is nothing that is not mysterious, if they are.

Collective reference can seem to be magical, creating countless things out of next to nothing, and thereby enabling me to prove the probable existence of something that many of us do not believe does exist. And there is an ancient paradox of things that cannot be referred to: Although there clearly are things that we cannot refer to—such as a rocky planet in a distant galaxy—we clearly can refer to them. We were just doing so. The solution to that paradox is that although we cannot refer to them individually, we can refer to them collectively. It would be paradoxical if we could not.

There is nothing mysterious about a big collection of things being a lot of things. Now, suppose we have a lot of cats and a lot of dogs. The question of whether or not it is logically possible to pair up all the cats with all the dogs is a question that does not rely on collections being things, over and above the things in them. But it is clearly a question about two collections. Each collection is in that sense one thing, but there is nothing mysterious about that sense. And there is no other sense in which I shall be taking collections to be things. Although because our three items of food are in a bowl that contains nothing else (except loose molecules and bits of dust), they are unified in quite a physical way by that bowl.

Now, I pick up the apple and I have selected A from our collection. Note that I am able to do that because it was already a possible selection. We can call this “the possible selection of A from {A, B, C}”, or just “the possible selection of A” (the context being obvious). We can write this symbolically as {A}. There are seven other possible selections from A, B and C, making a total of eight:

I could have picked all of them out,

or just A and B,

or just A and C,

or just B and C;

although I did pick out A;

but I could have picked out B,

or C,

or I might have picked none of them out.

You might want to count that last selection as not making any selection at all, rather than as a possible selection. Not counting it would not affect the substance of following proof, only some of the incidental numbers. We would then have 7 things in that list, instead of 8. But the main thing here is that both 7 and 8 are bigger than 3 (the number of things we began with).

The mathematics of this is much neater when 8 is used instead of 7 (because eight is three twos multiplied together, with that “three” coming from the three items of food), and so I have been able to write out the logical argument of this chapter more neatly by having all eight selections. And if we think of what might be left in the bowl after various items are rejected, then having nothing in the bowl is more obviously the same kind of thing.

Those eight things are not things in exactly the way that apples are things, of course. But the main thing here is that those possible selections are things in the sense that there are eight of them—each is one thing. Each is one possible selection, distinct from the other eight. Indeed, they are just as distinct from each other as the items of food in the bowl are, because they differ from each other in just such ways (as you can see from the list above). And these are real possibilities, not fictional possibilities. Fictional apples can be any way an author wants them to be, but real apples have to obey the laws of physics. And similarly, these possible selections have to behave logically.

Since I picked up the apple, rather than the banana (or any of the other combinations), I implicitly selected that selection. You can look at the list above, and pick one or two of those eight for yourselves. So, these possible selections are themselves things that can be selected. It is that fact that will lead us inexorably towards a contradiction (given assumptions that we would not ordinarily think to question), by getting us more and more selections from collections of selections.

Our collection of eight possible selections, from our three items of food, inherits a certain unity from the bowl: they are the possible selections from the food in that bowl. It is exactly the same unity that our three items of food have. And these eight possibilities are exactly as distinct from each other as the food was, so they also inherit that. Each time we consider selections from collections of selections, that unity and distinctness will continue to be inherited, in exactly the same way. And all the possible selections will be similarly possible.

From 3 things, we have 8 possible selections. Given those 8 possibilities, there are 256 (which is 8 twos multiplied together) distinct possibilities for selections from them. A couple of possible selections from the bowl are, for example,

the possible selection of A and C;

and the possible selection of just A.

That pair is just one of those 256 possible selections. You can see, from the list of the eight possibilities above, how this pair is already there, implicitly, in the list. It is the presence there of both the selection of A and the selection of A and C that makes it possible for me to make the selection of that one of those 256 (and the same kind of thing is clearly going to be true for the other 255). If someone else selected those two selections, the result would be just the same: that pair. It would not matter how that other person selected them. And if no one selected them, they would still be a logically possible selection.

Moving on, the number of possible selections from those 256 possibilities is, similarly, equal to 256 twos multiplied together. And so on, endlessly. These numbers come from a branch of mathematics that deals with combinations of things, combinatorics; but we are not particularly interested in the actual numbers, just in the way that they get bigger and bigger.

We started with some items of food, in a bowl, and I have selected the apple and the carrot. I was able to do that because A (the apple) and C (the carrot) are already a possible selection. Logically, it does not matter how I make the selection, or whether I even make the selection. A and C are a logically possible selection, from that bowl, because A and C are already there, in that bowl. It is only this kind of logical possibility that we are interested in here. Had it not been for the ensuing contradiction, no one would ever have found it questionable. What could possibly be questionable about it?

These logically possible selections are distinguished from each other by being different combinations of the things being selected. It is physiologically possible for me to pick up the apple and the pear, because of how my body is. But even if the bowl and its contents were suddenly encased in concrete, it would still be the case that were the concrete removed, the apple and the pear might be selected. That selection is logically possible because those are two of the three items of food before me. Logically possible selections exist because the things in that selection exist in the original collection, not because a possible selector exists. Although, since logically possible beings can be arbitrarily powerful, there would always be a logically possible selector.

Consider again the list of eight things that I might do. It is the presence there of the third and the fifth things—the possible selection of A and C, and the possible selection of just A—which means that that couple might be chosen. Their being there is, in effect, the existence of that logically possible selection.

Note that such possible selections are exactly as distinct from each other as our original things were, because they differ from each other in just those things. For example, the logically possible selection of A and C (from A, B and C) and the logically possible selection of A (from A, B and C) differ in just that C.

And note that they are all already there. Anything that is ever possible was always a logical possibility. As we consider possible selections from those 256 things, we are again considering things that are well defined and just as distinct from each other, for just such reasons, and which are similarly already there (it makes no sense to ask where, of course).

And similarly, for possible selections from those possible selections, and so on, ad infinitum. We end up with infinitely many things, each of them well defined. They remain just as distinct from each other, as their descriptions get more and more complicated. And being possibilities, all of them were already there. So, there are infinitely many of them.

We have got an awful lot of things, given only three items of food. In itself, that is not too odd. Given any three ordinary objects, we would also have all of the atoms in them, and all of the particles in those atoms, and so on. And from such items of food we might grow plants that could give us more of them, and hence more plants, and so on indefinitely. Nevertheless, you might wonder if all of these logically possible selections really do exist. You might, for example, wonder where these possibilities are, if they are real.

It was clearly possible for me to pick the apple up, though. That is “where” the possibility was. It was a physical possibility because the apple was a small ordinary object that was nearby; and it was a logical possibility because the apple was one thing. You cannot point to where the possibility, rather than the apple, is, but the logical possibility still has to behave logically. Similarly, where is time? You cannot point to it; but of course, time is real enough for it to be important how we spend it. Not being able to point to it does not justify wasting it. Similarly, where are feelings? You cannot point to them, but that does not mean that we need not take other people’s feelings seriously.

Various doubts will be considered in more detail in chapter 5. For now, let’s simplify our terminology a bit. Let’s use [1] to name the logically possible selection of all three of A, B and C, and [2] to name that of just A and B, [3] to name that of A and C, and so forth. For ease of reference, we can refer to those infinitely many possibilities collectively, as M.

From M, I can select [3] and [5]. Indeed, I just did. Note that such selections are as distinct from each other as the things in M. It is easy enough to imagine a logically possible selector making infinitely many decisions about the things in M. A logically possible imp, for instance, might decide whether or not to select the first thing, [1], in half a second, decide about the next, [2], in a quarter of a second, about the next in an eighth of a second, and so on, so that in one second decisions have been made about all the [n]. But whether anything like that imp is really possible, or not, the collections that might be selected by such an imp, were it possible, are distinct possible selections. They are precisely as distinct from each other as those [n] are. And just as [3] and [5] are already there, in M, to be selected or not, so each of those possible selections is already there, implicitly, in M. Those possible selections are distinct from each other because the things that may or may not be selected are distinct from each other.

Let's call the complete collection of all the logically possible selections from M, the power-collection of M. In symbols, P(M). The following two diagonal arguments are like Cantor's original arguments (more so than the standard set-theoretical arguments). The first is essentially the same as the argument that P(N) is bigger than N, which we saw at the end of the previous chapter, but with M in place of N. The second generalises that argument, and is at the start of the next section. It is basically Cantor's generalised diagonal argument, applied to logically possible selections. It is quite short, but rather abstract, and so this first argument, which goes quite slowly through the application of the general argument to the case of M, may help to clarify it.

So, you are going to see that:

(i) P(M) is bigger than M

That will be shown by way of the untruth of the following hypothesis. So, we begin by hypothesising that the possible selections from M can all be paired up with all of the things in M (each of which can clearly be paired with a counting number).

(ii) Everything in P(M) can be associated with a unique counting number

Consider any such counterfactual pairing. Corresponding to [1], in M, there is some possible selection from M, and corresponding to [2] there is some other possible selection from M, and yet another corresponding to [3] and so on, with every possible selection from M—everything in P(M)—somewhere in that infinite list.

Now, while (ii) is untrue, it is clearly true that each possible selection from M can be uniquely represented as an endless sequence of "In"s and "Out"s. That is because it is obvious that each of the possibilities in M—[1], [2], [3] and so forth—is either in that possible selection or else it is not, it is out. The possible selection of just [3] and [5], for example, can be represented as: Out, Out, In, Out, In, Out, and then nothing but "Out"s. The only "In"s are in the third and fifth places of that sequence. For simplicity, we can give it the following name: "(O, O, I, O, I, O, …)". To see that more clearly, it may help if I lay out that name for the possible selection of [3] and [5] from M:

M:	[1]	[2]	[3]	[4]	[5]	[6]	. . .
[3] & [5]:	(O	O	I	O	I	O	. . .)

Those symbols are just a relatively neat name for the selection (from M) that is, firstly, the possible selection of A and C (from the bowl) and, secondly, the possible selection of just A (from that bowl).

It is similarly obvious that, conversely, any such string of "I"s and "O"s, in any combination whatsoever, represents some logically possible selection from M. Now, according to the untrue hypothesis (ii), we could list all of those sequences of "I"s and "O"s as follows: first the one associated with [1], then the one associated with [2], then [3] and so forth. We would then have something that—in its top left-hand corner—would look like the following.

[1] is paired with	(O,	I,	I,	O,	O,	I,	. . .),
[2] is paired with	(I,	O,	I,	O,	O,	O,	. . .),
[3] is paired with	(O,	I,	I,	I,	O,	I,	. . .),
[4] is paired with	(O,	I,	I,	O,	O,	I,	. . .),
[5] is paired with	(O,	O,	O,	O,	I,	O,	. . .),
[6] is paired with	(I,	I,	O,	O,	I,	I,	. . .),
…	…	…	…	…	…	…	. . .

As a contradiction will shortly show, no such pairing is possible; but were there such a pairing, it could be shuffled around until the upper left-hand corner did indeed look just like that table.

Observe how the leading diagonal of that table—going from the top left-hand corner, downwards and to the right—is: (O, O, I, O, I, I, …). A completely different sequence would have "I" in place of that diagonal's "O", and "O" in place of its "I": (I, I, O, I, O, O, …). We could call that sequence the anti-diagonal of that list of pairings, or ALP for short.

Since ALP is an endless sequence of "I"s and "O"s, it should represent a possible selection from N. By (ii), it should therefore be somewhere in that list of pairings. But it cannot occur anywhere in that list, because it differs from the sequence corresponding to [1] in at least its first place, and from the sequence corresponding to [2] in at least its second place, and from the sequence corresponding to [3] in at least its third place, and so forth. We therefore have a contradiction: ALP should be in the list, but it cannot be in the list. Since that contradiction followed logically from the assumption that P(M) has as many things in it as M does, P(M) and M do not contain the same number of things (they are not equinumerous). And since P(M) is not smaller than M, it follows that P(M) is bigger than M. In other words, it is only possible to pair up some of the things in P(M) with all the things in M.

That diagonal argument is at the heart of the proof of this book, so let's review what we have just done. We found a sequence of "I"s and "O"s that differs from the sequence corresponding to [n] in at least its nth place, for all counting numbers n. And we found that sequence, ALP, by way of a pairing—the table of "I"s and "O"—whose existence followed logically from (ii). ALP being contradictory showed the untruth of (ii). It may help, when referring back to this logical heart, later in this proof, if all of that has been laid out in a clearer way. The next logical steps after (ii) are:

(iii) Let Table be some particular pairing, of each [n], with some unique sequence of "I"s and "O"s, in such a way that all logically possible endless sequences of "I"s and "O"s are thereby paired up.

(iv) Let Alp name the endless sequence of "I"s and "O"s defined as follows: for each possibility, [n], in M, if the sequence paired with it by Table has "I" in its nth place, then "O" is in the nth place of Alp, and otherwise, "I" is in that place.

Since Alp is an endless sequence of "I"s and "O"s, it follows that:

(v) Alp names a possible selection from M

But Alp was specifically defined, via (iii), so that it differed from everything in P(N), and so:

(vi) Alp does not name a possible selection from M

The contradiction between (v) and (vi) shows that there is no such thing as ALP. But were (ii) true, such a sequence would be defined by (iv). It follows that (ii) is not true; that is, P(M) and M do not contain the same number of things. And for each [n] in M there is, in P(M), the possible selection of just that [n], which means that P(M) is not smaller than M. It follows that (i) is true: P(M) is bigger than M.

That was a lot of symbols, but the underlying logic is simple enough. A collection of things is just those things, being referred to collectively. And a subcollection is just some of those things. The subcollections of A, B & C, for example, are A & B, A & C, and B & C. There are five more logically possible selections—or four, if you do not count the selection of nothing as a selection—but larger collections of things always have more subcollections than things.

Subcollections are things in the sense that they count as one thing, when we count them, or compare them with other things in the way that diagonal arguments do; but there are only ever the original things. Academic talk of sets can muddy our view of this, because axiomatic sets certainly are other things. And logically possible selections are similar to Cantor's original sets. However, what makes those selections possible is the existence of the things selected (that is what it means for them to be logically possible). There is nothing more to the existence of a logically possible selection from some collection than the existence of the things in that collection.

Given two collections, either it is logically possible to pair the things up, because they have the same number of things in them, or else it is not. The relation of equinumerosity—it being logically possible to pair up the things in a pair of collections, when they have the same number of things in them—therefore groups collections together. Every collection is in one, and in only one, of those groups (which mathematicians call equivalence classes). All the collections in that group (or class) have the same number of things in them.

If all the things in a collection X cannot possibly be paired up with all the things in a collection Y, but they can be paired up with those of a subcollection of Y, then it makes sense to say that Y is bigger than X. Arithmetic is familiar to us, so we naturally think of that "bigger" in terms of numbers. But what matters here, in this "proof," is that there is an ordering of collections: If X is smaller than Y, and Y is smaller than Z, then X is smaller than Z. We will be building up bigger and bigger collections, so there will be a natural ordering of them. The problem with the biggest collection is more logical than mathematical; basically, there should, for various reasons, be an even bigger collection. The biggest collection is, of logical necessity, in an equivalence class; but if it is an ever-growing totality, then it will not be some number of things.

To see how that would solve the logical problem, suppose that M was not a timeless collection—which is a logical possibility, for all that most mathematicians think that M and N are timeless collections—but was instead getting bigger and bigger without end (somehow). Then our diagonal argument would not have shown that P(M) was bigger than M. To see why, suppose that there was not a timeless collection of all of the counting numbers, but that the totality of all the counting numbers was forever growing in number. Some possible selections from those counting numbers could be fully described in just a few words, even infinitely big ones (such as all the even numbers). Some possible selections need longer descriptions than others; and some cannot be described in any finite number of words. We could say that those are random selections. A random selection can be specified in no shorter way than by listing all of the infinitely many things that are that selection. And because such a list would be ever-growing, if the counting numbers were, those random selections would never be distinct from each other. They would never be individuated (they would never exist as individual things).

3.ii Too Many Possibilities

The following argument will show that, for any timeless collection of things, T, the collection of all of the logically possible selections from it, its power-collection, P(T), is bigger than T. It is more abstract than the previous diagonal argument, but it should be easier to follow its logic, now that you have trudged through the special case of T = M.

You may well be feeling like you have climbed the alps, after all the symbols of the previous section, but there will be fewer and fewer symbols from now on. The following six-step argument shows that, for any timeless collection T, it is the case that:

(1) P(T) is bigger than T

For the sake of an argument that will conclude with a contradiction, we suppose something quite different:

(2) There are as many things in P(T) as there are in T

Were (2) true, each of the things in T could be paired up with a possible selection from T in such a way that every one of those logically possible selections was paired up with one of the things in T. Let Pairing be some such pairing. It follows from (2) that:

(3) Pairing exists

To show that (3) is untrue, Pairing will now be used to specify a collection called Alpine:

(4) For each thing in T, if the possible selection that Pairing pairs that thing with includes that thing, then that thing is not in Alpine, but otherwise it is, and there is nothing else in Alpine

So, the only things in Alpine are things in T, and so Alpine is—given (2)—a possible selection from T, or in other words:

(5) Alpine is in P(T)

However, (4) also means that Alpine would differ from every possible selection that Pairing pairs the things in T with. According to the definition of Pairing, that is all the logically possible selections from T, and so:

(6) Alpine is not in P(T)

That contradiction, (5) and (6), proves that Alpine does not exist, which means that Pairing does not exist. Since Pairing could have been any of the pairings whose existence was implied by (2), it follows that (2) is untrue.

P(T) and T are not equinumerous, and P(T) is not smaller than T (because for each of the things of T there is, in P(T), the possible selection of just that thing), so P(T) is bigger than T. In other words, (1) is true.

When T is M—the timeless collection of [1], [2], [3] and so on—we have the argument of the previous section. Taking T to be P(M), we have an argument that P(P(M)) is even larger than P(M). Similarly, taking T to be P(P(M), we have that P(P(P(M))) is even larger. And so on. We can keep going, in that way, until we have infinitely many increasingly large infinite collections.

All the things in all those collections are just as distinct from each other, as our original three things were. That is because each time we consider possible selections from a collection, those possible selections differ in the things of that collection. For ease of reference, we can refer collectively to all the things that we have been considering so far as U (which stands for union). U is larger than any of the other collections that we have been considering, because for each of them there is another of them (its power-collection) that is larger and whose things are all in U. So, U is bigger than M, because all the things in P(M) are also in U, and U is bigger than P(M), because all the things in P(P(M)) are also in U, and U is bigger than P(P(M)), and so on.

Since there are all of those things—the things in U (all the things in M, and in P(M), and P(P(M)) and so on)—hence there are also all of the logically possible selections from those things, P(U). Each thing in P(U) is just some—or all or none—of the things in U, and so the things in P(U) are just as distinct from each other as the things in U. And taking T to be U shows that there are more things in P(U) than there are in U. Similarly, taking T to be P(U) shows that there are more things in P(P(U)) than there are in P(U); and so on, and so forth. We have again got infinitely many increasingly large infinite collections.

We can again refer collectively to all the things in all those collections, this time as V. And there is similarly P(V), P(P(V)), and so on. We again have infinitely many increasingly large infinite collections, and we can again refer collectively to all the things in all those collections, this time as W. And so on. There is always a larger collection to be found to be already there. Either there is another collection of all the logically possible selections from the previous collection, or else there is another union, another collection of all the things in all the collections that we have, in this way, found to be there.

All these things are equally distinct from each other. And each was already there, because they are basically logical possibilities. It seems to follow that there must already be all the things that such steps could possibly get to. If they could possibly be got to, then they are already possibilities. And with that words “all the things that such steps could possibly get to” we have already referred to them all collectively. And they have inherited a certain unity from that bowl; so let’s, for ease of reference, call that timeless totality X.

The things in X are all as distinct from each other as our three items of food were from each other. And we can, of course, consider a pair of things in X, say the possible selection of [3] and [5] from M, and the possible selection of that selection from P(M). That pair is a selection from X. And each logically possible selection from X is, similarly, just some—or all, or none—of the things in X. It therefore makes sense that we could consider all of those possible selections from X. And yet, to do so would be to take a step beyond X of precisely the same kind as the steps that led us to X. Whereas, X was the result of considering all such steps to have been taken.

We can, and yet we cannot, consider all logically possible selections from X. So, we have a contradiction. We must therefore have been assuming an untruth. Is it true that we can consider all the logically possible selections from X? Well, to say “each logically possible selection from X,” as I did in the previous paragraph, is implicitly to be considering all of them. And they are all timeless possibilities, which are as distinct from each other as A, B and C were. They are all there, to be included or not in the various subcollections of X—or to be all included in X, or all excluded—and so all those equally distinct possible selections really do seem to be there too.

It would be different if not all the things of X were already there; and that is, surprisingly, a logical possibility, as you will see in the next chapter. It is only a logical possibility if there is a God; but that just means that we have a perfectly logical proof that there is (probably) a God, unless some other assumption was untrue. (The chapter after the next one considers how likely that is.)

4

Extraordinary Existence

4.i The Nature of Time

Why is everything in X already in that collection, so that it could, just possibly—logically possibly—be selected? It is, firstly, because each of the things in X is defined in terms of things that were similarly already there, to be possibly selected from—these things being possible selections—all the way back to our initial selections, from A, B and C.

So, secondly, it is because if something is ever possible, then it was always possible. Is it, though? Suppose that I have an apple on my desk. It is possible for me to eat this apple—but a few years ago, it did not even exist; so, how could my eating it have been a possibility then? Well, the reason is that these are logical possibilities. A few years ago, it was possible for there to be this apple here now—as the apple being here now shows—and so there was, even then, the possibility of my eating it. Indeed, there was always the possibility of my eating it. Quite generally, all logical possibilities were always logical possibilities. So, they do seem to be timeless possibilities.

Nevertheless, in light of the contradiction that chapter 3 concluded with, that last logical step—the step from them having always been possibilities, to their being timeless—ought to be examined. Let’s begin with what we know for sure. Clearly you exist, and so there was always the possibility of you. And there was also the possibility of someone just like you—which is not necessarily the same thing. It may, or may not, be physically possible for there to be other universes, as well as this one; but such parallel universes are logically possible. And some of those possible universes are just like this one. In each of those identical possible universes, there is someone just like you. And of course, none of them are you. So, the logical possibilities of those people are not the same as the logical possibility of you.

Now, suppose that the Big Bang had never happened, so that this universe did not exist. You would still have been possible, and so the logical possibility of you would still have been part of the logical possibility of someone just like you. Further suppose that there was instead an identical universe. Let’s call the person who is just like you in that universe—your double in that universe (who has your name in that universe)—dou. The possibility of dou would also have been part of the possibility of someone just like you.

But what if dou’s universe had also not existed? What if there had instead been a different identical universe, containing, say, dou2? Dou2 would then have been part of the possibility of someone just like you. But would the possibility of dou have been part of that more general possibility?

The possibility of you would still have been part of the more general possibility, of someone just like you. And dou is a lot like you. So, you might think that the possibility of dou would still have been part of the possibility of someone just like you.

However, we began by supposing that, instead of this universe—your universe—dou's had existed; and we then then supposed that, instead of yours and dou's, dou2's universe had existed. So, the words "suppose that there was instead an identical universe" would have been about dou2's universe, instead of dou's. Both at first, and then later, we were simply imagining a different but identical universe to this one existing, instead of this one. In effect, we changed the name of that universe from dou to dou2.

In the second scenario—in which dou2's universe existed instead of both dou's and this one—nothing was being named by dou, and so the possibility of dou would not have been part of the possibility of someone just like you. To see this more clearly, suppose that, on the contrary, the possibility of dou would still have been part of the possibility of someone just like you. In that case, how many other possibilities—like the possibility of dou and dou2, but of different people—would there similarly be? There seems to be no logical limit to how many there could be.

But the only thing distinguishing those possible but nonactual universes—that of dou, that of dou2, and so on (without limit)—from each other is that if any of them were actual, then they would be different but otherwise identical universes. Whereas, there is a significant difference between all of those possible universes and this one (of yours). We can refer to this universe by pointing to it. Those other possible universes are defined in terms of this one. There is, in short, this significant difference between dou and you.

There is no logical limit to how many identical copies of this universe there could be. But when we imagine one of them, there does not seem to be any real question about which one it is. It is simply the one that we are imagining. It therefore makes sense to suppose that it would only be when there was a universe just like this one that the person in it who was just like you would have a logical possibility that was distinct from the more general possibility of someone just like you.

Since that makes sense, this is, at the very least, a logical possibility—or is it? If this universe had never existed, there would still have been the logical possibility of you, as distinct from the logical possibility of someone exactly the same as you. Does that not conflict with it being the case that if this universe had never existed, then there would not have been that particular possibility, distinct from the general possibility? Although it seems to, there is no real conflict. That is because if this universe had never existed, then we would not be discussing this. Others might be; but our discussion of the possibility of this universe not existing has been using a word—you—that no one would be able to use, in the same way, if this universe had not existed. In an identical universe, they would use a word like dou. In this universe, you can be pointed to while "you" is said. But if this universe had never existed, you could only have been described in more general terms.

And all other objections to this being a logical possibility fail, similarly. So, if more and more copies of this universe were coming into being, then there would be more and more individual logical possibilities coming into being. Now, we could say that they were being individuated from the original logical possibility of a universe just like this one. So, it is conceivable that logical possibilities might, just possibly, become more numerous. It is therefore conceivable that chapter 3 is showing us that logically possible selections do become more numerous.

Of course, if we exist within an unchanging spacetime, then no possibilities could become more numerous over time. However, although time may well be just one of the dimensions of an unchanging spacetime, there are other logical possibilities for what time is.

When we think of reality, many of us think of the world around us—as we are thinking of it—and of ourselves, thinking. On that rather natural view of reality, it is all that space contains, and it clearly does change over time. A calendar and a clock, for example, change. While they remain the same two things, today's date moves along the calendar, and the current time moves around the clock. Now, it is easy enough to imagine those dates and times laid out horizontally, in a timeline. We could then put the names of events above that line, in between their start and end times. That would give us a picture of reality that included time as a dimension. However, we would not necessarily have been imagining time itself to be anything like a spatial dimension in reality. Not too dissimilarly, the weight on a beam is one of the dimensions of a graph that shows how that load varies along the beam; and weight is of course not like a spatial dimension.

Anything that varies in a continuous way can be represented as a dimension. And the temporal dimension would be like that, if presentism was true. Presentism—according to my preferred version of presentism (there are other versions in the literature, but they need not concern us here)—is basically the view that the past and future do not exist. The past is just the way reality was, the future the way that reality is going to be. In short, only the present moment exists. So note that presentism is not about living in the present, in the sense of our not taking the future seriously. It is of course important how things are going to be, whether presentism is true or not. And if presentism is true, then the future is even more important, because the present—as you read this—has already become the past—as you read this—and the past is no longer real. It is just that if presentism is true, then the future does not already exist. Some of the things that will be around in the future do already exist, but most of them are not as they will be; and there will be some new things. And in particular, some logical possibilities might become individuated over time.

The sorts of puzzles that arise when philosophers think about time have never amounted to the logical impossibility of presentism. For example, the question of how long the present moment lasts can be presented as a dilemma:

Either the present moment has a duration, or it does not.

If it has no duration, then we could hardly do anything in it. There would be no time for us even to perceive anything. So, how could that moment be the only real one?

Whereas, if the present moment had some duration, then it would have parts that were earlier and later. It would be some mixture of past, present and future times, not just the present. So, how could the present moment have a duration?

But although the question of how long the present moment lasts is a legitimate question, the presentist present does not have an intrinsic temporal extension, not even one of length zero. It only has extrinsic temporal extensions. In other words, although it does make sense for us to picture time as a timeline of events, the presentist present is not a natural part of a temporal dimension that is as real as the spatial dimensions.

As we think about things changing, we naturally conceive of a timeline of events. Regular changes, such as those of a clock, and the movements of the solar system, are compared with each other, primarily by means of records of past changes. And in that way, we naturally conceive of time as a temporal dimension. But the present moment comes first, if presentism is true. The presentist present is the whole of reality. It includes such things as clocks, the sun and the earth—and you. And whether presentism is true or not, you are clearly the same person that you were an hour ago. You are wholly present in the present moment. It is events that are spread out over time.

The present moment is full of things, while spacetime is full of events. But there is a legitimate question about how long the present moment lasts, even if presentism is true. We can think of the present moment as an event on a timeline, with start and end times and some distance between them—its extrinsic duration. For example, we might take the bit of time that we are aware of to be the present moment, regarded as an event. That awareness could then be measured by neuroscientists in various ways, and so we might get different answers for its duration. But those would not be contradictions that refute presentism. They would just be different measurements of the activity of the human brain.

When we think as logically as this about ordinary aspects of reality—such as time (or our minds, or numbers of things)—they seem to be much stranger than they really are. And that makes those ways of thinking about them seem unrealistic. It is therefore a good idea to keep returning to the obvious facts of the matter, when doing metaphysics. So, consider the following question:

Where is the present, in the temporal dimension?

Whether presentism is true or not, the present is everywhere in the temporal dimension. To see why, it may help to imagine an enormous history book in which everything that happens is recorded. The present is when the book is written. As time passes, the book gets bigger and bigger. One book gets written in that way. That book is a timeline of events. It is a linguistic model of spacetime. And wherever you look in that book, what you read there is what was happening when that was the present time.

The main alternative to presentism is four-dimensionalism, which is named after the three dimensions of space plus a temporal dimension that is closely related to them. It seems to suit the physics of space, with its non-Euclidean spacetime, and so philosophers who want to be scientific may find themselves wondering which bit of the temporal dimension presentism is identifying with reality. Such misunderstandings are perfectly natural. When we try to answer questions like “what is time?” our words naturally fall over themselves. That is because our words evolved to help us to communicate about very different matters. Consider how connoisseurs describe wine. They know perfectly well—by direct acquaintance—what it is that they are trying to describe. But they say things like “charcoal and cameral aromas, suggesting tea and jam tarts, and the texture of toffee apples,” which sounds to me like wine gums dipped in vinegar.

What is time? Well, the meaning of the word time comes from our everyday experiences. Time is clearly something that is measured, so it is in that sense a dimension. And it is clearly something that passes. As you read this book, you read one sentence, and then you read another, the same person reading each sentence. You are, at present, wholly present—in an apparently three-dimensional space—just as you are now, and will be in a short while. Part of your life is past, a tiny part is present, and part remains to be lived—but you are not like that. When you change, you will still be you, just with slightly different properties. From the perspective of spacetime, events are fundamental, but we naturally think of beings like ourselves as fundamental. It seems more natural for us to think of events as composed of participants, than it is for us to think of people as composed of their lives. And beings are fundamental, if presentism is true.

Beings are also fundamental if all things were created (as you will see later in this chapter). And while there are other arguments against the logical possibility of presentism, they are especially weak if all things were created. For just one more, some philosophers think that if presentism was true, then there would be nothing to make true the truths about the past.

Let me explain what that means. What, to begin with, makes it true that you are now reading this question? It was clearly your reading of that question, a moment ago. But if presentism is true, then the event of your reading that question no longer exists. Nevertheless, it is still true that you did read that question. So, if presentism is true, then what makes that truth true?

Most mainstream philosophers would reject the God hypothesis as unscientific. Nevertheless, God is a logical possibility. And if there is a God, then God's memory could make true all the truths about the past. There may well be other answers to this question, but the logical possibility of presentism only needs there to be one. The main criticism of presentism is not that it is not a logical possibility, though. It is that it is shown to be probably false by modern physics.

However, even the Einsteinian part of modern physics can be expressed in presentistic ways. Such expressions are not yet the most elegant, but it is not as if the four-dimensionalist picture of modern physics is more elegant. According to Einstein, the rate at which time passes is, for each object, relative to other objects, and depends upon the relative speeds of those objects—as though each object had its own temporal dimension—which is why Gödel was able to show (in 1949) that time travel was possible in Einsteinian physics. And experimental results connected to Bell's inequality, in another part of modern physics—quantum mechanics (the physics behind chemistry)—currently have relatively inelegant four-dimensionalist descriptions, compared to their presentist versions. To cut a long story short, Einsteinian dynamics and quantum mechanics currently add up to a simple lack of evidence either way. And it is anyone's guess what will be added to that balance by new sources of data.

There are already signs that there could be extra facts, which are not being investigated because they do not conform to the current ways of doing physics. They fit inelegantly with materialism, and they may have no obvious application, of interest to those who fund physics research. But the ones that I have seen do seem to support presentism, rather than four-dimensionalism. Modern physicists tend to be broad minded when it comes to such metaphysical questions. Most of them are materialists, of course. But this book's proof that there is (very) probably a God might change that. And problems in physics that remain unsolved, despite much work on them, throughout the twentieth century, may soon be seen to be problems for materialism.

Fifty years ago, there was some interest in the physical side of parapsychology, for example. Little came of it, but cultural factors—such as the common association of the supernatural with folklore and religion, and the materialism of the scientific modelling of the mind as a computer—could have accounted for that. But this book might herald a metaphysically theocentric revolution in physics. And the brain is primarily a biochemical structure, in which quantum-mechanical processes are fundamental. Even fifty years ago, this was well known—a big pink book on my shelves is The Self and Its Brain, by Karl Popper and John Eccles (Springer International, 1977), for example.

Physics is all about the mathematical modelling of physical changes. In physics, changes that can be repeatedly measured, accurately and reliably, are naturally more important than other temporal aspects of the world. There may well be more to the metaphysical nature of time than the variable t in the equations of modern physics; not too dissimilarly, there is more to the movements of people than their masses.

Imagine a huge alien computer—an artificial intelligence—looking at us through a powerful telescope. It sees buildings and cars, over long periods of time, and it constructs a mathematical model of the motion of the cars. But it is missing an essential ingredient: the drivers. Much of what it will dismiss as noise is actually quite relevant to why the cars are doing what they are doing. Its theory of what those movements are all about, and what they might be in the future, will be wide of the mark.

The fundamental particles of modern physics would not be the most basic stuff of the world, if there was a God. Not too dissimilarly, the basic stuff of a work of fiction is often the central characters, and an outline of what they will be doing. Other aspects of the fiction are fitted around those core elements, as it is written. In light of the proof of this book, the most basic things may well be people like ourselves. The metaphysical nature of time may therefore have less to do with which theories of physical particles are the neatest, and more to do with such things as free will.

But in any case—even if relativistic physics does eventually account for all the quantum-mechanical facts, in some surprisingly elegant way—we would still have little evidence for time being four-dimensionalist. Consider how the elegance and utility of Newtonian physics never actually amounted to much evidence for space being Euclidean. Nineteenth-century physicists naturally thought that it did; and yet, a few extra facts, together with Einstein's equations, did transform it into evidence for space being non-Euclidean.

And furthermore, the argument of this book does not necessarily need presentism to be true. It needs a God who is able to change. But those changes do not necessarily have to be in our dimension of change, time.

Suppose that four-dimensionalism is true, so that it would be totally appropriate to call time the temporal dimension. Presentism would be false, but it would still be logically possible, and so this four-dimensional spacetime might, just possibly, exist "within" a temporal "dimension" that was presentistic—which is just to say that if there were people just like us "in" that "dimension" then what they would call presentism would be true.

Let's call such a "dimension" hypertime. The argument of this book does not need presentism to be true, it needs only the logical possibility of presentism. The argument of this book is a perfectly logical argument that there is (very) probably a God who can change. The question of whether presentism is true or not is only the question of whether that God would exist "in" time or hypertime.

If a God who was able to change had had a reason to make a four-dimensionalist spacetime, containing people like ourselves, it would presumably have been made so that it lasted forever, in hypertime. And the changes needed to explain chapter 3—the endless individuation of extremely abstract possibilities—are so subtle, they would not affect the physics of such a spacetime. And since all of the parts of such a spacetime—past, present and future—would be changing together, in hypertime, these changes would not be what we would call changes, were we in such a spacetime. Those changes would be very subtle. They would be the same kind of thing as the following, but even more abstract.

Suppose that this God makes a second spacetime, exactly the same as the first. There was always the possibility of that second spacetime, but there was only always the possibility of that particular spacetime, as distinct from the more general possibility, once this second spacetime had been created. That is strange use of words. It is as though hindsight could actually change the past. But in this hypertime, the past does not exist, there is only the changing present.

Change in hypertime is hard to imagine. But the main thing here is that presentism is a logical possibility. Consequently we cannot yet rule out the logical possibility of our possible selections becoming more and more finely differentiated (although it is hard to imagine how it could happen). On this hypothesis, there could be distinct possibilities that were originally indistinct. There could, in other word, be more and more possibilities. Which could solve the puzzle of chapter 3. If our possible selections—or at least, the very complicated ones—were the result of possibilities becoming individuated, then their totality would not be a timeless collection. Consequently the diagonal argument would not apply to them, and so there would not be the contradiction that chapter 3 ended up with. If their totality was not a timeless collection—if their totality was getting bigger and bigger forever—then not all of the subcollections from it would exist (for reasons like those sketched at the end of the first section of chapter 3).

Is it conceivable, though, that anyone would be able to individuate logically possible selections? Saying that a God could do it in some miraculous way is not good enough. Imagine a God splitting a possible selection into two, somehow. Each of those two parts was always a logical possibility. Every one of our possible selections was very clearly there already, in chapter 3. Still, that does not mean that a God could not be endlessly individuating them. If a God was doing that, then they might well look as they do to us. There similarly seems—now that you exist—to have always been the possibility of you, not just the possibility of someone exactly the same as you. But it is, as we have seen, conceivable that the former possibility only exists because you exist. So, if a God was forever individuating the logically possible selections of chapter 3, it might very well seem to us that they were all there already.

There are significant differences between copies of physical things like humans, and differentiated selections. But one of them is that, while it is conceivable that such copies could just happen to come into existence, it is very hard—I would say impossible (you might see why I would if you try)—to imagine possible selections just happening to become more finely differentiated. The next section considers at some length the conceivability of a God doing it.

Before we begin, we might expect scientific objections to that conceivability to be a bit like religious objections to the conceivability of a God creating right and wrong, rather than simply knowing what is right and wrong as part of being all knowing. Some religious believers think that God being able to do anything means that God could create a round square shape. “Why not?” they will ask; but to any possible answer, they can counter, “That may well be why creatures cannot, but this is God we are talking about.” Some religious believers think that that is not good enough, and those that think that God could create right and wrong give better answers to those who think that God merely knows what is right and wrong. The next section is all about trying to give good answers to reasonable doubts.

4.ii An Authorial Authority

If the world was created by God, out of nothing but God’s creative power, then those things would have a relationship to God that was, to some extent, like the relationship that fictional things would have to their authors (were their authors also the bulk of the readership). And fictional things are, as a rule, a matter of what their authors had envisaged and described (and what the readers of those fictions thereby imagine). There may therefore be a way in which the most unimaginably complicated of chapter 3’s logically possible selections could, just possibly, be individuated.

For definiteness, let’s begin with a simple example of a fiction. Consider a two-dimensional space containing square people and circular people. The following very short story is based on Flatland, a novella written by the British schoolmaster Edwin Abbott (1838–1926). And just as Abbott’s was, this two-dimensional space is surrounded by a three-dimensional space. In it, there is a squat cylindrical person, Cyril, who can appear to my flatlanders as a square person, or as a round person, depending on how he passes through my flatland, because his height is equal to the diameter of his circular cross-section.

At first, my flatlanders cannot imagine how that square person and that round person could be the same person. Even in flatland, round squares are impossible. Nevertheless, Cyril can appear anywhere in flatland, turn from being square to being round or from being round to being square, and then disappear. The flatlanders call him Cyril the Incredible, and some of them even think that he is God—although of course, they ought to think of their God as being like more like me.

Or Abbott; although his story was about a sphere, not a cylinder. And like nineteenth-century school mathematics, it was full of triangles. But what neither of us would have written is too much detail. Abbott would not, for example, have described his spaces axiomatically, despite having been able to (he won a prestigious mathematical prize when he was a fellow of St John’s College, Cambridge). Fictions are deliberately sketchy. Whereas, reality is far from sketchy. So note that when God is being likened to an author (and the bulk of the readership), reality is not being said to be sketchy. Rather, reality is being said to be fundamentally what God thinks it is.

And a God would naturally be as mysterious as we would be to flatlanders. But we do have this analogy to work with. And since the proof of this book needs to find nothing more detailed than a logical possibility, we will not need to go much beyond this analogy in this section. In any fiction, there are some details that could have been changed by the author very easily, while others will have been central to the intended story. So, if there is a God, you might expect there to be something like plotlines and outlines of key characters, at the most fundamental level of reality (instead of the conservation laws and fundamental particles of physics, which you might expect were materialism true). The purpose and hence the nature of our souls might therefore be a significant factor, when it comes to the nature of reality.

More simply, suppose that a God had wanted to create an apple. There might have been the idea of a particular apple first, and then some act of creation out of nothing, resulting in there being that particular apple. Or there might first be the idea of apples, and other fruit, followed by the creation of a starry space that would eventually contain an evolving biosphere in which there would eventually be apples. There might be many other ways in which a God could create an apple, though. We really have no idea what a God could do.

What if I wanted to put an apple into a story about flatlanders? I might think of an idealised apple, and write “a small red disc” in the text of the story. That would put that description into my story, but it would not actually create anything, of course. Writing and reading only facilitate the communicate of ideas. And even movies only make those ideas more like dreams. Furthermore, our ideas are rarely, if ever, completely original. If there is a God having an original idea, then that would be a totally original idea.

Suppose that a God had wanted to create people who could see red things. There had first to be the logical possibility of redness. And there would also have been the logical possibility of orangeness, and the logical possibilities of all the other qualities that we know—together with some range of logical possibilities of qualities that we do not know, but which are logically possible and are, in other ways, very similar to redness. It is quite conceivable that a God would have known all about all of those, of course. But it is also conceivable that not even a God could have known all about all of them—it all depends upon the nature of those qualities that we do not know.

Consider how, if the human eye had evolved slightly differently, then redness might have been the same, or only slightly different. With a very different evolution, there might, for all we know, have been, instead of redness, a quality so different it was not redness. And in between those two extremes, there might, just possibly, be qualities that are as much redness as not. Redness as we know it does shade into orangeness through colours that are only vaguely red. And were there a God, there could be many other ways in which redness might shade into other qualities. Now, a God would presumably know all those different possibilities for creation. But we naturally think of that in terms of things—individual possibilities—even if they are so numerous that they form continuous spectra of possibilities.

Whereas, it is conceivable that God’s creative powers are greater than that. Having chosen to create, God may eventually have chosen to create a world of things as we know them, having begun with a range of possibilities that also included worlds of stuff that is only similar to things as we know them, is perhaps only as much things as we know them as not, in various ways. And if so, then although God would presumably have known all about those possibilities—presumably by way of self-knowledge—that might not have been like knowing a massive book, but more like seeing an enormous picture, parts of which would, were they described well, give rise to such massive books. And even for a God, it might be logically impossible to know all that could be in such metaphorical books. Somewhat analogously, free will might, just possibly, involve an inability of a God, to do something that only a less powerful God could do. Suppose that God, having free will, can choose to create beings with free will. Their freedom would conceivably mean that not even God could know what they would be doing in the future. But that would be a small part of God's power to create such beings. Conversely, a God who could only make beings without free will could know all about the future, but would clearly be a less powerful God.

The question in this section is the question of how a God might create a world of things, as we know them. And the creation of a world of such things could conceivably involve God having to choose—with respect to stuff that is only as much such things as not—whether they should count as such things or not. Somewhat analogously, if we were looking at a colour that was only vaguely red, it is conceivable that we would be able to decide whether it was red or not. Now, I do not think that we could always do that. That is because if we could, then there would be very similar colours, one of which was red, and one of which was not red. But some philosophers think that we could always do that, and that there would always be a right answer. And it is relatively easy to imagine how some people would be able to decide that some such colour was red, while others decided that it was not red. A God, by contrast, would have decided what sort of thing redness was (with at least the authority of an author who was also the bulk of the readership). But a God could conceivably have decided what sort of stuff things—as creatures would know them—would be by making choices about stuff that is only as much such things as not.

What we know is that the physical universe probably began in a Big Bang. There was originally one infinitesimally tiny universe, which ended up containing a huge number of particles. And it may have been created out of nothing, in that Big Bang, by a God who can change. And although we can hardly imagine what such a God might have had in mind, it would certainly have included something like human beings—and something like apples, bananas and carrots—in a world that was approximately Newtonian. The quantum mechanics of particles gave rise to a chemistry rich enough for there to be such biological structures; and relativistic spacetime is approximately Euclidean, for us. We experience a relatively simple world, which may well have been the original intention.

Some of the most important things about us—such as our having free will—do seem to defy logical analyses. And if there is a God, then they may well be fundamental. Such things may well exist because such things have always existed, as aspects of our creator, and because their being aspects of us was part of the point of creation. Difficulties in describing them would then arise, not because of their nonexistence, but because our languages developed as they did primarily in order to help us to communicate more mundane knowledge.

You and I are certainly two quite distinct individuals. And while we are people, not things, the sense that this book is concerned with is the sense in which there can be one thing, two things, three things and so forth—and in that sense, we are things. But what about freedom; is freedom a thing in that sense? Is God? Some religious people describe their God as a Trinity, so it is certainly conceivable that God is above and beyond the concept of a thing. And while is more natural for us to think of God as one thing, does that make that thought a more logical thought? The standard response to Cantor's paradox had mathematicians trying to conceive of the collection of all the mathematical collections as being above and beyond the mathematical conception of a collection. And mathematicians are certainly logical thinkers.

Imagine that there was originally a God, and that there was only one being of that kind. Indeed, let's imagine that there could not be other beings of that kind. That would then be a very simple kind of thing. But it would be quite unlike the things that we are used to. It is easy for us to imagine numbers of such things. You might think that the whole universe is an exception, but we can imagine a second universe and a third, before and after this one, or alongside it in some extra spatial dimension. But imagine that there is only one possible God. For such a being to create other things, a different concept of a thing would be needed. And for such a being to create things as we know them to be—things with the Cantorian property described in the previous chapter—it would have to be a relatively complicated concept. So it is not illogical to think of God as being one thing in a sense that is not our sense of thing, and which is objectively simpler than our sense. We have our sense, of course, and so such a God might, for example, look like one thing and three things. But it would be our concept of a thing, not God, that was relatively complicated.

The concept of a thing is a very simple concept. It is the simplest concept in mathematics and logic. But when it comes to such extraordinarily transcendent "things" as God, it is a relatively complicated concept. And analogously, Flatlanders would think of Abbott as two-dimensional. In short, it is not illogical to speculate that the concept of a thing may originally have been chosen by God (from some unimaginable range of possibilities). And the logical possibility of God being well described as both one thing and two or three things is certainly much simpler than the theological concept of the Trinity. It is a bit like how Cyril is well described by flatlanders as both round and square. There are no round squares in flatland; and similarly, nothing in the world could be one thing and, in the same way, two things (although one pair of things is just those two things). But a God might be above and beyond the concept of a thing, in a way that is a bit like how Cyril is above and beyond the shapes of flatland.

Another analogy for creation is dreaming—we might think of God being, with respect to creation, a bit like how we are to our dreams—and while this analogy is no better than the analogy with writing a story (and is in many ways worse), it does include an analogy for God incarnating as a human: Might that be a bit like our being in our own dreams? A more abstract point of view is usually created for the readers of a story, by its author. Now, we might do things in our dreams that we would not do, were we in comparable situations in the real world; so, we might have several personalities. As well as our waking personality, we might adopt different personalities when we are dreaming. Indeed, we might do something similar when we read stories, or watch movies.

Anyway, analogies aside, imagine that there is a God who is above and beyond the concept of a thing, as we know it. Such a God would presumably know of other concepts, of things like things, so to speak. Each would suit its own kind of creation, of a world of such things. We can hardly imagine what other kinds of stuff a creation could contain. But it is conceivable that the concept of a thing, as we know it, is a range of very similar concepts, from God's perspective. After all, we can imagine the author of a story about flatlanders writing "a small red disc" and thinking of Red Delicious, Royal Gala and Braeburn apples.

Is this a logical possibility? Could a God have had a range of concepts in mind when creating things? Well, if each of those narrower concepts of a thing would have facilitated the creation of exactly the same things, with exactly the same properties, then why not? The main thing would have been the sorts of things that God wanted to create. Presumably God wanted to create things as we know them. Perhaps such simplicity suits thinking creatures. Most of us can think logically enough about sufficiently simple things. And if having a range of very similar concepts of a thing was needed, in order to have such a creation, then not only would that not necessarily have been a problem, it might have been one of the things that made such a creation a good idea. If this book is essentially correct, then it would have facilitated a relatively accessible proof that there is probably a God. A God could not, more simply, have put lots of signs saying that there is a God, all over creation, because such signs could have been put up by aliens with godlike powers. God is not like a god. Analogously, while I could have put a pink rectangle saying "I am the author" into a story about flatlanders, my appearing there in person is logically impossible.

It is only natural that the concept of a thing should seem simple to "things" like ourselves. But much the same is true of space and time, and Einstein's theories changed our minds about those. And Cantor's paradox made people think again about things. Those thoughts did not usually involve a

God, but they have involved a lot of convoluted complexity (axiomatic set theory being the tip of that iceberg). So although, if I am right, the concept of a thing involves a process of unimaginable complexity, that is par for this course. And if the concept of a thing was, from God's perspective, a range of similar concepts of a thing, then the differences between those concepts might at some point cause created things to have incompatible properties. They might not be properties that we would notice, from within creation, but they might be such as would eventually affect some of the logically possible selections in chapter 3.

Suppose that the things that God would be selecting would, at some point in the endless process sketched in chapter 3, have to instantiate a slightly different concept of a thing. At that point, God might choose a different range, of these similar concepts, so as to preserve this creation. Alternatively, this presentistic God might simply end this creation. The intention may have always been to end it at some point. The essences or souls of the created people might be preserved, if their underlying concept could be made less thingy—more like the concept of a Trinity (or whatever God actually is). We might then enjoy closer relationships, with each other and with God, than would otherwise have been possible. Indeed, something like that might, just possibly, have been the original reason for creating things. Things might be an interim stage in the creation of people.

I am not saying that this is particularly likely. God is essentially ineffable, after all. I am simply observing that there are such conceptual possibilities. The thing is, this particular metaphysical possibility does support a logical solution to the puzzle of chapter 3. It is, to begin with, relatively easy to imagine that even someone with the power to create things (including people) from nothing but that person's own creative power might have some intrinsic limits to that creative power. There could, in particular, be an intrinsic limit to God's ability to simultaneously make infinitely many decisions either to keep this creation going—by changing the range of such similar concepts of a thing, in some appropriate manner—or else to end this creation.

There would always be too many such choices for God to ever make them all, and so there would be no timeless totality of logically possible selections, from selections from … selections from the original collection. Just by thinking of that collection, God would be thinking of all the logically possible selections from it, and then of all the selections that could be made from them, and so on and so forth. But because that process involves arbitrarily many choices, it would take an arbitrary amount of time. All of the logically possible selections considered in chapter 3 are logically possible, whether there is a God or not. But on this metaphysical possibility, most of the unimaginably complicated ones will only ever have been possible as indistinct parts of their general possibility. What individuates the others is God thinking of them as individual possibilities. From our perspective, it naturally seems as though all of them would already be individuated. That is how they seemed in chapter 3. But it is God's perspective that matters—given that there is such a God—when it comes to the question of what they really are.

It is relatively easy to see this in the case of physics. Many things are bound to be a certain way—such as the sun rising tomorrow. Our experience of time is therefore one of moving down a pre-existing timeline. But if there is a God who is able to change, then the sun would rise tomorrow only were God willing to sustain the rule of the laws of the physics of the sun's rising tomorrow. For us, the laws of physics are timeless. But they would hardly appear timeless to a God who had freely chosen to create this physical world out of nothing, and who was able to change. Such a God could change those laws at will. And what the laws of physics really are is how they seem to God (given that there is a God). In a sense, the laws of physics have changed, as ancient worldviews became medieval worldviews, and then modern worldviews; but there is an obvious sense in which the laws of physics are timeless. Nevertheless, that is only from our point of view. The laws of physics being timeless in that sense does not obstruct the logical possibility of there being such a God.

The same kind of thing with mathematics is naturally harder to see. Logically possible selections are essentially subcollections, from our point of view. But if there is a presentistic God, they are fundamentally selections that God could possibly make. And a collection of things is objectively—if there is such a God—some things that have been collected together, in the mind of God, to make another thing. And it may have been the case that, while there are various totalities of all the narrow concepts of a thing, together with various ranges of very similar concepts, there is no totality of all and only those narrow concepts.

All the logically possible selections of chapter 3 were always possibilities. Even if they are individuated endlessly, those individuations were always possible. Nevertheless, it could have been that there was no totality of all of those possible individuations. It depends on the original possibilities for creation—on what it was that the narrow concepts were originally part of. To see why, it may again help to consider what we do know. Red apples, for example, are possible, but there is no totality of all possible red apples. Not only is there no totality of all and only the various shades of red, there is no totality of all possible apples. The original apple tree evolved very gradually from a tree that was not an apple tree. And who knows how apples will vary in the future. So, there is no totality of all and only the possible red apples. What there are is various totalities of all possible red apples together with various possible fruits that are only vaguely red and/or only vaguely apples; and various collections that do not include all possible red apples.

When we choose between two possibilities, so that we have two possible futures, a presentistic God would know both possibilities. But if we have free will, then it could be that not even God would know which one was the actual future. There will be just the one future, but if presentism is true then it would not already exist. And a God would know all the possibilities for fruits and other stuff, of all colours—including stuff that was as much things as not. So, it could be that not even God would know a totality of all and only the things, as we know them. On this theoretical possibility, there would originally have been a God, a thing in various narrow senses. For each sense, there would be other possible things in that sense. But God may well have been two, or three things, in some similar narrow senses. And for each of those senses, there would then have been other possible things. And all sorts of other narrow senses might be possible, such that our concept of a thing could, just possibly, be an ever-varying mixture of narrow senses.

There may well be other theoretical possibilities for how it could be that a God could be forever individuating the logically possible selections of chapter 3—and, similarly, individuating the whole numbers themselves. The main thing, so far as this proof—that there is (very) probably a God—is concerned, is that it is logically possible for a God to be doing such things. No great detail is required for the plausibility of that. To see why not, it might help to compare it with the plausibility of the logical possibility of minds being fundamentally properties of brains. The plausibility of that materialist conception does not depend upon having any mathematical model—or even any conception of the basic form—of the way in which brains give rise to minds. An atheist does not need to have any idea of how such a mathematical model might be derived from the equations of modern physics, in order to avoid being unreasonable, when finding it plausible that such a model might be so derived, and maybe even conjecturing some general possibilities.

Details are a good way of giving meaning to conjectures, though. And one of the simplest of the details of the theoretical possibility considered in this section concerns the possibilities that are so complicated, even a God would have to spend some nonzero amount of time (or hypertime) individuating them—how complicated would they be? We could hardly know about such matters, but there would presumably be some natural stage in the endless process of chapter 3—that consideration of possible selections from previous selections—which is such that God would have known all of the earlier stages immediately upon thinking of the possibility of creating things. We can have no idea what that stage is, but it is easy enough to get a glimpse of those earlier stages.

In chapter 3, we began with three things, A, B and C, and then we got to an infinitely big collection, M, by considering possible selections from previous selections. We then moved on from M and P(M), to U, V and W. Now, we would soon have run out of capital letters. But instead of U, V and W, we could have called those unions U(1), U(2) and U(3). And then, instead of continuing on to X, Y and Z, we could have continued on to U(4), U(5) and U(6), and so on and so forth. When we have infinitely many increasingly large unions, we can take the union of all of those unions.

We might call that a super-union, SU(1, 1). We could then consider its power-collection, P(SU(1, 1)), and the power-collection of that, and so on, endlessly. We could then consider the union of all of those power-collections, calling it SU(1, 2). And so on. When we have infinitely many increasingly large unions, we could take another super-union, which we could call SU(2, 1). And so on.

When we have infinitely many increasingly large super-unions, we could take the super-duper-union SDU(1, 1, 1). And so on and so forth. We get, in that way, longer and longer symbols. And when we have an infinite sequence of those, we can again consider their union—the collection of all the things in all of those collections—and call it, say, Q.

We have just gone from M to Q, and we can do all of that again, starting with Q instead of M. Let's say that we get to Q2. Doing that again gets us to Q3, and so on, ad infinitum. We could then consider the union of all those Qns, for counting numbers n—and we could keep on doing that kind of thing indefinitely, with much neater symbols. And perhaps when God originally thought about making numbers of things, some unimaginable range of possible things was also thought of, along with such individual possible selections from that range—and from those possible selections, and so on and so forth—as might be in the sort of collection that many trillions of lifetimes spent playing with such symbols might result in.

That range would originally have been an indistinct part of the simple logical possibility of created stuff of all kinds. The idea of making people like ourselves—people who might contribute to their own creation, through the choices they make—would have involved the concept of a thing, and hence God thinking about making numbers of things. Things being as they are, that would then involve God individuating whole numbers endlessly. Fortunately, the proof of this book does not rest on the details of this particular metaphysical possibility. It rests on the plausibility of there being some such logical possibility, however unimaginable most of them are. It also rests on more than a century of failures, by a host of mainstream philosophers, to find an atheist explanation of Cantor's paradox that is more logical than there being some natural imperfection in human logic. Mathematicians prefer to think of numbers as being created by mathematicians, whether by stipulating set-theoretical axioms (with classical symbolic logic) or in the more obscure manner of the mathematical constructivists (who use an intuitionistic symbolic logic).

But there is definitely something very objective about there being one thing, two things (such as you and I), three things and so forth. It therefore makes some sense to think of such natural whole numbers as being created by the creator of all things. And it is precisely because such a being would be essentially ineffable that we can hardly rule out—on logical grounds—such a being having the ability to create numbers out of nothing. It is relatively easy for us to think of this God as creating whole numbers endlessly, by adding units to unimaginably big whole numbers. That would correspond to the individuations of the possibilities of such numbers of things. We might think of that process as akin to a clock, telling the age of creation (rather than as being akin to the efforts that we make, when we do mathematics). And it does make intuitive sense that someone who could create things—not just physical particles, but people like us too—might also be able to create numbers.

We are naturally unable to see how that could possibly be done, but that is hardly the same as seeing that it could not be done. For an analogy, consider the theory that the moon causes the tides. The movement of the moon obviously correlates with the timing of the tides. But the idea that the moon caused the tides was rejected by Galileo. How could the moon pull all that water towards itself? There is clearly nothing, in between the moon and the seas, upon which the moon could pull. Newton is said to have wondered, upon seeing an apple spontaneously fall from a tree, what had pulled it down from the tree. Something had got the apple moving straight down towards the earth, even though there was nothing under the tree, pulling at the apple, except the earth itself. So, the moon might, just possibly, be pulling at the seas, for all the evident lack of anything in between the moon and the seas.

That it was inconceivable how the earth pulled the apple down, and how the moon pulled at the seas, did not matter. What mattered was that there was no better explanation. And not too dissimilarly, the logical possibility of God creating numbers out of nothing is not undermined by our natural inability to envisage very clearly how that could be done. The logical possibility of a God—in the sense of the creator of the world out of nothing—is certainly not undermined by our having no idea how things could be deliberately created out of nothing. That they, along with space itself, might come from nothing is implicit in the scientific hypothesis of a Big Bang. And that hypothesis is clearly not undermined by our natural inability to imagine how or why a Big Bang might occur.

Anyway, given that a presentistic God had decided to create such things as our souls, such a God would presumably have immediately grasped all (humanly) imaginable arithmetic—and presumably a lot more—and would immediately have started individuating the more complicated possible selections, collections and associated whole numbers. That sounds like hard work; but for a God, it may well be less of an effort than the beating of your own heart as you rest between jobs. Perhaps it would be more of a chore; how could we ever know? What we can know is that a being who could create all of us out of nothing—who might transcend the concept of a thing by being more like a Trinity—might, just possibly, be individuating subcollections just by thinking of those particular things as one collection of things. When we notice a subcollection, we see a particular selection that was always a possibility. But it is relatively easy to see how that could, just possibly, be due to our position within creation.

Suppose that a presentistic God determines to create a ring of equally spaced, absolutely identical objects (as described in the second section of chapter 2). None of those objects is individuated, until the ring has been created. God chooses a counting number greater than 2, and makes that many of these objects. Before that moment of creation, there is only the general possibility of such an object. Afterwards there is, for each object, the individual possibility of that object in particular, in addition to that general possibility. That is just what happens, in this scenario.

Once a particular object exists, there seems always to have been that particular possibility, because that particular object was always possible. We can refer to it, and know that it was always possible. But from the description of this scenario, we also know that it was only the general possibility that always existed. The God who made them—who is above their physics—can see those numbers increase, can recall increasing them. And because presentism is true, in this scenario, it is certainly not the case that there is a past that is changing. Those possibilities are increasing in number only in the ever-changing present. They only seem to have been there in such numbers in the past.

Each of these (theoretically) actual objects was always possible—there was always that particular possibility—but it was an undifferentiated part of the general possibility until the object existed. It was there, but undifferentiated. We can know that it was there, and that it was undifferentiated. It is in the nature of individuated possibilities, that they would seem to have always been individuated. But we see the world from the inside out. With the idea that God creates the world we have somehow to imagine it from the outside in.

Not too dissimilarly, the logically possible selections of chapter 3 seemed to already be distinct possibilities. But we can also see how that could, just possibly, have been due to our thinking of them from within creation. Now, I do not know how realistic is my idea of narrowing ranges of similar concepts of a thing. We will never be able to know much about how a God creates. But it is relatively easy for us to see that a God is a logical possibility. And we can get a better idea of how reasonable it is to think of a God individuating such logical possibilities as those of chapter 3—and similarly, the logical possibilities associated with whole numbers—by considering just how wide is the range of what it is reasonable to suppose that a God might be able to do.

To begin with, it is logically possible for a God to create physical things out of nothing. How are we to imagine that? It is quite mysterious how matter could be created out of no pre-existing material. We can create poems, and we can shape clay into pottery, and we can even create matter out of energy. So, those things are plausible. But we might reject the very idea of creation out of no pre-existing material, on the grounds that it is nonsensical. However, it is also mysterious how an electron can go through two gaps at the same time. It is mysterious how molecules, atoms, ions and electrons can be arranged so that there are living beings, with feelings, thoughts, ethics. Given electrons, we should not rule out as logically impossible their travelling through two slits at the same time. Given living beings, we should not rule out as logically impossible their being composed entirely of molecules, atoms, ions and electrons. And similarly, if we observed something that required the authorial authority of a creator of real things out of no pre-existing material, then that should trump the mysteriousness of such creation.

And if there is no God, then it might just be a brute fact that the Big Bang happened. Or perhaps there was originally a range of possible universes, with that range being a brute fact. Either way, the most fundamental laws of physics could not have been otherwise. We can imagine them being otherwise, but that is a different matter (akin to the way in which physics has changed over time). Whereas, if there is a God, then the laws that physical things actually obey really could have been otherwise. For a start, creation might not even have been physical originally. There might have only been heavenly spirits, who invariably obeyed moral laws. And if we imagine that there is a God, then it is also easier for us to imagine that morality might be an objective matter.

We all find it hard to imagine a world where ethics is different. We tend instead to imagine a world that is wrong. For example, in a story in which aliens had evolved to consider, as good, things that would be evil were we doing them, it might be hard for us to avoid thinking of those things as evil, or at best as the actions of amoral animals. In the real world, different human cultures do sometimes have different conceptions of what is right and wrong. But it is only natural for us to think that they are, in such cultures, getting some things wrong. Although, we might consider it unlikely that we just happened to be born in the correct culture. So, some of us try to imagine that such differences are more like different speed limits. Once a speed limit has been set, we are legally obliged to comply with it, and had it been set differently, we would have had a different obligation. Although that is hard to do when we think of things that are evil.

If there is a being who created things in general out of nothing, then it would make sense for that being to be the ultimate judge of what is the case, and in particular, of what is right and wrong. This would follow from the almighty power of such a God—but not in the manner of an extremely powerful dictator. It is more like how a very good amateur artist would be the best judge of his own work, if he had absolutely no interest in fame or making money.

There are basically two ways of thinking of God as the ultimate judge. Firstly, there might a fact of the matter of what is right and what is wrong, which God knows because God knows everything, and which God agrees with because God is good. Secondly, it might be that what is right is whatever God approves of, and what is wrong is whatever God disapproves of. This second way of thinking about God being the ultimate judge is usually called divine command theory. But a wide range of theological theories go by that name, so let's here call this basic way of thinking the divine preference view.

You should not think of the divine preference view as saying, of someone like Zeus commanding what should be done, that those things that He commands are all of them good. Even the ancient Greeks sat in judgement of Zeus, as they were told stories about him, and went on to invent democracy. No, the divine preference view is essentially the idea that the good is fundamentally what God likes about creation. If there is a God, and if the divine preference view is true, then it would be from those divine preferences for creation that our own ideas about what is good—about which things are good, and about what that means—would derive (insofar as they were realistic ideas). But whatever those ideas were, God would certainly be good in the most objective sense of the word. Now, the ways in which a God could pursue those preferences are innumerable. Most of them are unimaginable—as, indeed, the preferences themselves may well be. But that is not our concern here. This is a logical book, not a religious book.

Atheists tend to think of ethics as something that social animals imagine, within a fundamentally physical reality. They tend not to believe in objective facts of the matter of what is right and what is wrong. There is the law of the land, and there are cultural and biological facts about what humans tend to believe, but that is about as objective as it gets. Whereas, if there is a God, in the logical sense of this book, then although there is still all of that politics and psychology, there is also the matter of what God wants. When we think of God as logically possible, it makes sense to ask if God is good because God chooses only to do good things, or if those things are good because God chooses them. The former is how we naturally think of people being good, but the latter makes at least as much sense because God would have created everything out of nothing.

We can similarly ask if God is able to create things because they are possible, or if they are possible because God could create them. The former is how we naturally think of people doing things, but the latter makes more sense if things exist because there is a God. Logical possibilities are, insofar as they are objective, primarily a matter of what God can do, if there is such a God. It does therefore make sense to think that logical possibilities could, just possibly, be individuated by a God—for all that they do seem to be timeless, when we think of them as objective.

Atheists tend to think of logical possibilities as either subjective or else as timeless, and of numbers as belonging to science, rather than religion. But if numbers are standard sets, then numbers can be created by mathematicians. And the main alternatives to set theory, such as category theory and constructive mathematics, are similarly created by mathematicians. So, the idea that God might be able to create a more objective kind of number is not too odd. Numbers are not obviously going to be harder for a God to create out of nothing than physical energy or human souls. If we have accepted the logical possibility of a God, then the divine individuation view of such logical possibilities as whole numbers—and the logically possible selections of chapter 3—is not a great leap. Whereas, if there is no God, then such objective whole numbers would of course be timeless. Cantor's paradox is then a good reason to do science without such numbers.

Numbers have to be at least slightly different to how we most naturally imagine them. That is what Cantor's paradox shows. So, it may well be—give that God is a logical possibility—that the largest whole numbers only exist when they are individuated by God. The counting numbers could still have always existed. The most basic logical possibilities would presumably have always been part of the mind of God. But perhaps it was when God thought about creating us, that the larger infinities of modern mathematics were individuated. And why should inconceivably large infinities not be individuated forever? Doing that would be no more tedious for a God than breathing is for a healthy animal.

Although this is not how we naturally think of numbers, Cantor's paradox was paradoxical. And given theism, it is a reasonable possibility. So, since a dynamic Creator could, just possibly, be individuating logically possible selections forever, hence such possibilities could, just possibly, be growing ever more numerous. And since there is no other way of avoiding the contradiction – no way that is not too odd – hence those possible selections are indeed growing in number. Now, outside the context of the absolute dependency upon their creator, of things created out of nothing, there is no possible way – that is not too odd – in which those possible selections could be growing in number. So, since they are growing in number – according to logic – hence there is such a God.

Maybe there is not a God – indeed, maybe this is all a dream – but we do have a logical proof that there is a God. Maybe we are all unavoidably illogical; but if we assume logic, then we get a God. Note that it is quite possible that there was always as much arithmetic as humans will ever discover, and that the endless construction of humanly unimaginable arithmetic is not just irrelevant to the practice of mathematics – as embodied by the axioms of standard set theory – but is a positive pleasure for God. Why would God not be the sort of person who enjoys mathematics?

So, things that are as Cantor's famous diagonal argument shows them to be could, just possibly, exist within the creation of a God, were that God not classically changeless. And that fact yields a reason why there is – or is probably – such a God, because there is no other way in which things as we know them to be could exist – or at least, no other way that is not too odd has been found, after many years of many intelligent people looking for another way. There are lots of arbitrarily weird alternatives that we have not even considered, but only because they are all far too odd. A being that is not much like a God, and which would therefore be unthreatening to an atheist, but which could individuate possible selections, for example. Or such possible selections differentiating themselves, somehow. It makes no sense, of course. It makes some sense to think that if there was a creator of things in general out of nothing, then that being might be able to do it. And of course, in being logical rather than empirical it differs from such arguments as the human eye, or purported miracles. magic

The God of this chapter is not being invoked in order to explain something that has a realistic chance of having an atheist explanation, or which might not need explaining. A lot of very clever people have been looking for an atheist explanation for over a hundred years, and they all ended up doubting the very logic that led them to their atheist worldview in the first place. They ended up with atheist cultural beliefs, to defend rhetorically; but we do now have a lot of evidence that no atheist explanation is possible. And while some things may never be explained, it is certainly not the case that chapter 3 might be something that has no explanation. Rejecting an explanation when there is one is quite different to not needing an explanation. There was a time, for example, when few people saw any need to explain how there could be rainbows. Newton's explanation could have been—and was, by some—rejected for being unnecessary. But there was no good argument against Newton's explanation there. Nor was the fact that it was left unexplained how raindrops are able to refract white light. All scientific explanations create further mysteries.

Consider some unimaginably complicated collection of logically possible selections (of possible selections). All the things in that collection are there, and so all the instances of 'just some of them' are there, among the rest. Is it conceivable that not all those instances count as individual things? It seems not, because when we imagine such an instance, those things are there, and they are distinct from those of any other instance. And yet, to imagine such an instance is to imagine those things being already separated out from the rest. While each instance is a logically possible selection, for a logically possible selector, it might be that for the most unimaginably complicated selections, the only possible selector is God. We can imagine other selectors, but if there is a God then it may well be that that is the only selector possible. It might be that the most unimaginably complicated of the logically possible selections are not differentiated or individuated until such a creator of things in general differentiates or individuates them, simply by thinking of them, in the absolutely definitive way of such a being. To most us, such possibilities seem as immutable as the laws of physics – paradoxically – but to a God the laws of physics are mutable.

Each of the more complicated of the possible selections is a subcollection of a collection that is already there, and so they too are already there. The way we think of things, in general (as things), makes it difficult to imagine how such subcollections could be coming into existence. That is why all those possible selections were thought to be already there. But, if things in general were created by a God, then (i) their metaphysical foundation and their mathematics might, just possibly, have been created too, and (ii) there would be a reason for there being things, for example the creation of some finite number of souls, and a variety of environments for them (although it may well be something less easy for us to imagine, of course). In that case, there would be a profound difference in kind between the small infinities – those of the counting numbers, or equally N, for example, and maybe also those of finitely dimensioned spaces of points, or equally P(N) – and the others.

It would be difficult for atheists to deny that there might be such a difference, when standard mathematics embodies a difference in kind between small infinities (given by the axioms of standard set theory) and large infinities (given by extra axioms). But of course, I am talking about a different kind of difference. It is quite possible – if God is quite possible – that things are such that the more complicated of the possible selections based upon them (in the manner of chapter 4) are subtly but fundamentally different to the possible selections that we are more familiar with. The latter might be more like the things of our world. Perhaps such arithmetic as we might ever do always existed, as part of the timeless concept of a thing, with only the larger numbers being forever differentiated by a God (whose knowledge can increase), as part of the process of creating things made necessary by the nature of the concept of a thing, as conceived of by a God who can create things out of nothing.

Note that what matters here, for this proof, is whether that is remotely possible. It makes no difference to the part it plays in the proof, whether it is part of your philosophy of mathematics or not. The proof just needs that a God whose knowledge can increase in some such way (although perhaps not in this spacetime) is a reasonable logical possibility. The proof works by way of the alternatives being far too odd. So, if you think that such a God is far too odd (on a par with true contradictions perhaps), and that limiting our primate logic to tried and tested matters would be reasonable, then there is no proof here for you. This is only a proof, not propaganda.

There are basically four ways of thinking about numbers, and other mathematical objects. We naturally think of them as timeless, but secondly, modern mathematics explicitly creates its mathematical objects, such as those within standard set theory. And if there is a God, then we might think of mathematical objects as being timeless parts of either the environment or the mind of God, or fourthly, as being created. Those are four very different things: mathematical objects existing objectively in an atheist world, or their being our creations there, or their existing eternally in the mind of God, or their being God's creations (to some extent at least). Of course, in a sense all of mathematics is the creation of human mathematicians; but there are those four way of thinking about mathematical objects, nonetheless.

Insofar as the more complicated of the possible selections are possibilities, we can fairly easily imagine them being differentiated. However, insofar as they are subcollections we find that difficult to believe. But then, we can imagine that subcollections need to become one thing somehow – maybe by being collected by a collector – on top of the many things that the things of collections are. God would be doing something like making such selections, or making collective reference, in order to individuate those possibilities, but we do not really know the sort of thing that a God would be doing. Similarly, how would any God create anything out of nothing? Would it be like sculpting, from existing rock, but without the rock? Would it be like dreaming? Not much like either, presumably. But that does not mean that such a creator is very unlikely. Similarly, how do feelings arise in a world of randomly moving atoms? How does a Big Bang get started? How does gravity work?

Even the most vocally atheist of academic scientists feel obliged, by the demands of academic and scientific rigour, to say, not that there is no God, but that there is probably no God. That is pretty good evidence that it is logically possible that there is a God, a creator of things in general out of no pre-existing material. If it was not even logically possible, then they would presumably say that it was not. Given that it is logically possible, then what matters is how strong the evidence for it is.

It is understandable that scientists would not have thought much about creation out of nothing. Such things are talked about in religious circles, and seem to have little to do with science. But when we have a contradiction unless we were created out of nothing, then scientists do have a reason to think about it. And they have all their usual reasons for thinking about it logically. And it is not just the consistency of natural logic that real scientists actually assume in practice. They also, for example, assume the falsity of Cartesian-sceptical hypotheses. It is logically possible that there is an identical spacetime—off in some hyperspatial direction—and that bits of it are constantly swapping with bits of this spacetime. Nevertheless, we naturally refer to the things of this spacetime as though they were was no such swapping. We do not make that assumption because it is the simplest thing to do. Rather, it seems simple to us because we do it without thinking. And the "theory" that this is all a dream is much simpler than all of biology, chemistry and physics put together. Nevertheless, no scientist would conclude that this is probably all a dream, just because a favourite sports team lost the big game (which they would not really have done were this a dream). So, why would any scientist conclude that logic is probably inconsistent, just because the alternative is not their favourite metaphysics?

Whatever evidence you have that there is no God, it is more than counterbalanced, in any rational assessment of the evidence, by this very logical reason why there is, very probably, a God who changes. And many atheists do like to think of atheism as the product of a rational assessment of all the evidence. Atheists could keep atheism and replace logic with symbolic logic, but the atheism that they kept would be a cultural belief, on a par with a religious belief. Logic is a ladder that helps you to climb, step by logical step, from the evidence to the truth. It is very scientific. Who would throw that ladder away, just because it had got them to the evolution by natural selection of primates such as ourselves, and then to Cantor's paradox? Those who would seem to have been prevailing in academic science, throughout the twentieth century. But the twentieth century might have benefited from being less illogical.

It is quite possible – given that a God is a logical possibility – that the more complicated of the possible selections, or subcollections, are such that they are to some extent like those possibilities that, as we have seen, could – conceivably – be differentiated, and to some extent like subcollections that can be counted. Of course, such individuation is like nothing that we creatures could do. And as we saw above, we do not naturally think of the paradoxical subcollections like that. But they are paradoxical. And such individuation does seem to be the sort of thing that a God could do. And if there was a God, then it might – just possibly – be the case that all collections of things are such that they do all need a collector. It could just be that they do not seem to need one, when seen from within a creation that is such that normally it does not even seem to need a Creator.

To recap the argument so far, we started with some things (the food in a bowl) and found that there had always been more, many more things, all of them equally distinct. The method was to notice all the logically possible selections from—or, all the subcollections of—a collection. I selected, for example, a couple of ways of selecting items of food from the bowl: selecting the apple and the pear; and selecting just the apple.

Each of those subcollections is just some of the given things, and so of course they are already there. But since they are primarily there as some of the given things, not as one subcollection, we naturally wonder if each one really would be there, in the absence of a person who could refer collectively to them, as that one. That is why chapter 3 was about logically possible selections. A very natural solution would otherwise have been the observation that collections of things are not one thing but a number of things. That shows how conceivable it is that a referrer is needed, for subcollections to count as one thing.

Incidentally, my argument does not refute classical theism, no more than it shows that there are not true contradictions. But it does mean that classical theism does need such true contradictions. That is not so much a problem for classical theism, which has always embraced the idea that God transcends logic. Cantor's own resolution of his paradox was that the inconsistency of the infinitude of the whole numbers was a fitting symbol of God. On the other hand, what if people say that they do not want to go to hell, but that they will do bad things anyway because this is only inconsistent? It is God who would decide who goes to hell, and God is above logic. And how fitting is an inconsistent arithmetic, as a symbol of God? It is one thing to say that God is above and beyond logic, it seems to be another to say that God is inconsistent. But this is only a bit of a problem. A religious believer does not have to care about the logical details. Furthermore, a believer could accept the force of logic, and the force of my argument in particular, and still reserve judgement on the grounds that there might be subtle mistakes in it. And moreover, even granted this proof that God is not classically changeless with respect to arithmetical knowledge, God could still be unchanging in most of the ways that matter religiously. God could be sure to keep promises, for example. And God could still know all about our futures (if presentism is false).

The biggest problem with this proof that God exists is that most scientists are atheists. Scientists are the modern experts on what there is. And we naturally defer to expert opinion on such matters. So, it would not be unreasonable to think that there must be a logical flaw somewhere in the above. After all, there is always the possibility of a subtle mistake, in any proof, of anything. But a hundred years of searching for a logical flaw in Cantor's paradox does amount to considerable evidence that there is no such flaw in the above.

There is always the possibility of a very subtle mistake. But that is true of every single proof (arguments for the earth not being flat, arguments for evolution by natural selection, arguments for man-made climate change, and so forth). So although that possibility could be given as a reason for dropping the parenthetical “very,” from before the word probably (when the real reasons are personal or cultural aversions), it would hardly justify replacing it with the prefix “im-” (to make the word improbably). That would just be wishful thinking.

You might think that I have presented no actual evidence that numbers can become individuated.

Since Cantor, mathematicians have not had the same intuition that unimaginably big whole numbers are timeless.

Intuitively, numbers cannot be individuated because that is not the kind of thing that we do with numbers. But nor do we, for example, create energy out of nothing. Energy can be converted from one form to another, but it is always conserved. That does not mean that the Big Bang could not have been preceded by a complete absence of all the energy of the universe. All we know is that there is (very probably) a law of the conservation of energy (including mass-energy) within this spacetime. We do not know that a God could not create energy out of nothing. And similarly, we do not know that a God could not individuate numbers. That is a logical possibility. A creator of things out of nothing is a logical possibility, and so is presentism; and if a presentistic creator was thinking of collections of a thitherto unthought of size—unimaginably big collections that not even that creator had thought of before then—then why should those whole numbers not become individuated then? It is hardly possible to think of a reason why not. And to demand more evidence of logical possibility would be unreasonable.

It is admittedly strange that numbers become individuated, and that a Godlike being must therefore be individuating them—it is strange, but not logically impossible. For atheists, God is simply unrealistic though. The only realistic option for them is to take Cantor’s paradox to be showing logic to be not quite good enough for thinking about the totality of all the whole numbers. The observation that primates are unlikely to have a perfect logic really does seem more realistic. However, logic is a scientific necessity. What matters scientifically is that the evidence is assessed logically, not what one finds realistic. Soviet biology was rightly derided for putting soviet philosophy ahead of a logical assessment of the evidence. And the few creationist biologists that there are in America are similarly derided by their peers.

In between Cantor’s discovery of his paradox, and Russell’s infamously long proof that 1 + 1 = 2, physicists were surprised by Einstein’s theory of relativity. Einstein’s theory was verified by an observation of a detail of the orbit of Mercury, in 1919—but surely those physicists could have thought of such astronomical details as being a bit beyond the reasonable logical reach of primates, even highly evolved primates. In that way, they could have kept all their very successful physics, with its relatively simple metaphysics and mathematics. But of course, that would not have been a logical assessment of the evidence. Logic is a scientific necessity.

In chapter 3, we saw how a contradiction can be derived, in a seemingly logical way, from the concept of a whole number. In this chapter, we saw how that contradiction would not arise if the largest whole numbers were being forever individuated by a logically possible God. Insofar as we can see that there is no other way of avoiding a contradiction logically, we will have a good reason to think that there really is such a God. The proof of this book rests upon a very hard look for another logical way. Only following our failure to find any other logical way, having properly looked, will we have a proof that there is (very) probably a God of the kind described in this chapter.

5

Unreasonable Doubt

5.i Mathemagical Thinking

Mathematicians and scientific philosophers looked very hard for a logical explanation of Cantor's paradox, and in over a hundred years they found nothing that panned out properly. They constructed theories that looked very scientific, theories full of symbols and equations. But if any of those theories had been what the general reader would call logical, it would be obvious. Either mathematicians would have replaced the standard set theory, as the foundation of mathematics—had standard set theory not been a very good model of that explanation—or else, were it a good model, the philosophy of mathematics would be little but an account of how it was. Neither of those is the case (you could try googling "foundation of mathematics" and "philosophy of mathematics," to see for yourself, or look at introductions to the philosophy of mathematics). The general reader is therefore spared my describing those theories and their logical failings (if you are interested, such descriptions are to be found in the philosophy of mathematics literature).

In this chapter, I shall address the logic of chapters 3 and 4 more directly. And to begin with, there are certainly things of various kinds. Consider just some of them. To talk about those things as I am doing now, is, arguably, to have already done the most questionable thing about the proof of chapters 3 and 4. Referring to them collectively—with that "them" (the third word of this sentence) or with "those things"—is not obviously at all questionable. But it is the nature of paradoxes that they make us question the unquestionable. And those concerned with Cantor's paradox did identify, early on, the universal validity of collective reference as one of the weakest assumptions that that paradoxical reasoning was implicitly making.

A good analogy might be with Euclid's fifth postulate, his axiom of parallels, which was identified by mathematicians as being conceivably false—long before it was found to be actually false for relativistic spacetime—even though it did seem to be obviously true at the time.

But suppose there are some things, each of which is clearly distinct from the others. Each exists, in some way or other. They all exist. That is simply what we are supposing. Now, suppose there are also some other things. There was nothing questionable about our initial supposition, so how could there be anything questionable about this? But now we are considering two collections. As though we now have two extra things, one collection and another collection. But of course, it is just that there were all the original things, and there are also all these new things. Calling one lot of things “original,” and the other lot “new” is essentially the same as having those two extra things.

How could it be problematic to speak of there being different kinds of things? If there are some things—say, an apple, a banana and a carrot—and some other things—say, Andrew, Betty and Christopher—then that is what there is. And Andrew liking bananas just is a different kind of thing to Andrew liking Betty. If we chose not to talk as though there are two groups of things there, that difference would still be there. It would only be our ability to talk about it that was less than it had been. Still, a lot of theories, expressed using a lot of symbols, can be built on such a foundation, by those who know how to construct and defend such theories.

The problem that we reached at the end of chapter 3 involved a much bigger collection, which I called X. But a collection is just a lot of things. Each of the things exists, in some way or other, however many they are. And X was not even that big, compared to such hypothetical collections as the totality of all logical possibilities.

The problem with X was basically that, given that a God is a logical possibility, it is logically possible for a God to consider just some of the things in X, where that “just some” can include absolutely any of those things, simply because each of the considered things exists. A diagonal argument shows that there are more of those logical possibilities than there are things in X, which is a logical problem because X includes all the logical possibilities of that kind.

The problem at the end of chapter 3 was, then, a problem about logical possibilities. It was a logical problem. But as mathematicians replaced all their collections with axiomatic sets, collections like X looked like things that a scientific philosopher might construct a theory about. (Such a theory might say that, for example, that “just some” was only able to include some of the things in X.) The more formally those theories were written up, the more scientific they looked. And the more of them there were, the more there seemed to be a logical pursuit of the truth of this matter. But all these theories had moved the question away from the logical problem with collections. The logical problem that had originally surfaced as Cantor’s paradox was swamped by a tide of formalisms.

The overwhelming majority of mathematicians work entirely (if implicitly) within a standard set theory. In that theory, Cantor’s paradox is used to prove the nonexistence of sets whose members are as numerous as chapter 3’s logically possible selections.

Ignoring chapter 3 on the grounds that it was not good mathematics could therefore seem reasonable. Much more reasonable than regarding chapter 3 as evidence for the actual existence of the very strange entity described very sketchily in chapter 4. But although the number of all the standard sets is not defined to be a number in standard mathematics, it can be defined to be a number in a mathematics based on a set theory that has all the axioms of standard set theory plus an axiom asserting the existence of a set of all the standard sets.

That axiom would be essentially the same kind of axiom as the standard axiom asserting the existence of a set with as many members as there are counting numbers. And then, an axiom asserting the existence of a set of all those non-standard sets could be added to those axioms, to give a number of all those non-standard sets. And so on, endlessly. So, the standard approach to Cantor's paradox is not so much in conflict with, but is rather compatible with a neat mathematical model of, chapter 4's solution to the logical puzzle of chapter 3.

And mathematicians themselves are unlikely to think of chapter 3 as bad mathematics. To see why, we need only consider other scenarios that are well-modelled by paradoxical mathematics, and ask ourselves what mathematicians would say about those uses of mathematics. And to begin with, note that because adding zero to any amount does not change it, we could keep adding zeroes forever and it would make no difference. In symbols, adding 0 + 0 + 0 + … is the same as adding 0, which gives us this equation:

$$0 = 0 + 0 + 0 + \ldots$$

Each of the zeroes to the right of that equals sign can be replaced with 1 – 1, because 1 – 1 = 0, which gives us:

$$0 = (1 - 1) + (1 - 1) + (1 - \ldots$$

And presumably those brackets can be removed:

$$0 = 1 - 1 + 1 - 1 + 1 - \ldots$$

Next, we replace all those subtractions of units with additions of negative units:

$$0 = 1 + -1 + 1 + -1 + 1 + \ldots$$

In the next equation, brackets have been reintroduced.

$$0 = 1 + (-1 + 1) + (-1 + 1) + \ldots$$

And in the next equation, each (–1 + 1) has been replaced by 0.

$0 = 1 + 0 + 0 + \dots$

All the zeroes on the right-hand side of that equation can of course be ignored, which gives us:

$0 = 1$

That is clearly not true, so we have a mathematical "proof," of an untruth. And if you look back up that list of equations, you will see that each is essentially the same as 0 = 1 until the point where negative units had been introduced.

Was it with the introduction of negative units that that "proof" went wrong? Negative numbers are fairly familiar. Increasing a debt, for example, can be thought of as moving in the opposite direction to increasing wealth. Debts are negative, where wealth is positive. But the use of negative numbers does mean that all the ones in that equation are really positive numbers. Now, positive and negative numbers have directions as well as magnitudes, they are essentially vectors. And so the 0 in that equation—and in all the subsequent equations—would have been the zero vector. That is a vector with no particular direction, so it is a pretty strange thing. Something went wrong in this "proof," and vectors are certainly not normal numbers.

However, mathematicians would identify the place where it went wrong as the equation before the introduction of negative units. Following the removal of the brackets, we got an equation with 1 – 1 + 1 – 1 + 1 – 1 + … on its right-hand side. That is called the Grandi series, and mathematicians say that it has no sum, for the following reason. Mathematicians do infinite sums by working through such series step by step and seeing what happens. Summing the first two terms of the Grandi series gives them 1 – 1 = 0. Summing the first three terms gives them 1 – 1 + 1 = 1. Summing the first four terms gives them 1 – 1 + 1 – 1 = 0. And so on. If those sums had been getting closer and closer to one particular value—with the difference from that value getting closer and closer to zero—then that value would have been the sum of the series (which is how such series are summed in the calculus). But they did not do that; they were alternating endlessly, between 0 and 1.

It was the paradoxicality of that sort of "proof" that led mathematicians to sum infinite additions and subtractions (called series by mathematicians) in that way. Whereas, the Italian mathematician Guido Grandi (1671–1742) thought that the way in which the series that now bears his name can sum both to 0 and to 1 showed that creation out of nothing was a reasonable belief.

Creation out of nothing is not an unreasonable belief. The entire universe (very) probably began in a Big Bang, and it would not be unreasonable to believe that it might have been preceded by nothing, because it would not be unreasonable to believe that time itself began with the Big Bang. However, Grandi's argument was more rhetorical than logical. It reminds me of Cantor's justification of his paradox on the grounds that it was symbolic of the ineffable infinity of God.

Anyway, a hypothetical creation of a particle "out of nothing" (actually out of the energy of the vacuum) can be modelled mathematically by the Grandi series, as follows. In modern physics, a particle and an antiparticle can appear together from the background energy of the vacuum. You may have heard of such pairs, because they give rise to Hawking radiation from a black hole. Once formed, the particle and the antiparticle move away from their point of origin, which we could picture like this: /\

Near a black hole, one of them might be swallowed by the black hole, leaving the other moving away from the black hole (as a bit of Hawking radiation). But in that picture, they are moving downwards, with the particle on the left-hand side. Now, an infinite line of such particle/antiparticle pairs might—just possibly (although it would be highly unlikely)—appear within an infinite space.

Space does not appear to be infinite, but an infinite space is certainly a theoretical possibility. And such a line of particles and antiparticles might—just possibly—move in such a way that each antiparticle collided with the particle immediately to the right of it, which we could picture like this: /\/\/\/\/\...

The top of that zig-zag represents pairs of particles and antiparticles spontaneously appearing, which is modelled rather well by this infinite addition:

(1 + −1) + (1 + −1) + (1 + −1) + …

That addition sums to zero, representing the empty space that those particles and antiparticles appeared within. The middle of the zig-zag represents an infinite line of particles and antiparticles, which could be modelled by this infinite addition:

1 + −1 + 1 + −1 + 1 + −1 + …

And the following addition models all the antiparticles colliding with particles, as represented by the bottom of the zig-zag:

1 + (−1 + 1) + (−1 + 1) + (−1 + …

Because that sums to 1, while the first addition summed to 0, we have a pretty good model of the spontaneous appearance of a particle in an infinite space.

Would a mathematician object to that very simple mathematical model on the grounds that the middle addition has no sum? But having no particular sum is what allows the adding of brackets to it, in two different ways, to give rise to two different sums. It is what makes this an appropriate model of this hypothetical scenario. Whether a particle might spontaneously appear in an infinite space, or not, is clearly a question for physicists, not mathematicians; so, why would a mathematician object to that model of that scenario, on the grounds that the Grandi series has no sum in the calculus? That model was not in the calculus.

And similarly, although Cantor was doing mathematics, he was not doing it within a set theory. And while Cantor's paradox is essentially a mathematical paradox, chapter 3 is based on the logical essence of Cantor's paradox. Chapter 3 is essentially metaphysics, and is part of a logical argument, not a symbolic-logical argument. Cantor originally had a mathematical need—for a certain kind of symbol for infinite amounts (mentioned in passing in the second section of the next chapter)—which he satisfied in a mathematically acceptable way. Part of that involved identifying ordinary arithmetic as one of the results of thinking logically about equinumerosity. By further thinking logically about equinumerosity, Cantor got as far as the paradox that bears his name. But chapter 3 is based on the logic of equinumerosity, not the mathematics of Cantor's paradox.

To see more clearly how chapter 3 was not bad mathematics—even though Cantor's paradox was mathematics, and the mathematicians' problem was resolved by axiomatic set theory—because it was metaphysics, not mathematics at all, consider the case of Archimedes' famous discovery. Archimedes noticed that you can measure the volume of an irregularly shaped object—such as a crown, or a human body—by immersing it in water and measuring the volume of the water displaced by that object. He was thereby able to work out that the crown in question was indeed made of gold, by weighing it and working out its density—its weight divided by its volume—which he found to be the same as the density of gold. He was doing mathematics. But his observation about displacement relied upon gold and human bodies being the sorts of substances that do not absorb much water, even when immersed in it, nor the sorts whose own volumes change much when surrounded by water. It relied upon facts of physical chemistry, which it said little or nothing about.

The argument of chapters 3 and 4 is logical, not empirical, so it has to be more rigorous than Archimedes' argument. But the logic of the argument of chapters 3 and 4 requires only that the God that chapter 4 describes is a logical possibility. Consequently, that kind of God only needed to be described very sketchily. The rigour comes from chapter 3 being so paradoxical, there is clearly no other logically possible solution to that logical puzzle.

More familiar to scientists would be a theory modelling mathematically some physical phenomenon or other. Which would be described in more mathematical detail. But such a theory would say nothing about how, for example, charges cause forcefields, or how forcefields move charges. To see more clearly how chapter 3 was not bad mathematics, it may help to consider a case that is more like Cantors' paradox.

Whole numbers, together with ratios of whole numbers, such as ½ and ¾, are collectively called rational numbers. Ancient mathematicians had done a lot of mathematics with rational numbers, when they found a proof that some numbers are not rational numbers. But by then, it was too late for that proof not to look like a "proof," of something unquestionably untrue. The idea that all numbers were rational numbers had become an integral part of the worldview of ancient mathematicians. In part that was because there was a numerological side to ancient mathematics, which associated the power of mathematics with everything having an occult number. Femaleness might have the number 2, for example, and maleness 3. Ancient mathematicians could work out the meanings of more complicated numbers by breaking them down into simpler numbers (5 = 2 + 3 might represent marriage, for example).

For such reasons, a simple proof that one particular number was not a ratio of whole numbers—mathematicians call such numbers irrational numbers—was for them a paradox. It is easy for us now to see that proof—which will be described shortly—as a proof, because mathematics was built on a foundation of such proofs. But we can imagine how they would have reacted to the discovery of that proof. Perhaps they demanded evidence that numbers could possibly be irrational, before they would accept that the proof proved what it seemed to be proving. After all, you might have wanted to see more evidence that the logical possibilities associated with whole numbers can, just possibly, be individuated endlessly.

But of course, if ancient mathematicians were not going to accept that proof, as evidence enough, then what could they possibly accept as evidence? An infinitely precise measurement of an actual length of material? That would have been impossible; and even if they got such a measurement, they would then have had to see that it was not a rational length. Going through all the infinitely many rational lengths, comparing each one with that given length, would also have been impossible. Although I suppose that those ancient mathematicians could have taken such talk of infinite impossibilities to be showing just how absurd irrational numbers are.

The best evidence there could be, that the logical possibilities associated with whole numbers can be individuated endlessly, is similarly Cantor's paradox, in the very logical form of chapter 3. That chapter is basically a proof that they are. The fact that such individuations only make sense if there is a God of the very logical kind described rather sketchily in chapter 4, to do those individuations, does not count against that. On the contrary, it explains why everyone in our scientific culture feels, to varying degrees, that there should be more evidence.

Now, this is complicated enough, so I will be considering a fictional culture, instead of the ancient Greeks, in order to keep things as simple as possible, by eliminating the need to consider what actual Pythagoreans might have said.

This fictional culture has been making great progress by teaching the importance of respecting teachings. To show any of their teachings to be false would therefore be to undermine all that progress. One of their teachings is that all lengths are ratios of whole numbers of a unit length. And they have a proof—which they take to be a "proof"—that some lengths are not expressible as such ratios.

Lengths were taken to be counting numbers of some unit of length. For any given length, taking the unit to be small enough enabled that length to be measured accurately enough for all practical purposes. Long distances are measured in miles, small lengths in inches.

At school, you may have measured the area inside a shape by copying that shape onto a grid of small squares (called "graph paper"), and then counting the squares inside that shape. That might sound slow, but for the bulk of the inside of the shape you would use larger squares, such as seven by seven of the small ones (adding another small square nearer the edge makes a total of fifty), or a large rectangle (five squares by ten makes fifty too). That can be quite accurate because around the edge you can pair up squares, such as one that is only a quarter filled with one that was three quarters filled, counting those two as one full square. By using small enough squares, you can easily get a result that is accurate enough.

Ratios are easy enough to understand. If I have twice as many sheep as you do (a ratio of 2 to 1) then you have half as many as I do (½ as many). And ratios were empirically adequate. Expressing lengths as ratios or fractions a/b, for counting numbers a and b, enabled any length to be approximated, as accurately as b is large, by taking a of those small parts. That might seem primitive to us; but to them, it was sophisticated. And they had some impressive architecture to prove it. Their culture was the greatest and most rational that the world had seen. They felt about their rational numbers much as our mathematicians feel about the axiomatic sets that they use to do the whole of standard mathematics with. They too could explain what "1 + 1 = 2" means. Although 1 + 1 is clearly not the same thing as 2, 1 + 1 represented two unit lengths placed end to end, and 2 represented the length of something with the same length as that.

Larger numbers can be regarded as sums or multiples of smaller numbers. And fractions – for example ¾ – are ratios of whole numbers (3 and 4, in that example). Trivially, whole numbers can be thought of as having 1 for the bottom number, in such a ratio, and so they can be counted as rational numbers too. If all numbers are, in that sense, rational numbers – ratios of whole numbers – then everything has a meaning, in a coherent system of meanings. There is this association with the other sense of "rational" (although we would not now think of numerology as rational).

Fractions were the first extension of the counting numbers. When we divide a whole cake into two equal parts, each part is half of that cake. In symbols, 1/2 = ½. And another fraction is three quarters, ¾. Each fraction can be represented by two counting numbers, one over the other, separated by a forward slash. The unit (such as the unit length) is divided into the bottom number of equal parts, and then how many of those parts make the fraction is given by the top number.

Extending the integers—which are positive and negative counting numbers, plus zero—in the same way as gets us the fractions from the counting numbers, gives us the rational numbers, whose name comes from the word ratio. Other possible lengths of geometrical lines (excluding infinitesimals) are therefore known as irrational numbers.

In my fictional culture, there is a similar commitment to all numbers being rational numbers. Although these fictional mathematicians have also been developing proper arithmetic and geometry, within a civilisation that similarly prides itself on its rationality. As we join them, they have just discovered a proof that the length of the diagonal of the unit square is not a rational number. To them this was unthinkably irrational. They felt about this much as psychologists might feel if they had to try to imagine the human mind as a spirit in the complex machine of particles that is the human brain, just because of some reports of some experiments in some psychology laboratory. Our psychologists would look long and hard for where those experiments had gone wrong. And similarly, these fictional mathematicians looked for where the so-called 'proof' could conceivably have gone wrong.

The length of the diagonal of the unit square is the square root of 2. The square root of 2 (or root two, for short) is the number that squares to 2. The square of a number is just that number multiplied by itself. It is called "squaring" because we calculate the area of a square by squaring the length of its sides. To see this, imagine a hundred little squares arranged in ten rows of ten squares each, the rows sitting on top of each other to make one big square.

Now, the diagonal of a rectangle or square divides it into two triangles, each of which has one angle which is like the corner of such a rectangle or square. In other words, they are right-angled triangles.

The longest side of such a triangle is opposite that corner (or right angle), and according to a well-known theorem of Pythagoras, it has a length whose square equals the sum of the squares of the lengths of the other two sides. Proofs of that theorem are simple enough, so we can assume that it was known to our fictional mathematicians. The best-known example of the theorem is a triangle whose sides have lengths 3, 4 and 5 units. Such a triangle has a right angle opposite its longest side, because 25 = 9 + 16.

And the square of the length of the diagonal of the unit square is similarly 2 = 1 + 1. So, the length of that diagonal is the square root of 2. To prove that this length is not a rational number, we begin by hypothesising (falsely) that this length is a rational number. We are going to get a contradiction by taking only logical steps, which will prove that our hypothesis was false. But for the ancient mathematicians, this hypothesis was an axiomatic—an unquestionable—truth. They got a contradiction from logical alone (apparently).

In symbols, we begin by supposing (falsely) that a2/b2 = 2, for some counting numbers a and b. Here ‘a’ and ‘b’ are standing in for some counting numbers that we do not know (indeed, cannot know, since our supposition is false). But if a was 2 and b was 3, for example, then a2/b2 would equal 4/9, which is roughly a half.

We can also assume that a and b have no factors in common.

That is because any common factors could have been cancelled out. A factor of a number is another number that divides into that number. One factor of 6 is 2, for example, because there are three 2s in 6. (We do not count 1 as a factor.) So, if a and b had had a common factor of 2, for example, so that a = 2c and b = 2d, for counting numbers c and d, then a/b = 2c/2d = c/d. And quite generally, the common factors would cancel out. To continue with that last example, if c is 2 and d is 3, then a/b = 4/6 = 2/3. We could therefore have started with c and d, since they would have had the same ratio. And in general, either c and d have no common factors, or else we could do the same cancelling out with any factors that they did have in common. So, we can simply assume that a and b have no factors in common, that such cancelling out has already been done.

Having supposed that a2/b2 = 2, where a and b have no common factors, we now proceed logically, step by logical step, until we get a contradiction, which will show that our original supposition was false (although the ancients had to find another explanation).

Firstly, we multiply both sides of that equation by b2. That gives us another equation, and if the one that we have already got is true, then so will this new one be. It is like putting equal weights on both sides of a balanced pair of scales. So, multiplying both sides by b2 gives us: a2 = 2b2.

That means that a2 is twice some other counting number. In other words, a2 is an even number.

An odd number times an odd number is an odd number (because an even number times an odd number is clearly an even number, and if we add another of our odd numbers to that, then we get an odd number). So, if a square of a number is even, then that number is even.

So, since a2 is even, hence a is even. In other words, a = 2c for some counting number c.

Putting that value for a back into our equation, a2 = 2b2, gives us 4c2 = 2b2, or 2c2 = b2, which means that b2 is an even number, which means that b is even.

And that means that b = 2d for some counting number d, contradicting our assumption that a and b had no factors in common.

From that contradiction it follows logically that this length, whose square is two, is not a rational number. In other words, there is no fraction which, when multiplied by itself, results in 2.

That was a mathematical example of a proof by reductio ad absurdum. It shows that some lengths cannot be expressed as fractions. In other words, it shows that there are irrational numbers. Adding to that proof an unquestionable claim that all numbers are rational numbers would give us a paradox.

They had to find out where their paradoxical argument had gone wrong—a task akin to the task that our mathematicians set themselves when faced with Cantor’s paradox. Some of these fictional mathematicians wondered if they had a proof that there can, occasionally, be true contradictions. Perhaps some contradictions describe reality. After all, some of them wondered, should we not assume that everything the gods say is true, even if it is contradictory? If so, then there are true contradictions, and maybe this is just another. Other mathematicians disagreed, naturally enough; but they could not quite see where the argument had failed to be a proof.

A mathemagician saw their confusion, and pointed out to them that they did not really know that such precise lengths existed. When lengths are measured, they are never measured so precisely. For all we know, the mathemagician observed, things have lengths only as accurately as we measure them. In themselves they might be quite fuzzy until we look at them. We do not know that they are, the mathemagician observed; but nor do we know that they are not. We simply assume that they are not, when we calculate with the lengths that we have measured. Lengths are properties of physical things, as colours are. But colours are not as objective as the coloured objects that have those colours. So why should we assume that lengths are?

The decimal expansion of the square root of 2 is an endless sequence of digits; but perhaps that just means that there is a rational number appropriate for any degree of precision and any size of square. This proof is really, the mathemagician suggests, a proof that this is indeed the case. Were it not the case, there would be absurd inconsistency with the indisputable fact that all numbers are rational numbers. When they discover the quantum-mechanical foundation of the chemical properties of physical objects, they will think that they were right (which is a bit like the reaction of mainstream philosophers to logical paradoxes).

Alternatively, why assume that very large numbers are just like the ones that we are familiar with, only bigger? We have no direct acquaintance with extraordinarily large numbers of things. The square root of two (the number that squares to two) could be a ratio of two extraordinarily large numbers. It could be that we have a reason why such numbers would not be like the ones that we calculate with: the assumption that they are just like them leads to a contradiction with a mathematical proposition that most mathematicians – who are reasonable people – believe to be true, the proposition that all lengths are rational numbers.

Eventually most of these mathematicians decided to have fictional lengths that were whatever they wanted them to be. They were the experts, after all. And they felt that, in that way, their future – and the futures of their children – would be secure. This resolution also meant endless employment for the mathemagician, because epicycle after epicycle of fictional properties had to be added to the fictional lengths, in order to keep them empirically adequate. And who better to do that than the mathemagician?

What should be clear to us is that denying that numbers – even just the very large numbers – exist is no way to rid ourselves of the meaningful contradiction, which actually shows that the square root of two is not a rational number.

Do ratios of lengths exist? You can no more point to a ratio of lengths than you can point to a possible selection of possible selections. But you can point to a possible selection of items of food from a bowl, though: to point to one of the items of food is to make such a selection. And we have just been referring to – and hence have selected – such a selection. And in a similar sense, ratios do exist. One third of the food in the bowl is an apple. Ratios do not exist in the same way as apples do, but ratios of food are no more fictional than they are. Ratios have to be logical, because they are not fictional. Such metaphysical questions as what kind of thing ratios and other numbers are should therefore have no bearing on the meaning of the contradiction in the proof.

Similarly, the logically possible selections were logical possibilities. The fact that we find it hard to answer the metaphysical question of what kinds of things logical possibilities should have no bearing on how the paradoxical contradiction of chapter 3 should be explained. Saying that there are no such things does not make standard set theory a better explanation.

The replacement of our natural conception of numbers of things, with axiomatic sets, was a similarly extreme reaction to a contradiction. Now, there is nothing illogical or unscientific about using such a mathematical model of mathematics as a temporary measure, while the real cause of the contradiction is sought for. But it would be wrong to regard it as close to a satisfactory solution of the mathematical problem of Cantor's paradox. Anyone could be forgiven for thinking of standard set theory as satisfactory, because it has been the standard foundation of mathematics for a hundred years. It was very familiar to those who taught our teachers. But it is basically the same kind of thing as that analogy.

You might think that a big difference between those analogical rational numbers and whole numbers is that there the argument was showing a mathematical fact about lengths – that some of them are not rational – whereas my argument about whole numbers will have been showing that there is a God (of some peculiar kind). However, my argument was in the realm of the presuppositions of arithmetic, and it will have been showing something about the same, rather metaphysical realm – not that there is a Zeus on some cloud somewhere, but that things were created out of nothing, so that constructivism in what we now see – from our perspective in the world of things – as the possibilities of things would be a realistic possibility.

The problem with our possible selections is not such simple collective reference as that calling of them 'our possible selections' (or that later 'them'). It is that although there would already be all of those possibilities (that such steps as we were taking could take us to), there was nonetheless a next step of exactly the same kind. That next step is there because since all of those possibilities are there, hence so are just some of them. The problem arises – via a diagonal argument – from all the different instances of that 'some of them'.

The problem is not our calling of them all, collectively, 'some of them'. That does not enter into the diagonal argument. The problem is their being distinct from each other, their being just as distinct from each other as our original things were. That is what means that there can be as many of them—in the intuitive sense of equinumerosity—as some other things, or more of them, or less, whichever is the case.

Those instances of ‘some of them’ can only be distinct from each other because they are things that can be compared with each other, because each of them is one thing. There can only be a whole number of them – in a non-standard sense – because each of them counts as one. This seems obvious, but we are running out of places where the cause of the contradiction could be. Now, we have seen how standard set theory is, of necessity, a little unrealistic; but there will, of course, be good reason why mathematicians originally chose to look at mathematical models of some things being collectively one thing, when they were looking for a way around Cantor’s contradiction. We should therefore entertain the possibility that for the biggest, most complicated collections of logically possible selections (of possible selections), not all the instances of ‘some of them’ are individual things.

It is of course hard to see how that could be a possibility. That is why we are only now considering it. But even the standard response to Cantor’s paradox contains echoes of this possibility. The most controversial axiom of standard set theory is the axiom that is used to get a set of just some of the things in any set of things – the axiom of Choice. Starting with any set of things, this axiom gives us another set that has a single member, one of the things from that original set. That single member could be any of those original things – it is as though the axiom has chosen one, or picked one at random.

Repeated use of the axiom of Choice will give us an arbitrarily wide range of subsets. The axiom of Union – which gives us, from two original sets, a set whose members are all the members of the original two sets – can then be used to stick those single-membered sets together. The axiom of Choice thereby enables standard mathematics to model such collections as that of all the logically possible selections from some given collection.

The major alternative to set theory, as a mathematical response to Cantor’s paradox, was mathematical constructivism, according to which mathematical objects are constructed by mathematicians. Constructive mathematicians avoided the threat of other possible falsities in mathematics – Cantor having shown that there was one – by being extremely minimal in their suppositions. And there were mathematical doubts about Choice because it is non-constructive: it says that there is such a set with a single member without saying which one of the original things that member is. Of course, it is hardly questionable that, given a set of things, there is one of those things. Nevertheless, it is the case that even in standard mathematical proofs, where the use of any of the other axioms would go without saying, the use of Choice will often be mentioned in a footnote.

Nevertheless, there are reasonable doubts about the existence of all the instances of 'some of them', for our most complicated possible selections. If they are all there automatically, as we would naturally assume, then we do get an untruth (a contradiction), apparently logically. And for them all to be there automatically, we would need it to be the case that, in general, many things are one thing. In English, it is hard to talk about a number of things without referring to them collectively (it seems to be impossible to do so unambiguously). But there is also the philosophical or metaphysical question, of what makes a many a one. The obvious answer would be some act of collecting or selecting or referring (and so forth). When we imagine them all just sitting there, they are just as many as they are; there are not then all of the subcollections (and so forth) too. Nevertheless, it is not as if those subcollections are not there. That is why we were so logically able to get to a contradiction. Now, our problem was not just caused by those instances of 'some of them' being individuals – individual instances – it was also due to their being possibilities, and hence being there already.

Many atheists—and some classical theists—will undoubtedly say that my proof was insufficiently rigorous mathematically. My use of non-standard numbers was not at all justified—they will say—by any proper philosophy of mathematics. But those are not reasonable doubts, to entertain about my proof. It is unscientific to stipulate evidence away—which is what the standard set-theoretical foundation of mathematics does (in the case of Cantor's paradox, and chapter 3).

Regarding the philosophy of mathematics, consider the following three facts, F1, F2 and F3:

F1 The overwhelming majority of professional mathematicians are not going to be wrong about what numbers are.

F2 The overwhelming majority of mathematicians assume, in their professional work, that numbers are axiomatic sets.

F3 Numbers are not axiomatic sets.

Those three fact appear to be inconsistent. And F1 is common sense, while F2 is a matter of record, and so a lot of mainstream philosophers deny F3. But for thousands of years, most mathematicians took whole numbers—in the most basic sense (numbers of things)—to be what we all learn they are when we learn that part of our native language. Basically, whole numbers are properties of collections. They are instantiated in, and are abstracted from, collections of things, much as colours and shapes are instantiated in, and are abstracted from, ordinary objects. And it is, of course, only common sense that those mathematicians knew what numbers are.

The conjunction of F1, F2 and F3 would be a contradiction, were the mathematicians referred to in F2 not just assuming that numbers are axiomatic sets for the purposes of proving theorems from axioms. But as most mathematicians know, that is probably what most of them are doing. Proofs have to start somewhere, and the standard axioms are pretty good descriptions of numbers of things, especially when we are not considering such large collections as that of chapter 3.

If you ask mathematicians what numbers are, they may well be unable to give you an answer immediately. That is because there is something clumsy about the question. There are lots of different kinds of numbers. After a moment's thought, a mathematician might refer you to lots of lists of axioms. Axioms are what mathematicians work with. If you ask mathematicians where numbers are, they may not be able to answer at all. But that is obviously because of the strangeness of that question. It is not because it is the business of mainstream philosophers, what numbers are.

With all those axioms, mathematicians can seem to be creating numbers—much as writers of fiction create fictional characters. But mathematicians are primarily modelling actual numerical properties of reality, and occasionally varying those models in order to investigate those abstract structures. The abstract structures themselves are real abstract structures, not fictional abstract structures. Where are they? They are instantiated in—and are abstracted from—real or possible structures, much as colours are instantiated in, and abstracted from, coloured objects, real or possible. Numbers exist in a sense that is sensible enough for mathematicians to be able to study them logically. Such things have to obey the laws of logic. Fictional abstract structures, by contrast, are whatever their author says they are. Where are they? They are in the fiction, much as the colour of magic is in the fiction of that name.

Mainstream philosophers also ask questions like: Do numbers (or sets) exist? If they do, where are they? If they do not, then what does the symbol 2 refer to? The implication is that since numbers (or sets) are abstract objects, hence if they do exist then they exist in some Platonic realm of abstract objects, raising the question: How is it that mathematicians can gain access that realm, in order to know such properties of numbers as arithmetic? But similar questions can be asked of colours, shapes, value, truth and so on and so forth. Do such questions help us to understand such things? Do they further the mainstreamal task of describing accurately what such things are? Or do they just give rise to peculiarly "philosophical" problems, which might easily be solved by seeing how inappropriate the questions are?

Before the mainstream philosophy of mathematics could be used as an objection to the content of this book, the appropriateness of those questions about numbers (or sets) would have to be justified. And there is an obvious inappropriateness to this question: "Does value exist; and if so, what colour is it?” Clearly some things have value. And so value clearly exists, in that sense. But it makes no sense to ask what colour it is. It would clearly make more sense to blame any difficulties encountered in answering such questions on the inappropriateness of the questions, rather than on the postulated existence.

Mainstream philosophers also ask questions about truth that are like the questions that they ask about numbers. “If there are truths, where are they?” I might answer that they are to be found in the most logical parts of academia. But “that is where truths are expressed; where,” they might repeat, “are the truths themselves? If they exist in our minds, then how can they can be objective? Or, if truths are not in our minds, then, since truths are clearly not physical objects, is there an external world made, not of atoms, but of truths? If there is such a world, how do we have access to it, and how do such truths manage to be about the physical world?”

Fortunately, such convoluted complexities are irrelevant to the pursuit of truth itself. Where is truth? Is it somewhere in between our words and the world? It can be hard to point to it. But presumably you know what truth is. To tell the truth is to say, of something that is a certain way, that it is that way. Good descriptions are true (and not too detailed, and they are relevant to the conversation, and so forth). Truth should be pursued, and so it exists, in an important sense—it is that sense that we are primarily interested in here.

What you can point to is coloured objects. And it is obvious that colours exist, for all that they do not exist in the same way as objects. And it is obvious where they are (you can point to them). And yet, the question of where the colours are is a puzzling question, which can make philosophers wonder about what is real. Where is the blue of the sky, and the green of the grass? They seem, clearly enough, to be out there in the world. And yet, they are sensations produced by our brains. There is a very real sense in which they are inside our heads.

That may seem to be yet another paradoxical contradiction. But the solution to this logical puzzle—philosophers call it the puzzle of perception—is not hard to find. When people perceive the world around them, there is an interaction between them and the world—called perception—and there are of course two sides to that interaction. There is no threat to reality here. Or to our knowledge of reality. Indeed, the world is only known at all because it is perceived. As Russell emphasised, the existence of colours is even more certain than the existence of such coloured objects as scientific equipment. But scientists are completely certain of the existence of their scientific equipment, and hence reasonably confident of the measurements that they make with it. They naturally have a bit less confidence in the theories that encapsulate much of that data, and even less confidence in their understandings of those theories. And much less in our readings of those theories, of course. Still, the puzzle of perception can help us to understand materialism, and the God hypothesis, as follows.

What we naturally think of as physical—the hardness and coldness of metal, its smooth texture and shiny colour, and its resistance to the forces that have to be applied when bending or lifting it—such physical properties are all mental phenomena. The physical objects of materialism are the particles and forcefields that are described mathematically in the theories of physics. And we know very little of what they are in themselves (except for those descriptions, which justified by enormous quantities of measurements, and the fact that they give rise to such ordinary observations as those mental things that we naturally think of as the physical).

And similarly, what mental things are in themselves is not better known than those particles (or strings, or however else the theories are best expressed, mathematically). We each know one of them by being it; but that might amount to relatively little knowledge, by comparison with all the data that has gone into those mathematical descriptions, of particles (or strings). For all most of us know, such mental things as you and I might really be such physical things as our brains. And materialism can be thought of as saying that that is indeed the case. Idealism can be thought of as saying that the physical things, in themselves, are really mental things; and the difference between that and materialism comes from the God hypothesis. That might be thought of as saying that physical things and the familiar mental things are both creatures of—or creations out of nothing but—a God who is naturally thought of as mental, rather than physical.

Much more simply, colours are found in—or are instantiated in (and are abstracted from)—coloured objects. And you can, similarly, point to shaped objects. These letters, for instance, have shapes. Shapes are found in (and are similarly abstracted from) shaped things. Mathematicians have always studied shapes, as well as numbers. And whole numbers are found in (and are abstracted from) numbers of things. Abstraction can seem complicated, especially in the context of modern mathematics; but the basic process is simplicity itself. And this book is only about what the existence of things—one thing, two things, three things and so forth—says about reality.

No one expected it to say much. And many of our scientists and academics do not like what it says. But those two facts do not add up to a reasonable doubt. And there is surely no reasonable doubt about the existence of things. You and I are two people. The sun, the earth and the moon are three quite distinct bodies. These are four words. Nor is there any reasonable doubt about the existence of things from Cartesian sceptical scenarios.

Chapter 3 began with there being an apple, a banana and a carrot. But a scientist might say that what really exists are the fundamental physical particles of which ordinary objects are composed. And an mainstream philosopher might defend that view by pointing to other contradictions obtained, in apparently logical ways, from such assumptions as there being an apple, a banana and a carrot. Imagine, for example, a futuristic machine that is able to pluck molecules, one by one, from ordinary objects. The machine is set to work on an apple.

Taking just one molecule away from any apple would of course not stop it being an apple. So, after the first molecule has been taken away, there would still be an apple. That means that when the second molecule is taken away, it would be taken away from an apple. We would therefore be left with an apple, and so when the third molecule is removed, the machine would again be taking one molecule away from an apple. We would again be left with an apple, and so the fourth molecule would be taken away from an apple—and so on and so forth.

As this futuristic machine takes molecules away from this apple, it seems that it will always be taking them away from an apple. Except that eventually, of course, it will be taking the last remaining molecule away. So, we have a contradiction. And some philosophers have said that it shows that there are not really any such things as apples. They say that although apples are ordinary objects, they are not logical objects. They do not belong in our most scientific theories.

There are of course such things as apples, though. If anything, such an argument would have been showing the futuristic machine to be physically impossible. Although such high-precision machines are not that unrealistic. And in any case, there is a common-sense solution to that puzzle, which you can find at the beginning of the next chapter. As for the thought that what really exists are the fundamental physical particles, the "proof" of chapter 3 could have begun with three fundamental physical particles.

Had it done so, then it could have been objected that such things are only theoretical entities. Such particles have only such properties as our physicists' theories give them. So, if some of those properties were leading those theories logically into absurdities, then those properties could—and presumably would—be replaced with properties that did not do that. Still, the "proof" of chapter 3 could have begun with three pieces of scientific equipment—an altimeter, a Bunsen burner and a calculator, for example. It would be strange if scientists denied that there were any such things as the instruments that they use to get the data that shows how realistic their theories are.

But in any case, you only have to consider any coloured object to see that there clearly are colours. So, chapter 3 could have begun with red, blue and yellow. They clearly exist—in some pretty obvious way—and they are clearly distinct from each other. Alternatively, the "proof" of chapter 3 could have begun with three words. These words are clearly distinct enough from each other for us to be able to read them. Furthermore, all possible proofs use words, if only in the definitions of the symbols in them. So, blocking the conclusion of this proof by denying that there were any things at all—not even words (not really)—would also block every other proof. It would not be much of an improvement over throwing logic away.

Chapter 3 could have begun with the word blue, the colour blue, as perceived by you, and you yourself. Such things clearly exist, albeit in different ways, and so they have to behave logically (unlike fictional things, for instance). And those three things are very clearly distinct from each other. It should therefore be clear that A, B and C could have been some three things, distinct enough from each other for there to be three of them, and real enough for their being three in number to be a matter of logic. Consequently, there was clearly nothing wrong with taking A, B and C to be an apple, a banana and a carrot. And such items of food are very familiar. We all know that such things can be distinct and real.

Indeed, we may originally have learnt the meaning of the word three by being shown such things as three apples. Whereas, philosophers arguing that apples are not things usually have a very different sense of thing in mind. They might, for example, be thinking about the kinds of things that a future "theory of everything" could sensibly refer to. The sort of arguments that such philosophers bring to bear against apples being of that kind hardly get in the way of our being able to count three of them as three things for the purposes of the proof of this book.

Being three in number is primarily a property of all—and of only—those logically possible collections of things whose things can all be paired up with the first three counting numbers. Even clouds can be counted, if they are distinct enough from each other, and if we are counting them in a restricted setting, such as those that are visible in a particular picture of the sky. The fact that they are clouds of water droplets would be no more relevant than the fact that apples are composed of molecules. What is relevant is that they are real clouds—or realistic fictions. If I told you that, in this particular fiction, these clouds do not have to obey the laws of arithmetic, then that would of course undermine our ability to count three of them. But let's assume that these are realistic clouds. We might count three of the clouds that are sufficiently distinct for us to count them.

And we might pile sand up in the four corners of a room (a realistic room, of course). Then we could pair up three of those piles with our three clouds. Some or all of the things in one collection can be paired up with some or all of the things in another—that is a logical fact. So, there being different ways of making that association of piles with clouds is a logical fact. Our being able to count those ways is primarily a logical fact. It requires the things concerned to be sufficiently distinct from each other.

We could use ordinary arithmetic to calculate how many ways there are of making that association, of piles with clouds. And that calculation would of course not be affected by the fuzzy nature of clouds in general. So, why should the fact that apples are made of molecules affect the logic of chapter 3? Similarly, it does not matter what whole numbers are, in academic mathematics, or according to any philosophy of mathematics. What matters here is that there are these different ways—which is a logical matter. If a "philosopher" came up with something that he called a "whole number," which did not behave like the numbers of chapter 3, then it would be that ascription that was questionable.

Moving on, there are self-evidently lots of words. And words are certainly distinct enough for us to be able to read them. So, chapter 3 could also have begun, unproblematically enough, with the three words apple, banana and carrot. Now, those three words and our three items of food—A, B and C—are clearly two collections. And there is an obvious pairing up of the things in those two collections. Those collections are, to use the technical term, equinumerous. Incidentally, the concept of equinumerosity in modern mathematics is a set-theoretical model of that concept. Since we naturally defer to the professional point of view, it would be only natural for you to take that more professional concept to be superior to my informal description. It is not, though.

With respect to that word two—in "those two collections"—the three things are counting as one thing. Whereas, three things are three things, not one thing. Could this be where chapter 3 went wrong? No. This possible objection had been dealt with at the beginning of chapter 3. This is why we were looking at logically possible selections from collections, not subsets of sets. The paradoxical contradiction of chapter 3 followed from three things being three things: that was why there were eight (two to the power of three) different possible selections from them, and then 256 (two to the power of eight) different possible selections, and so on and so forth. The contradiction did not follow from our counting two things as one thing.

When I wrote this sentence, I selected the word apple from those three words. That was one way of making a selection from them. There is an obvious sense in which the following are two ways of selecting words. Getting you to choose, and putting my finger on the page or screen with my eyes closed. They are different methods of selecting. Whereas, in chapter 3 we were interested in ways of selecting that were differentiated by the results of the selections.

When I select the two words apple and carrot from those three words, that is another way of making a selection, in that sense. It does not matter how I made that selection. Apple and apple-and-carrot are two logically possible selections. What makes them different is what makes the word apple different from the pair of words apple and carrot. But what makes each of them one thing is its being one logically possible selection. This is a subtle distinction; but this is a paradoxical bit of mathematics, so it does all come down to subtleties.

The kind of thing that was being counted as one thing, in chapter 3, was a logically possible selection from a collection, not a collection. So, the contradiction at the end of chapter 3 did not follow from collections of things having been counted as single things. Still, once that has been eliminated—as where we might have been going wrong—we can simply observe that there is of course nothing wrong with counting a collection of things as a single thing. That is precisely what we do when we count collections. And we can of course count collections. For example, how many stamp collections are there? But Cantor’s paradox is paradoxical, and so atheists had to find something to blame. And it is a relatively simple fact that three things are not one thing. Three is not one.

Cantor’s paradox is usually presented as being about intuitive or naïve sets and subsets. Such a set of things is basically an imaginary bag of things. A collection of things is just some things being referred to as though they are one thing—they are being referred to collectively—and so this is naturally pictured as an imaginary bag. And a bag may of course contain only one thing, or nothing, and so there are sets with just one thing in them and empty sets. And a subset of a given set is a set whose members are only some—or all, or none—of the members of that given set.

Cantor thought of mathematical collections in that very natural kind of way. Axiomatic set theory then seems to address the underlying cause of the contradiction. It says which collections can count as one thing—those that are sets—and which cannot. However, that was not where the contradiction of chapter 3 came from. So, standard set theory does not solve this paradox. Chapter 3 did not show that collections are inconsistent, and should be replaced with axiomatic sets. Nor, similarly, did it show that set-theoretical “whole numbers” should have been used instead of the sizes of collections that chapter 3 used. Such “whole numbers” are defined axiomatically, so they are more like cartoon bags than realistic bags. That is their strength. Fictional entities can be given any properties whatsoever. And the standard axiomatic sets have been given such properties as enable the whole of standard mathematics to be done using them, without any of the contradictions that can arise when actual numbers of things are used. But it is not as though there are not actual numbers of things, in the real world.

There are, in particular, three items of food in the bowl described at the start of chapter 3. Facts about the real world might be observed in the nature of those things. I can pick up the apple: I can decide to actualise that possible selection. There are eight different selections that I could have made—if we include not picking any of them up as a selection; otherwise, there are seven. If you are looking at real items of food, in a real bowl, then it is easy to see how the molecular structure of apples makes no difference to that arithmetical fact. Of course, ordinary objects exist. But the main thing, so far as the reasoning of chapter 3 was concerned, was that our items of food were clearly distinct from each other. Had we started with particles instead, it is that property that we would have needed them to have, not some more abstract notion of existence not possessed by ordinary objects. I can pick up one of the items of food—for example the apple—leaving the other two in the bowl. That is all that is needed for the mathematical precision of such counting numbers as 3 to apply to the items of food in the bowl.

Much as I have simply assumed logic, so I assume that there are such things as those logical possibilities. A philosopher might observe that there may well be no such things as logical possibilities, and that if there are not, then those of chapter 3 will not have been so impossibly numerous; indeed, chapter 3 might be said to be some evidence that there are no such things. But that is akin to a Cartesian doubt. We have to assume logic, and logic is all about logical possibilities. And there clearly are seven possible selections from the bowl, plus the possibility of making no selections. There being senses in which such things might not exist is another matter entirely.

Logical possibilities exist because they are what we think about when we think logically. We have to assume that we can think logically, and so we have to assume that there are logical possibilities, for us to think about. They do not exist in the way that items of food do, but nor do shapes, numbers or words. And there clearly are shapes—O is round, for example—and there clearly are numbers: These are four words. Such things do not have to exist in the way that ordinary objects do; but because they do exist, they do have to be logical. That is also what logical possibilities have to be—that is why there is this logical proof. A logically possible selection is a selection that is logically possible; and given that a God is a logical possibility, there is a logically possible selector for each of the logically possible selections of chapter 3.

Could chapter 3 have been, in effect, a reductio ad absurdum of the logical possibility of such a selector? No, because, as chapter 4 described, there might, just possibly, be such a selector and no contradiction. What if the logical possibility of a God is being denied for other reasons? Well, in that case, the proof of this book would fail, because this proof has to assume that God is a logical possibility. Would denying the logical possibility of a God solve the puzzle of Cantor’s paradox, though? No. If it was that simple, then mathematicians would not have had to redefine everything in mathematics, in order to work around the paradox. I suppose that if God is not even a logical possibility, then most of the things in chapter 3 will not be logically possible selections. But they would still be subcollections, still be just some of the things in a given collection. I will address that aspect in more detail shortly. But first, I should emphasise how unscientific it would be, to deny the logical possibility of a God.

Those who do not believe in God might find that surprising. But those who do not believe in ghosts, for example, or alien abductions, would not normally deny the logical possibilities of ghosts or alien abductors. If extraordinarily good evidence that alien abductions had been taking place did turn up—because aliens really were abducting people—then anyone who had denied the logical possibility of alien abductions would have to regard that evidence as deceptive. And similarly, while it may seem even less likely that good evidence for ghosts will turn up, if it did turn up, then it would certainly not be scientific to deliberately misconstrue it. But logically, you would be committing to doing that, by denying the logical possibility of ghosts. Now, some people would say that the difference between alien abductions and ghosts is that, whereas our best scientific theories say that aliens are quite likely to exist somewhere in the universe, and only highly unlikely to be abducting people from this planet, there is no place in those theories for ghosts. But while those theories are certainly scientific, it would not be scientific to draw the line of logical possibility there. Our best scientific theories are works in progress, and it looks like they always will be. If the line of logical possibility had been drawn by the best scientific theories of earlier days, then we would never have got to our best scientific theories. Still, there are lots of things that people will say, and I can hardly address all of them here; so, let's return to the logic of my proof.

Big physical objects are different to small ones: stars are different to moons. And space is Euclidean locally, non-Euclidean on a larger scale. Might mathematicians have discovered that collections are different when they are bigger? But a collection of things is just those things, however many of them there are. There is clearly no difference between a big collection and a small collection. There are differences between collections of things due to differences in the kinds of things in the collections, but that is an entirely different matter.

Perhaps we run out of possible selectors, for the possible selections. If so, then for some collection of such things, not all of the logically possible selections from that collection will exist, there being no possible selector for all of them.

However, all of those things would be there. We might picture that as follows:

... thing ... thing ... thing ... thing ... thing ... thing ... thing ... thing ... thing ...

Each of those things being there, there would appear to be all of the subcollections of that collection, even if they could not all be selected, there being no possible selector for all of them. Recall how a collection of things is just those things, being referred to collectively; similarly, a subcollection is just some of those things, being referred to collectively. So, even if not all of those subcollections could – not even possibly – be selected (there being no possible selector for them), they would still all be there. If we used ‘Q(T)’ to name, collectively, all the subcollections of any given collection, T, then a slightly revised diagonal argument would show that Q(T) was bigger than T, for all T.

If a lot of things exist, then we can call them ‘a lot’ (or ‘them’). But given that contradiction, it would now be natural to wonder if there really is something that makes some things this extra thing: a collection. Is it simply the case that for each of those subcollections, that one is there? Are those things one thing? Primarily they are many things, not one thing. That is why Cantor’s contradiction originally led to theories of sets: theories (or models) of when some collections (or pluralities) are one thing and when they are not.

Our collections (of selections) all inherited a certain unity – Q(T) inheriting it from T, for each T – which originated with the bowl that contained our three items of food, A, B and C. But what about the individual selections in each of those collections, what makes each of those selections of various things one thing? Primarily, what we have is just some of the things in some collection. We might picture that by italicising some of the words:

... thing ... thing ... thing ... thing ... thing ... thing ... thing ... thing ... thing ...

Let’s suppose that for some of the things in some collection – which we might picture to ourselves as “thing”s that have been italicised – those particular things are no more a subcollection than they are a logically possible selection. That might be because there is no possible collector – no one to refer to them collectively – just as there is no possible selector. Still, those things are there (although we are assuming that they could not actually be italicised). They are, in the following sense, a value of the variable “some of those things”. Wordss like “some of those things” (or “a lot”, or “them”) are like variables because this is just like using “n” to refer to some particular counting number without saying – perhaps without being able to say – which one. Most counting numbers are far too big for us to be able to refer to them individually. We could say that n is one of those. If the number was so big, and unround (ten trillion being a round number), that the universe was not big enough to contain its description, and if there was only this spacetime in existence (materialism being true), then there would be no possible referrer. There would be a number that could not possibly be referred to. And similarly, there could (in theory) be some of the things in some collection that could not be referred to, could not be selected or collected (or have all and only their names italicised). Those things would still be there, in that collection. “Some of those things” just means one of those things, and another, and yet another, and so on, but not all of them. There is no sense of collecting or selecting in that “some of those things”. We are imagining that for some of those things, there can be no collecting or selecting of all and only them (presumably because of the absence of any possible collector or selector). There would similarly be no italicising of all and only their names; but clearly we could, in our picture, have capitalised different “thing”s:

... thing ... thing ... thing ... thing ... thing ... thing ... thing ... thing ... thing ...

We have two different italicisations of words, in these two pictures. And while such italicisations are just like collections or selections—in that they similarly require an italiciser—this is just a picture of how these possible collections are not about putting a bag around some of the things, how these possible selections are not about plucking them out.

Because there are at least two different values of “some of those things,” which we can picture as two different ways of italicising words, each of those values—each of those ways—counts as one (one way, of italicising or selecting or collecting). It is because it counts as one thing that we can imagine a bag around it. It is not because we can imagine a bag around it that it counts as one thing. So, when we think of all the different ways of having just some of the things in some collection, it does not matter whether there are possible collectors or selectors, or not. Each is one thing—one way of doing such things—because all those ways are different from each other. Indeed, they are exactly as different from each other as the apple, banana and carrot were.

That is why the contradiction of chapter 3 was not caused by there being no logically possible selector for most of the logically possible selections. So, the contradiction of chapter 3 does not show that we must run out of possible collectors, selectors and referrers. It shows that there are too many logically possible selections. A logically possible selection from some things is just some—or all, or none—of those things. All that is required for their selection to be logically possible is for those particular things to be there, among the original things; for them to be some—or all, or none (if there are none of them)—of those original things. But of course, if there is also a logically possible selector, then it is a logically possible selection in that sense too. And God is logically possible, so there is a logically possible selector.

You may well be wondering how there being a logically possible selector could help. You have just seen how the problem is not caused by there not being such a selector, but by all those logically possible selections being already there. Nevertheless, if things only exist because they are created by God, then the nature of those possibilities would differ from what we are naturally assuming.

It is only natural for us to think of God as being like Zeus, or someone with greater superhuman powers, who could know all about all of those logical possibilities. That is because we naturally imagine things based on our experiences. However, it is not so much that we are thinking of an extremely big person here; it is more that we are thinking about the creator of things in general out of nothing. So, our picture of God should be tempered with the analogy that, much as fictional characters are to their author and to all of their readers, so we are to God. For us, things are as we find them. But for God, things are essentially creations. If there is a God, of such a logical kind, then the existence of possible selections does depend upon the existence of that particular selector. A logically possible selection from some things is, fundamentally, something that God is able to select. It therefore makes sense that, by thinking of them individually, such a God could be individuating the logically possible selections. They would still all seem to us to be already there; but that may well be like how, now that you exist, there always seems to have been the possibility of you, as well as the possibility of someone just like you.

Chapter 3 was all about numbers of logical possibilities. And quite generally, when we think logically, we entertain various logical—and illogical—possibilities. But there are certainly not real things that are those illogical possibilities. And chapter 3 did seem to be showing that we were considering illogical possibilities. Is that a logical hole, in the argument of chapter 3?

But the contradiction of chapter 3 arose from comparing numbers of distinct possibilities, not from anything else. So, the things of chapter 4 only needed to exist as distinct possibilities – which distinct possibilities clearly do – for there to be a contradiction. And the possibilities were non-fictional. Now, it is conceivable that while each of the possible selections was real – starting with the real possibility of my selecting the apple, A – they became less and less real as the possible selectors became more and more remote from our everyday lives. It is conceivable, but it does not pan out. They remain just as distinct, and they remain just as possible because they are possible selections. And that is what led to the contradiction.

Philosophers do wonder whether there are such things as logical possibilities. And there are certainly ways in which they do not exist. They do not exist in the way that our items of food do, for example. They are not tangible, physical things. But we should not let that fact throw us. We can count anything, even fictional objects (in a sufficiently realistic and detailed fiction). And some philosophers also wonder whether ordinary objects like apples really exist, as definite things, because of such puzzles as the one this section began with (which is considered in more detail in the next and final chapter). Outside of such academic philosophising, what would it mean to say that apples do not exist, as definite things? That I could not pick A up? No, such philosophers clearly mean something else. Clearly A could be selected from that bowl.

The argument of chapter 3 depended upon the possibility of selecting A. Nothing else about that ordinary object was being assumed. To say that there is no logical possibility of selecting A would ordinarily mean that this apple could not possibly be selected. Perhaps it had been eaten (or the bowl encased in concrete). Apples – and similarly chairs et cetera – are real enough. And chairs are real enough for us to be able to sit on them, real enough for them to have to not be rickety. Chairs exist – in whatever way they exist – because we can sit on them, because they are not too rickety. They can be moved from room to room.

Logical possibilities are real enough for us to be able to think with them, real enough for them to have to not be contradictory. We think most logically by considering a broad range of logical possibilities. That is the only way they have to exist. If they did not exist in that way, then we could not think logically. We could calculate using symbolic logics, but we would be unable to think logically about what our calculations meant. And this proof does not need all logical possibilities to exist, only logically possible selections. They need to exist only in the sense that they are clearly distinct from one another, so that they can be counted. It is in a similar sense that fictional characters need to exist as distinct characters if they are to be legal property.

Some philosophers think that chairs are imaginary, that the atoms that make them up exist, but that the chairs themselves do not. But how could that be right? A chair made of Lego bricks would still be a chair. It would still exist, wholly composed of Lego bricks. Had it been made one brick at a time, with one brick not being a chair, and with no addition of one brick making a chair out of a non-chair, it would still exist. Similarly, even though orange fades smoothly into yellow and red, with no sharp boundary at either end of that part of the spectrum, that hardly means that carrots are not orange. There would be a puzzle – see chapter 7 for more about such puzzles – but it would make no sense to conclude that there were no such things as chairs, or that carrots were not orange. Chairs exist because we learn the meaning of “exist” in a world of tables and chairs, cars and bicycles. Bicycles do exist, epicycles do not exist. Similarly, protons do, pixies do not. When we say that protons exist, we mean that they exist like chairs do. So of course chairs exist. The basic contrast is with imaginary beings such as pixies. We are right to think that we can spray protons onto surfaces; we were wrong to think that epicycles were actual things in the sky.

Ironically, a Matrix-style world gives us a context in which “chair” could have a perfectly precise meaning (given to it by the programming of the computer that controls that virtual world). So if we did need such precision in order for a thing to really exist – or for it to be counted as one thing – then it might be only such chairs that really existed. Of course, it is just such chairs that we usually think of as not really existing. So, existence as a thing is clearly not very closely related to having a definite description.

In any case, that sense of “thing” is not the one that we are interest in here. Here we are only interested in whether they can count as one thing. And even in a fictional context, we do not need a perfectly precise description in order to have something that counts as one thing. We can count fictional unicorns just so long as the fiction in question includes good enough descriptions of when there are unicorns.

If our logical possibilities were fictional, they could have any properties whatsoever, even contradictory ones, without that necessarily being a problem. And there is a sense in which they do get more and more unreal: They get further and further removed from the tangible things that we started with – the three items of food in a bowl. However, the properties that give rise to that paradoxical contradiction do not get more and more illogical. They derive from that mutual distinctness, which remains the same.

To see this more clearly it may help to consider how, if the bowl was encased in concrete, it might be that I could not select any of the food in the first place. My possible selections would then be less possible – they would be very unlikely (I would have to make the effort to break through that concrete, and why would I?) – but they would be no less real. The main thing, as far as the reasoning of chapter 4 was concerned, is that the possible selections would be exactly the same, exactly as numerous. Indeed, even if they were not real, they would still be that many unreal possibilities. But the actual selections would still be possible for some possible being (if only God).

Nevertheless, if you wanted a solution that did not involve God, or anything illogical, then you would have to look for it somewhere else. And there are not many places where it could be. And there is this intuition that the possibilities get less real, the more removed they are from what could actually be done by anyone except a god, and eventually only a God. There are also analogies that make it seem possible that those degrees of unreality could all add up to unreality in a relevant way when we consider the totality of all the collections of such things at the end of chapter 4, even if they do not make any of those things less real in any relevant way.

In the heap paradox, for example, we have smaller and smaller heaps until we end up with too few things for them to be a heap. The heap that we start with gets smaller and smaller (becomes less and less of a heap, you might say, although they are all heaps, to begin with). And our collections of possible selections, from possible selections, get further and further away from the tangible reality of the few things that we started with. You might say that they get less and less real, in some sense, as we get collections of bigger and bigger size, until we end up with too many things for them to have a standard size. So it seems to be a conceptual possibility that they end up as too unreal to have the usual properties. Now, I have already argued that what we end up with would have those properties. But those were intuitive arguments – involving considering pictures like “thing ... THING ... thing” – and it might be thought that this conceptual possibility could be improved by formalising it.

We begin with a very difficult logical puzzle, Cantor’s paradox, with no logical solution apparent after more than a hundred years of looking for one. If such individuation is logically possible, then we have a logical resolution of that paradox. It does not matter that it is a counter-intuitive solution—that just goes some way to explaining why it did not appear in the twentieth century. What matters is whether it is indeed a logical possibility. If it is, then the puzzle detailed in chapter 3 and the very clear absence of other logically possible solutions add up to a perfectly logical argument that such individuation very probably does occur. If it is—but perhaps it was not really a logical possibility.

It was certainly a very counter-intuitive possibility. Does the possibility of you in particular (as opposed to the possibility of someone just like you) really come into being when you do, even though it will then seem as though there was always that distinct possibility? Or is that nonsense? To show such individuation to be logically impossible, its counter-intuitiveness would have to be turned into contradictoriness, which could only be done by assuming a background metaphysics. For example, if presentism could be shown to be logically impossible, then such individuations would be too. But as mentioned in chapter 4, there are plenty of reasons for thinking that presentism cannot be shown to be logically impossible. At the end of the day, we think most logically by not ruling anything out of being logically possible, until it has been shown to be logically impossible.

The logic of Cantor’s paradox—as exemplified by the “proof” of a contradiction in chapter 3—does not, then, allow for any other logical solution than the endless individuation of logically possible collections of things. Because there clearly seems to be no other way for that to happen than by logically possible collections of things being thought of by a creator of things in general out of nothing, who is able to change, this solution was not taken very seriously by the most scientific academics. As no other logical solution could be found, mathematicians replaced mathematics with a good mathematical model of mathematics (standard set theory), and they also developed a good mathematical model of logic (classical symbolic logic). But the main thing here is that, in all that time, no properly logical solution to Cantor’s paradox was found. Still, standard set theory is a very good model of actual mathematics. So, those looking at modern mathematics could be forgiven for assuming that mathematicians, a hundred years ago, discovered that numbers are standard sets.

Since Cantor’s paradox does not arise in modern mathematics—as anything except a simple proof that there is no standard set of all the standard sets—I ought to say a bit more about why that fact does not mean that chapter 3 is simply bad mathematics. And whether two collections have equal numbers of things in them or not depends entirely upon those two collections. Either the things in them could all be paired up, or else that is not possible. It is not a question of whether we could pair them up, or whether we could show that we could not; it is clearly a more objective matter than that. Either it is logically possible for all those things to be paired up, or it is not. When it is logically possible, we naturally think of those two collections as having the same number of things in them. But those are not the numbers of modern mathematics, and their use does not amount to a controversial philosophy of mathematics. I was only using them because they are the most natural way to talk about logically possible collections.

Still, such natural whole numbers do give rise to Cantor’s paradox. The numbers of modern mathematics are the way they are because no such problems arise with them. So, if I had used them when talking about the possible selections of chapter 3, I would have had no problem for my God hypothesis to solve. And they are the numbers used by professional mathematicians, so they are, as a rule, the proper numbers for us to use. The problem of chapter 3 could therefore seem to have resulted from my not bothering to do things properly. The next section will look at that objection in a bit more detail.

In Euclidean geometry, circles are defined in terms of the axioms of Euclidean geometry. Do such circles exist? Well, they exist within Euclidean geometry, much as fictional characters exist within their fictions. That is not much of an existence, but it is that kind of existence that most mathematicians are primarily interested in. Mathematical theorems about circles are proved to be true by deducing them from the axioms of Euclidean geometry.

Of course, circles also exist as shapes. And shapes are clearly non-fictional. The shape of the capital letter O is clearly a real thing, much as the colour of a coloured object is clearly real. And insofar as such a shape is described well by the axioms for a Euclidean circle, theorems about circles in Euclidean geometry do say something about such a shape. Now, it seems likely that there is nothing exactly like a Euclidean circle in actual spacetime. But there may well be many things in reality that are approximated well enough, for scientific purposes, by Euclidean shapes. Euclidean geometry may therefore be part of a good mathematical model of aspects of physical things.

Standard numbers exist primarily in order to resolve the problem that mathematicians had with Cantor's paradox. So, they exist much as a fictional character exists. They enable theorems to be proved. They also exist much as shapes do: they are a particularly well-defined pattern. However, their interest to mathematicians comes primarily from the fact that they are good models of actual numbers—they enable theorems involving actual numbers to be proved—and only secondarily from their own pattern. Are standard numbers the actual numbers? That seems unlikely; but it is another tangential question here, for the following reason.

In chapter 3, we were interested in possible selections. We were not interested in how possible selections exist, we were interested in how there were more of them in one collection, than in another. That is like being interested in how a square has more corners, than a circle does. The nonexistence of a round square is not a question like the existence of the circle shape. It does not really matter how circles, or logical possibilities, exist. What matters is whether these are non-fictional shapes. If so, then they have to behave logically. In a fiction, shapes can have any properties whatsoever, and we can similarly have axioms saying anything at all. But in a realistic fiction, nothing could have a round square shape. The "number" of the corners of something modelling a round square shape in an axiomatic theory of shapes could be defined to be 0&4. But that would of course not make round squares possible. And similarly, no axiomatic theory of collections like X (at the end on chapter 3) would amount to a logical solution of chapter 3. Insofar as it "solved" the problem—that is, insofar as it solved a mathematical model of the problem—it would be an unrealistic model of such numbers of possible selections.

The difference between something being stipulated to exist and it just happening to exist is like the difference between a legal individual, such as a commercial company, and the group of workers that actually do all the work of that company. The workers have to obey the laws of physics, the company ought to obey the laws of the land. They are very different kinds of laws, and very different ways of existing.

There are companies, just as there are fictions. They are what they are. Legislators stipulate what that is, to some extent. There are limits though. They cannot stipulate that their own companies are the most natural kind of company. Such matters are for psychologists and other scientists to study. And here we are only interested in one very simple aspect. Two companies are, at any point in time during which they both exist, two things. Note that these companies might be two companies because of a single legal words whose meaning is in dispute—that does not matter. There is one logical possibility: there is really only the one company (it was a legally bad words); and there is another logical possibility: there are indeed two companies.

The nature of that “two” is not a matter for lawyers to argue over. There is one thing, and there is or is not a different thing; that is all there is to this. The properties of the English word “two” are essentially logical. Could failing to solve a logical puzzle in a convenient way possibly contribute anything positive towards a reason for changing that word “logical” to “symbolic-logical”? You would naturally think not. The history of Cantor’s paradox has turned that into a reasonable question. But that is only a reflection of the extraordinary nature of the evidence that is Cantor’s paradox. It does not really mean that doubt is cast over what “two” means.

Two workers are similarly two things. They are social animals, and the nature of their social interactions is sometimes studied scientifically by psychologists, and is to some extent stipulated by legislators. Here we are interested only in how that “two” is the same whole number, with exactly the same properties. It can appear to be different, because the laws of physics are investigated by people who defer to mathematicians, when it comes to what whole numbers are. Lawyers tend to defer to common sense when it comes to such things; but it is only common sense that the views of physicists should trump common sense, when it comes to the nature of physical reality. There could therefore seem to be good scientific reasons for changing the word “two” to the standard mathematical stipulation. Nevertheless, the logic is of course the same.

How could failing to solve a logical puzzle in a convenient way possibly add to a good reason for changing that word “two” to some particular symbolic version of something like “the standard set whose only members are the empty set and the set whose only member is the empty set”?

Chapter 3 was based on Cantor’s paradox, which was that there is—presumably, although also there cannot be—a set of all the other sets. Before Cantor’s paradox, sets were basically collections, and there are of course all the collections that there are. So, there is presumably their collection. Sets being basically collections, there would presumably be the set of all the other sets. I say “other sets” because such a set would of course not include itself, not as a member of itself (given that it is basically a collection). Let’s call that hypothetical set of all the other sets “Y”.

There is a paradox because of the following reason why Y cannot exist. Given Y, there is a set of all the subsets of Y, P(Y). That is contradictory because Cantor’s diagonal argument shows that P(Y) contains more sets than Y contains, whereas Y contains almost all the sets. We saw in chapter 1, with

Galileo's paradox, that the cause of such a contradiction might be semantic ambiguity. Here there is nothing like that, just the very elementary logic of things, of numbers or collections of things. Ironically, it was Cantor, who had used his concept of a whole number of things to solve Galileo's paradox, who introduced something like it, by calling Y something other than set. He called it an inconsistent multiplicity. Nowadays it is called a proper class. Within axiomatic set theory, the contradiction became a proof by reductio ad absurdum that Y is not a set: were Y a set, we would have a contradiction.

Cantor took sets to be collections—including degenerate or improper collections—which could be consistently conceived of. He took the totality of all the sets to be conceivable but, because of his paradox, only inconsistently conceivable. He thought that that would be alright because such a totality could be symbolic of a God who transcends human logic. He was, after all, looking back, from the end of the nineteenth century, at plenty of precedents for such metaphysics. Infinity had always seemed paradoxical, and for many centuries before Cantor it had been taken to be a symbol of the ineffability of God. Cantor's new mathematics of the infinite was going to face less opposition from historically important quarters precisely because it had its own paradoxical infinity. Cantor also regarded his mathematical constructions as to some extent fictional—as "free creations of the human mind."

What mathematicians noticed was that Cantor's sets were perfectly consistent. Furthermore, his mathematics contained a rigorous proof that there was no set of all the other sets (that there is no Y). Could it be that the step from X to P(X)—the consideration of all the logically possible selections from X (the totality of logically possible collections that chapter 3 ended up with)—just gave us things that we already had? X is a fantastically huge collection, of unimaginably complex things, so it is impossible to see what is going on in that step.

But applying our general diagonal argument—steps (1) to (6)—with T = X shows us that there would be more things in P(X) than there were in X. So, the step from X to P(X) could not only have been giving us things that we already had. Could there be something wrong with such diagonal arguments? That is highly unlikely. They have been used by mathematicians for over a hundred years. And academics who thought that they had found something wrong with them always turned out to have been mistaken. If you are interested, the British mathematician Wilfred Hodges wrote about such mistakes in his "An Editor Recalls Some Hopeless Papers," which was pages 1–16 of volume 4 of the journal Bulletin of Symbolic Logic, published in 1998.

Of course, there is always the possibility of something having been overlooked. This book is evidence of that. But much the same can be said of each and every proof that we have, about anything. The mere possibility of there being some unnoticed mistake cannot be used to undermine the force of any of them, not without them all being undermined. Logically, in order for Y to be logically possibly different to the sets, sets had to differ from collections. And mathematics is all about logical consistency. It is all about proof. That is why collections were replaced with axiomatic sets.

The axiomatic approach itself was not unscientific, no more so than replacing intuitive geometrical entities – lines and points – with Euclidean ones had been. And it enabled Y to be something else while keeping sets very similar to collections in all their other applications. But while we now have good scientific reasons for thinking of space as non-Euclidean, we have never had a good reason to think of numbers of things as sets of things rather than collections of things. For an analogy, suppose I say that Robin Hood's actions were doings, except for his crimes, which were not doings. Then in all of his doings, he committed no crimes. But of course, that would not fool anyone. So, the use of standard sets did not solve Cantor's paradox, it only enabled it to be ignored by mathematicians.

Collections were replaced with axiomatic sets, though. And names of counting numbers became names of particular axiomatic sets in order to avoid even indirect reference to collections. But we still have all of those things. When using a set theory, mathematicians will choose some particular, very big set, and they will, in that application of the set theory, use only the members of that set. But to begin with, there are all the sets of the set theory to choose from. Let's call their collection "Z". In the theory, there is no whole number associated with Z. But those sets can all be paired up with all the things in some other collection. A different set theory, for example, could include a set with precisely that many things in it. In that theory, Z would be some whole number of things. That makes sense because, intuitively, there is a whole number associated with every collection. The concept of a whole number was only extended into the infinite in the first place because that is precisely what a whole number—in the sense of some number of things—is. Standard sets are a mathematical model of collections, and there can of course be other models.

Note that Cantor's paradox could not have been showing that our natural conception of a collection is a bit vague. There is nothing vague about taking a collection of things to be just those things. And in any case, the paradox of chapter 4 was due to there being too many things, not to our collective reference to them all. Despite many attempts, nothing paradoxical about collective reference to very large collections of things has ever been shown. The idea of things that we cannot refer to does come close though. There must be a lot of things that we cannot refer to—some bit of dust in a distant galaxy, for instance—and yet, have we not referred to all of them, collectively, with that "things that we cannot refer to"? That could seem like a contradiction caused by the concept of collective reference. But then, we also refer to them all collectively with "things". That does not contradict our being unable to refer to them individually, by way of definite descriptions, of course.

Suppose we are describing bags of groceries. We can have descriptions like "there is an empty bag." That can be regarded as an axiom for a bag theory. It stipulates that there is such a thing, called "an empty bag," within this bag theory. The theory might be a good axiomatic model of bags of groceries. And it might also be a good model of boxes of groceries. Theorems about bags of groceries that were provable within this theory might always be true of boxes of groceries. If so, then we could say that an empty box was bag-theoretically an empty bag. But of course, a box is not actually a bag. Standard numbers are like those bag-theoretic bags. We were interested in something analogous to those boxes. We were not stipulating them into existence, we were finding them in the world, much as earlier generations of mathematicians would have found actual numbers to be part of the shape of the world.

The axioms of standard set theory do amount to a mathematically precise conception, of something like a collection; but that does not mean that there was anything vague about our natural conception. Epicycles are mathematically precise, and that says nothing about heliocentrism. Sets are like epicycles in other ways. In particular, there is, given one set theory, no end to equally well justified set theories. Starting with all the standard sets, it would not be unreasonable for a mathematician to want to talk, professionally, about them all. And the easiest way to do that is to add a non-standard axiom to the standard axioms, so that there is a non-standard set of all the standard sets. We could call that set "Z", naturally enough. There would then be all the sets within this new set theory, defined by all the standard axioms plus this non-standard one. And it need be no less reasonable for a mathematician to want to talk about all of them. We have another axiom, and similarly another, and so on and so forth. These axioms are created by mathematicians to satisfy a range of criteria, depending upon the sort of mathematical work that the resulting sets will be doing. And with each set of axioms there will be a corresponding conception of "whole number".

Still, it is some evidence, in favour of the validity of our intuitive notion of whole number, that the standard mathematical axioms are a good model of that notion, when it comes to ordinary collections of numbers, such as counting numbers. They do not appear to be modelling any other notion until we get to such collections as that obtained in chapter 3. Adding an axiom that says that that collection is a set, in the new axiomatic set theory, amounts to saying that the intuitive notion of whole number should also apply to those collections too.

Note that each set of axioms is a number of axioms. That number is just the normal, natural number. Similarly, when we think about "logics" we have to think about them using logic. So, it is obvious that the axioms deliver models of numbers, if not fictional numbers. There are, for another way of looking at this, fourteen words in this sentence. That is a fact (as you can easily verify by counting those words). And the fact that we can, if we like, define a fictional entity that we choose to call a number, such that the so-called number of words in that sentence is fifteen; that is a fact that can be added to that fact. It is not a fact that could possibly contradict it, let alone overrule it. There would be an equivocation, not a contradiction, and a fairly obvious one at that. But modern mathematics uses set theoretic numbers, instead of numbers; and mathematicians are the experts when it comes to numbers.

Cantor's paradox became a proof within axiomatic set theory, but the original problem with collections remained. And so there remained such contradictions as that of chapter 3. The use of standard sets cannot solve that problem, because that problem concerns collections, not axiomatic sets.

There was a revolutionary transformation of mathematics in the twentieth century. Nowadays, in mathematics departments all over the world, numbers are defined to be standard sets. Were numbers discovered to be standard sets? The philosophical question "where are numbers, if they are real?" can make it seem as though there is a serious question mark hanging over the reality of whole numbers. But there is a similar question mark hanging over the reality of time. "Where is time, if it is real?"

There was a similarly revolutionary transformation of physics, following the discovery that spacetime is non-Euclidean. Was time discovered to be ct, where c is the speed of light, which is being multiplied by t, a real-number-valued variable? No, time was discovered to be well modelled by a four-dimensional vector which includes such a fourth component, in addition to three real-number-valued variables for the three spatial dimensions (like the three-dimensional Cartesian coordinates that you might know from school mathematics); and similarly, whole numbers were discovered to be well modelled by standard set theory. They were not discovered to be sets.

Space is all around you. That is what space is. Physicists talking about gravitational waves in terms of ten-dimensional strings are seeking to understand that. Axiomatic structures include Euclidean and non-Euclidean geometries, and standard set theories, and a lot more. Space is better modelled by non-Euclidean geometries; but when physicists discovered that space is non-Euclidean, they did not discover that space was an axiomatic structure. And when mathematicians discovered that whole numbers were well modelled by standard set theory, they did not discover that whole numbers were sets. Arithmetic is an axiomatic structure; but then, so is Newtonian physics, and string theory.

In each axiomatic system, there is some number of axioms. And there are, similarly, nine words in this sentence. Some axiomatic structures make better models of those numbers than others. And we can discover interesting new facts about them. Most of those facts are well modelled by standard set theory, but it is conceivable that one extraordinary fact might amount to a metaphysical proof. And it makes sense that, were that proof of an unpalatable truth, then that fact would have led to some confusion over what numbers are.

Could the numbers of axioms in each axiomatic system be numbers defined within that axiomatic system? That is like thinking that the way to think about a symbolic logic is defined by that symbolic logic. Whereas we want to think logically about symbolic logics, and axiomatic sets, and physics and all the sciences. The question of what logic is, that is an interesting question but it is clearly not one that can be answered by stipulating a symbolic logic to be logic, however authoritatively the stipulation is made. As for what numbers are, most mathematicians passed that buck to philosophers, at the beginning of the twentieth century. But most philosophers have since taken that question to be the question of what sets are. They justified that on the grounds that, in mathematics departments all over the world, numbers are defined to be particular standard sets. But that is some very circular reasoning. It is clearly rhetorical, rather than logical (even if they do go on to redefine "logical").

It is not that standard sets do not give us a very good model of numbers. They do. A very good case can be made for their use throughout the sciences. That is a good reason for mathematicians to concentrate upon standard set-theoretical proofs. But the paradoxical contradiction of chapter 3 is precisely where such mathematics fails—deliberately—to be a good model of reality. So, the flaw in my proof cannot have been that I should have been using such "scientific" numbers.

What makes my proof a proof is your own efforts, looking for all possible solutions of the paradoxical contradiction of chapter 3. Only you can know how hard you have looked. But to some extent, a century of brilliant minds looking at such paradoxes, and ending up with nothing more concrete than a mass of incompletely satisfying symbolic logics, and other axiomatic theories, is a strong indication that there is nothing more logical to find. That will become clearer after the next chapter, where you will see how simply a lot of the other logical puzzles can be solved, despite their symbolic-logical versions remaining paradoxical. Those solutions come from the natural foundation of logic: languages. Not too dissimilarly, the solution of chapter 3 is to found in the wider metaphysics of what things are. That metaphysics is primarily informal, just as languages are. The metaphysical speculations of those using symbolic logics to competitively construct consistent atheist models of large chunks of reality, where those chunks are lifted from cutting-edge science by non-scientists, and the competition takes place within the world of academia, is unlikely to result in any improvement over rigorously logical thinking about the actual problem.

Having said that, standard set theory has been a successful foundation of mathematics for a hundred years, and this has been one of the golden ages of mathematics. It would therefore not be unreasonable to persist in wondering, might the paradoxical contradiction of chapter 3 have been caused by my not using proper mathematics? Chapter 3 was all about whole numbers of possible selections, so it was essentially mathematical. Had I used the mathematics that is standard throughout the sciences, the problem would not have arisen. That mathematics—defined within standard set theory—was developed precisely in order to avoid such problems. In standard set theory, Cantor's paradox amounts to a proof that the biggest collections are not standard sets. In effect, standard set theory simply ignores such paradoxical collections as X, the paradoxical collection obtained and then shown to be contradictory at the end of chapter 3. According to the solution of chapter 4, there is no such collection as X, because God will be constructing such collections forever. That means that standard set theory is actually quite a good model of what we have found.

Whole numbers were discovered to be such that there is no timeless collection of them all. Were there a timeless collection of them all, there would be a whole number of them all. That fact is well modelled by standard set theory. The difficulty of explaining how that could be was added to the difficulty of saying where numbers are, to create the impression that whole numbers had been discovered to be unreal. It then made some sense to think that they could be what they are by stipulation. And that seemed to be enough sense for highly evolved primates. But that is not what was actually discovered.

The axioms of standard set theory encapsulate symbolically just such properties of whole numbers as we have been using. There is an axiom of infinity, which basically says that there is a set of all the counting numbers. There is an axiom of powersets, which says that given any set, there is also its powerset. There is an axiom of unions, which says that given any sets, there is the set whose members are the members of all those given sets. And so forth. Consequently, there are standard sets that are quite good models of all the non-paradoxical collections of chapter 4. The empty set and singletons, in particular, provide good models of the possible selection of no things and possible selections of one thing, respectively. Even the way in which standard set theory can be extended, by adding an axiom that asserts the existence (within the new theory) of a set of all the standard sets is a pretty good model of how God endlessly individuates possible selections, as the new theory could itself be similarly extended, and so on. Mathematicians call the totality of the set-theoretical whole numbers a proper class, the term class deriving from philosophy.

Nevertheless, it could be said that my proof is unscientific. The scientific view of numbers of things is, by default, defined within standard set theory. So, I have not been using a scientific sort of collection. Still, I have been using a very logical conception of a number of things. Science is primarily logical, and X is arguably where the standard definitions stop being very scientific. The way in which a proper class of sets can become a set, in a theory with an axiom to that effect, seems to show that proper classes are a lot like sets. And intuitively, equinumerosity does apply to proper classes of sets. For example, we could have started with three particles, an electron, a proton and a neutron—using E, P and N instead of A, B and C—and ended up with a paradoxical Y; and all the things in X can be paired up with all the things in that Y in an obvious way.

There are other ways in which the standard definitions do not seem very scientific, when it comes to our paradoxical X. Standard set theory is fundamentally a mathematical model. Its axioms give an artificial "language" to classical symbolic logic, which has first-order and second-order versions. We do not need to go into the technical details of that distinction here. To see roughly what it means, recall how the powerset of a given set is the set of all its subsets. In second-order set theory, that word all does mean all, but in first-order set theory—which has the nicest mathematical properties—the word all means some of them. If we used a first-order theory, there would not necessarily be too many things. In that sense, first-order set theory "solves" the problem.

But of course, those things—those ways of selecting—would still be there. All the things that might be selected would be there, and all the possible selections for selecting from them would be just as distinct from each other. By using a first-order theory, we would only have limited our ability to describe them all. We would just have a worse model. No paradox is actually solved by our closing our eyes to it. Ignoring a paradox in order to get on with a lot of other work is fine. Standard set theory has proved itself to be very useful, in that sense. But that utility of standard set theory gives it no authority when it comes to the metaphysical question of what this property of whole numbers is saying about things in general. It is logically fallacious to think that the mathematicians' use of first-order set theory makes the proof of this book unscientific.

If we are to have confidence in our ability to reason logically, about anything, then the paradoxical contradiction of chapter 3 needs to be accounted for in a logical way. We cannot simply replace logic with symbolic logics, and numbers with axiomatic sets, and think that the problem is solved. Still, we do not need to solve every logical problem, in order to reason logically. While we are trying to solve this problem, we need only be extra careful in similar situations. Classical symbolic logic (which is usually just called classical logic, somewhat misleadingly) and standard set theory have enabled us to do that. And using such modelling to look for solutions could have been scientific enough. It would have been unscientific to use the opacity of symbolic logic and set theory to obscure a logical problem whose only logical solution was unpopular. But perhaps that is not what has happened.

For another analogy, imagine a medieval scholar watching the sun rise over a stable earth. "How could we all be flying around that sun at high speed? Would we not have noticed?" Had it been pointed out to him how some new observations would soon be showing it to be very likely that we were indeed orbiting the sun (in such a way that we would not have noticed), he could have countered that epicycles, though hard to master, would be able to accommodate any new observations. Furthermore, the new observations would just be more indirect evidence. Common sense says that the sun orbits a flat and stationary earth, because that is what we can directly observe. And logic supports that common sense view: Today is not yesterday; but if daytime moves slowly around the globe, then it is the same thing when it has gone all the way around. "Would not be more rational to wait until someone goes up in a balloon, high enough to see which body is going round which?" that scholar might have asked, thinking that even if it did then look like the earth was orbiting the sun, it would still be more rational to consider how the devil could have created that appearance, so far from most good people, than to doubt the common sense that one has when one has one's feet on the ground.

But of course, science was built on a foundation of rejecting such arguments, of following instead the evidence in a more logical way. The logic that our logicians' symbolic logics merely model mathematically is more important to science than the materialist sort of evolution that most scientists believe to be true (and far more important than belief in a God who is totally unchanging, of course).

5.ii Doubting Reason

Include there being no things (since logic as we know it needs things).

The idea of logicians not trusting logic is an odd idea—but is it any stranger than mathematicians not being interested in whole numbers (only in sets, or some other axiomatic structure)?

It is certainly not too odd to suppose that the logic of primates—even highly evolved primates like ourselves—might not be perfectly consistent. Whereas, our "proof" that there is a round square was almost too odd. It was so short, we could all but see that there was nowhere for any illogicality to be hiding. And most logicians would say that they do trust logic—although they would be taking the word logic to mean symbolic logic. And symbolic logics are essentially mathematical models of logic. They govern the use of artificial languages like axiomatic set theory (much as grammatical rules govern natural languages like English). Being essentially mathematical, such “languages” are perfectly precise; and they are almost certainly consistent. But they are no more languages than computer programming languages are. Artificial intelligence could, similarly, seem to be a kind of intelligence, in the context of a materialist view of the human mind; but of course, a computer is just a machine.

The basic symbolic logic is called classical logic. It is a good mathematical model of what we ordinarily call logic. And by varying the axioms of classical logic, logicians get various non-classical logics—much as mathematicians get non-Euclidean geometries by varying the Euclidean axioms. And since actual space was discovered to be non-Euclidean in 1919, it was not obviously unreasonable for the logicians of the early twentieth century to think that they might discover—from such puzzles as those of the appendix—that the best symbolic logic is actually non-classical. Now, a lot of mainstream philosophers hope that symbolic logics will eventually prove to be better than logic when it comes to our most scientific reasoning. However, although it is certainly scientific to use mathematical modelling in pursuit of the truth, revisions of logic naturally lead to revisions of truth—because logic takes truths to truths—and if truth is redefined, what happens to the scientific pursuit of truth?

The best symbolic logics are good mathematical models of logical thought, because their very precise definitions were originally abstracted from more intuitive arguments that were—despite their relative informality—rigorous enough for them to have been widely regarded as proofs. Were such a proof to include a step that did not correspond to any of those symbolic definitions, we would have a reason to add such a definition to our symbolic logic. But we would of course have no reason to reject the proof. Conversely, were any symbolic logic to give us a result that was counter-intuitive, we would just have that reason to think less of that symbolic logic. So, there is nothing to be gained by beginning with symbolic definitions, instead of an intuitive but rigorous proof, when we have one. And it would of course be quite unscientific if, having found a logical solution to a puzzle that seemed to have no other logical solution, the undesirability of that solution was taken to be any evidence at all against logic, and in favour of replacing it with some symbolic logic.

The logical steps of standard mathematical proofs—which begin with the standard set theory (which is in that sense the foundation of mathematics)—are classical symbolic-logical steps. But the approach of mathematicians to Cantor’s paradox was quite unlike the approach of physicists to such observations as those of 1919. Upon reflection, it is conceivable that space is only locally Euclidean, that globally it is part of a non-Euclidean spacetime. But when we have some things, there is not some other thing—their set—that we also have, as well as them. Mathematicians did not discover that there are sets of things, which are well-modelled by the standard axioms. They simply started to use axiomatic sets, instead of collections of things. A collection of things is just those things, being referred to collectively. So there is a collection of things when there are those things, because it is those things. How could mathematicians have discovered that things are such that their collections are really something else?

Logic is all about rigorous proof. Of course, academic mathematics is too, and its axiomatisation has been a great success. Nowadays, mathematical proofs are constructed within an axiomatic model of mathematics, usually the standard set theory. Axioms and definitions encapsulate symbolically the definitive properties of the mathematical objects that the proofs are about. (More natural proofs would have used those some definitive properties in their deductions.) And mathematical models help scientists, technicians, statisticians and others to do their jobs; and if those mathematical models use mathematical models of numbers, instead of numbers, then it hardly matters. Whereas, it might matter if, instead of being logical, the reasoning of those professionals was merely well modelled by some symbolic logic or other. Such professionals pursue the truth of the matter. And logic is all about proofs of truths.

Could twentieth-century logicians have been discovering the true nature of logic, as they developed their symbolic logics? Or were they just investigating such questions as how artificial intelligences might best be programmed? The two would seem to be connected, given that our brains probably evolved by natural selection from the brains of more primitive primates.

The consideration of non-classical axioms for symbolic logics does resemble the consideration of non-Euclidean axioms for spacetime. But there are significant differences. Note that we have to use logic to think about symbolic logics properly, just as we have to use logic to think about space, models of space, other physical theories and so forth. If we are not thinking logically, then we are thinking illogically, or just not thinking.

Suppose that it were otherwise. Suppose that in order to judge a symbolic logic, we could only use that symbolic logic. What symbolic logic would not look good by its own standards? So, we might easily get something that we would rather reject, for some reason that was logical, but not symbolic-logical. Why would we regard the internal consistency of the symbolic logic, and its ability to deliver results that we liked the sound of, as better than the logic that could help us to find out which of those results were true? There is of course no logical reason. It would be absurd to use a symbolic logic to judge itself.

We should of course think logically—not symbolic-logically—about symbolic logics, just as we should think logically about physics, or biology. Still, the scientific fact that we—or at least, our bodies—evolved does seem to raise quite a reasonable doubt about the proof of this book. We may be highly evolved, but primates are unlikely to have evolved to be perfectly logical. Would it not be reasonable to draw some realistic limits to what a logical argument could possibly tell us? There are two main reasons why it would not. Firstly, where would we draw those limits? If logic is not telling us what is realistic, what will? It would be only natural for us to find our own cultural prejudices to be realistic. But to use them to limit logic would hardly be scientific.

Secondly, the possibility of logic itself (that is, our natural logic) being imperfect—whether because of evolution, or (under some theologies) because of the Fall—is essentially a Cartesian sceptical scenario. Given a proof of some fact, to say that things might not be that way for this reason is not to say that we should not believe that they are, with that word should having any rational force. It is to deny the importance of such rational force. It does not give us any reason not to believe the proof. Other reasons might be given—social disapproval, for example—but they would have to be essentially irrational. Let’s look at this is a bit more detail.

We all dream, some of us very vividly, and some of us fall into comas, so it would be hard to justify ruling out the possibility of someone in a coma experiencing something akin to what you are now experiencing. If your experiences were very strange, the idea that you were dreaming could therefore seem reasonable. And yet, that is when you should take things more seriously.

Similarly, there is a lot of evidence that we evolved by natural selection, which would have been a rough and ready process. So, it is easy enough to imagine that our natural logic is likely to be imperfect. Indeed, that might seem to give us a plausible explanation of the logical paradox that we met in chapter 2, which was an English version of Curry’s paradox. Such paradoxes might even seem to be evidence against the perfection of our natural logic. And then, the fact that Cantor’s paradox can seem to be a bit like them would amount to evidence against chapter 4 as the only logical explanation of chapter 3. But logical paradoxes are just puzzles that some people have yet to solve. The final chapter solves those logical paradoxes straightforwardly enough.

When would it be reasonable for us to question the consistency of our logic? Should we, for example, question the consistency of our logic whenever our reasoning went much beyond the kind of reasoning that had been subject to evolutionary pressures? But then the reasoning that takes us from all the evidence for evolution by natural selection to our having probably evolved by natural selection would become suspect. Whereas, it may only have been our bodies that evolved. So, without questioning any of that evidence, just by adding a bit more evidence to it—evidence that we are primarily souls, only currently in these bodies—we would have evidence that we could be sure about. And that would be evidence that our bodies evolved by something a lot like natural selection. Our minds might then be perfectly logical, if only we make the effort.

And we should of course assume that there is nothing wrong with logic itself, when we look at the world scientifically. Finding an inconvenient truth is not a good reason to doubt the scientific method. And being unable to solve a logical puzzle is not a good reason to doubt logic. But such matters did become very complicated in the twentieth century.

For thousands of years, God had been invoked to explain all sorts of things. But as successful scientific explanations of nature mounted up, God seemed to be an unnecessary hypothesis. It seemed to many academics that God could not possibly have any explanatory power. There would only be "God of the gaps" arguments. Such arguments invoke the existence of a God in order to explain a current gap in our scientific knowledge, a gap that might eventually be filled properly by scientists. The argument that this book has been making is not so much a "God of the gaps" argument, as a "God of the unfillable gaps" argument. If scientific academics had not regarded this particular "gap" as unfillable, they would hardly have taken it to be a good reason to revise the mathematics and logic that had led them to their scientific worldviews in the first place.

The best-known example of a "God of the gaps" argument is the invocation of God to explain the harmonious structure of the human eye. The occasional scientist still argues that the eye is likely to have been designed. But proper scientists have been working towards a good understanding of how the eye could have evolved by natural selection. And even before such an understanding was close to being achieved, it was not that unlikely that it would eventually be obtained. Furthermore, even if there had been some evidence that the eye had been designed, aliens interfering with our evolution would surely have been the most likely explanation for such design. What counts as a reasonable explanation depends on one's existing worldview, and for most scientists God was never going to be a reasonable explanation.

But my proof is not based on physical evidence, but on mathematical evidence. Aliens could hardly be individuating the possible selections of chapter 3. Only if things in general were created out of nothing but the power of a creator who was able to change could such possible selections—and the associated whole numbers (of possible things)—be endlessly individuated. The God hypothesis is therefore an appropriate hypothesis, when it comes to Cantor's paradox. My proof might seem to be a "God of the gaps" argument, because the explanatory gap that my God hypothesis aims to fill is being addressed, professionally enough, by mainstream philosophers. And they are producing prodigious amounts of stuff, much of which is mathematical. But then, so were epicycles. It is not unscientific to seek to fill an actual gap in our scientific knowledge by hypothesising something quite alien to the existing worldview.

When nineteenth century scientists observed the planet Mercury being where they did not expect it to be, they hypothesised that it had been affected by a previously unknown planet, Vulcan. That was in keeping with their Newtonian worldview. But they did not find Vulcan where their theory said it should be. And early in the twentieth century, their data became part of the evidence that space is non-Euclidean. At that time, very few scientists understood Einstein's mathematics, but no scientists thought to defend their Newtonian worldview by invoking the possibility of our human brains not having evolved to be able to do such astronomical sums. Doing so would just not have been scientific.

My argument is not a "God of the gaps" argument, but a "God of the unfillable gaps" argument. If this particular gap had not been widely regarded as unfillable—by mathematicians, mainstream philosophers and other scientific academics, working within worldviews that were, if not atheistic, then classical-theistic—then far fewer of them would have taken it to be further evidence of our logic's imperfection. The idea that such paradoxes amount to some reason to doubt reason is therefore solid evidence that the only logical solution to the puzzle of chapter 3 is indeed the God described in chapter 4.

There is always the possibility of a mistake being made. We are only human. And there are certain logical mistakes that we all tend to make. See, for example, the book by the British psychologist Philip N. Johnson-Laing, "How We Reason" (Oxford University Press, 2006). But by describing those imperfections in a very clear way, Johnson-Laird has shown how logically we can think about them, when we try to. The likelihood of our making mistakes, when we try to think logically, is quite different to the possibility of logic itself being imperfect. (Similarly, the prevalence of bad intuitions about statistics gives us no evidence at all against the consistency of arithmetic.)

Now, it is not implausible that the explanations for some things will be forever beyond our human abilities. A suspect's fingerprints at a crime scene would of course stand in need of some explanation, but harmonies in things that have been evolving for hundreds of millions of years may well have been another matter entirely. Although, when we have an excellent evolutionary explanation of the harmonious structure of the human eye, it is of course no objection to it to say that that structure needed no explanation. And disliking the materialism implicit in evolutionary explanations would of course not make that objection any better. Similarly, it is not a good objection to chapter 4 to say that chapter 3 did not need a logical explanation. Adding that scientific explanations should not involve a God would not help.

The God of this book, whose existence is being invoked to order to explain a gap in our mathematical knowledge, may seem to be a God of the gaps. But it could, just possibly, make sense for God to be a possible explanation of some very general facts—logical, mathematical or metaphysical facts—even if God naturally has no place in ordinary scientific explanations. That is basically because God is, compared to this world, like an author, compared to one of her novels. A resolution of Cantor's paradox that accords with ordinary scientific intuitions is not expected any time soon, to say the least, which means that it now makes scientific sense to regard Cantor's paradox as a telling impression left on whole numbers by the creator of all things.

However, once they had begun to think that the problem with Cantor's mathematics might be logic itself—the logic of highly evolved primates—mainstream philosophers started to look at some ancient logical puzzles in a new light, and also discovered several new puzzles, such as Russell's paradox and Curry's paradox. Ordinary logic was well and good for ordinary thoughts, these philosophers thought, but for scientific purposes the meanings of such words as "number" and "logic" had to be revised, primarily by mathematicians and logicians. But science is based on a very logical look at the evidence, in pursuit of the truth. Even philosophers had to think logically, or else be thinking illogically. There is no getting around the fact that logic is, or should be, in charge.

Why not keep the consistency of logic and have a proof of a God of a relatively logical kind? When we consider how our ordinarily unquestionable assumptions could, just possibly, have been false (in chapter 5), you will be able to see for yourselves just how odd the alternatives are. They are all far too odd to obstruct that proof. That so many of those illogical (in the ordinary sense of the word) possibilities have been taken seriously by logicians who hardly ever mention God shows how much of a subcultural bias against that hypothesis there is, in science and academia. Atheists often claim explicitly to be agnostics who simply see no need for the hypothesis of a God. But they have collectively acted as though such a hypothesis made less sense than true contradictions.

As they attempted to understand Cantor's contradiction, philosophers such as Russell devised their own logical puzzles. And many more have been developed since Russell's day. But in the appendix you will see how easily those puzzles can all be solved. That gives us further evidence that chapters 3, 4 and 5 do amount to a refutation of atheism. Why else would such puzzles be cited by so many philosophers as reasons why 'logic' and 'truth' need formal redefinitions? Not only are such redefinitions unnecessary, as you will see, we can hardly redefine what it is that science pursues. And we cannot redefine any illogical rhetoric, so that it becomes, by definition, logic; we cannot do that and be scientific.

To solve the logical puzzle behind Cantor's paradox, there has to be a God. Logically, it does not matter that there are a lot of bad "God of the gaps" arguments too. It does not matter that "God wanted it to happen" could be used to explain anything. Nor, similarly, does it matter that we can always find, or hope to find, an evolutionary explanation for anything biological. What matters is that there is an obvious pattern in all the biological and paleontological facts, which can most easily be explained by evolution by something a lot like natural selection. And similarly, the hypothesis of there being something like the God described in chapter 4 explains the mathematics of chapter 3. The big difference is that the only thing that could be sufficiently like that God is a God.

The concept of God is not necessarily irrational. Some religious beliefs might be irrational, but the mere hypothesis of a God is reasonable enough. Given that it is a sensible possibility that we evolved from uncreated lifeless matter, why should it not be a sensible possibility that we were created by an uncreated person? The two hypotheses are on a par (note that we have direct experience of, and therefore knowledge of, personhood, and only indirect knowledge of the physical world that lies behind the phenomena that it presumably gives rise to). God being logically possible, then the argument of chapters 4, 5 and 6 shows that there is – or is insofar as logic applies, or is very probably – a God. That use for the hypothesis of a God is a scientific use because without such a hypothesis the contradiction of chapter 4 makes a lot of scientists throw their own logic away – to be replaced with a convenient symbolic logic – which is incredibly unscientific of them.

Cantor's paradox did not show that the logic of arithmetic was—in its natural, pre-formal state—inconsistent. Nothing could possibly show such a thing. We should always assume that there is a logical solution, to any mathematical puzzle, just waiting to be discovered. But the real problem may, I think, have been that although there was a logical solution, it was essentially a proof that there was a God. That would explain the building of a case against logic, using a variety of logical puzzles, few of which are that difficult to solve, as you will see in the next chapter.

That picture of primates evolving to be imperfectly logical is certainly realistic. However, we have already seen examples of realistic pictures failing to legitimately threaten knowledge that they would, at first glance, appear to threaten. They gave rise only to Cartesian doubts. We do dream, for example, and some of us dream very vividly; and some people are in comas—but none of that undermines our ordinary knowledge of the world around us. We should of course assume that the world around us is the real world. What could we do with the thought that it was not the real world? It hardly matters if we are wrong—unless we have very good evidence that we are dreaming, but then this would no longer be a Cartesian doubt. And similarly, we should assume that primates have evolved to be capable of being logical enough, when they try hard enough (and when they are among the most intelligent primates). What could we do with the thought that none of us could possibly think logically enough about certain matters? How would we draw the line around those matters? Not logically.

Atheists are going to have to choose between logic and their belief that there is probably no God. Now, most atheists think that their atheism results from looking at the facts in a very logical way. They would therefore deduce—from their atheism—that there is no such proof. By justifying their atheism in terms of logic and the facts, they would seem to have prioritised logic. You might therefore think that if they were to find out that there was such a proof, then they would deduce that atheism was false. However, those facts are naturally organised into their atheist worldviews, which include our evolution by natural selection in a purely physical world. So, if they were to find out that there really was such a logical proof—unlikely as that would seem to them—they might regard that proof as some evidence that we did not evolve to be perfectly logical. They might prioritise atheism, over logic. And that is what they seem to have been doing, for over a hundred years.

Scientific theories are naturally justified in terms of logic and the facts. But perhaps our logic evolved as it did because it helped language-using primates with their teamwork. In such ways, an account of logic might be given in terms of an atheist worldview and the facts. The details of such an account are not important here; what is important here is that an atheist account of logic raises such questions as this: why would we ever have expected primates—even highly evolved primates like ourselves—to be perfectly logical? Why expect logic to be perfectly adequate when it comes to such extraordinary proofs as that of this book? Scientists are big fans of logic; but if the logic used to justify scientific theories could be distinguished from the logic of the proof of this book, then this proof's unbelievability—as committed atheists see it—could be regarded as evidence against the reliability of the latter.

The logical problem of Cartesian doubt is an interesting problem in logic, and it is a similarly interesting problem how we should think about such pictures as those evolving primates; but what makes the latter so interesting is the proof of this book—its unbelievability—and so I shall be focussing on that proof in this book. And the proof itself is relatively straightforward. And I would emphasise that I am not asking you to believe that God exists. I am only asking you to read the following proof that God exists. Of course, that may well sound like someone saying that, although it is of course hard for you to believe that two plus two equals five, I am only asking you to read my proof that two plus two equals five (plus a rambling preamble). You would of course assume that that would be a waste of time, which is why I emphasise the remarkable fact that if atheists were to find out—as you will find out, if you continue reading—that this proof is not illogical, then they would be very likely to conclude that there must be something wrong with logic itself.

Who would have expected the biggest fans of science to be such fair-weather friends of logic? Most atheists would not have expected that. But this proof being so logical might embarrass our experts at thinking. They may well be considerable relief when professional logicians assure us that there is indeed something a little imperfect about the ordinary meaning of the word "logic".

Throughout the twentieth century, logicians took a wide range of logical paradoxes to be showing logic to be in need of modernisation. Modern logicians have constructed a wide range of "logics" using such scientific tools as mathematical modelling. Being essentially mathematical models, they do look very scientific. They are full of mathematical symbols. And when they are thought of as models of logic, rather than as replacements for logic, they are very scientific. They are pretty good models of logic, for most scientific purposes, having been constructed to capture much of the logic of the scientific method. But most of them were designed to lose the logic of such arguments as the proof of this book. By losing some logic, they aimed to resolve various logical paradoxes. But the idea that logic could be replaced by one or more of these "logics" is an unscientific idea. Logic takes truths to truths, so what becomes of the scientific pursuit of truth if logic is redefined? And although the idea that we did not evolve to be perfect is clearly a realistic idea, the idea that logic itself could therefore be imperfect is more like a Cartesian doubt than a reasonable doubt.

Symbolic logics differ from actual logic in much the same way as Newtonian physics—a mathematical model (or theory)—differs from the actual laws of the physical universe. Newtonian physics is a mathematical model. But while physics is about the external world—and so needs to be modelled, as part of the scientific process of testing hypotheses against observations—logic is internal to the human mind. It is logic itself that needs to be used when proceeding scientifically. Suppose we instead used a model of logic, instead of logic itself. If the model was a deliberately inaccurate model of logic, would you deliberately use it illogically? If you were replacing logic with the model, you would have to. And if it was not deliberately inaccurate, then why would you need to replace logic? Or, would you use it like you would use any other mathematical model? It seems that in their arguments about such things, philosophers do try to be logical (much as you will be doing, when you try to answer such questions).

A mathematical model of something is effectively a description of it. Schoolbook physics, for example, is a description of physical reality. By contrast, the bits of the model are – like the words in a description – defined to be what they are. In Newtonian physics, for example, force is defined to equal mass times acceleration. In symbols that may be familiar to you from schoolbooks, $F = ma$. In the real world, we find that there is a relativistic correction to that equation (whose details need not detain us here). In the real world, things are described as they are found. The more we know about the things in the real world, the more precisely they can be described.

Things in stories are as they are described by their authors. They are essentially defined into a fictional existence. The describing is not normally done with a lot of precision, because unnecessary detail makes for a dull read. In a model of reality, things are usually defined much more precisely. But models of reality are non-fictional even if they are deliberately inaccurate. Note that it can be very useful to have a simple approximation. Fifty years after Newtonian physics had been falsified, for example, that mathematical model was still being used to get men to the moon, and back. It would make little sense to call such a false model fictional.

Redefining “logic” to avoid the logic of an undesirable proof is a different kind of thing though. There certainly seems to be an element of deception there, and so that would seem to be more like a fiction than a model. In any case, there being a logical proof that God exists is not at all challenged by there being—for a lot of academic definitions of “logic”—no “logical” proof that God exists. Quite the contrary: the fact that there have been those redefinitions is some evidence that there was something extraordinary going on.

As the twentieth century began, words like “number” and “logic” were being given new definitions by academic mathematicians and philosophers. As a result of the redefinition of “number”, the paradox no longer arises within the mathematics that is routinely used throughout the sciences. But of course, redefining the numbers of academic mathematics did not change the underlying logic. And nor does redefining “logic” change the underlying logic. Replacing logic with some abstract structures—which are then called “logics” by academics who call that replacement a precise definition of the word “logic” for scientific purposes—is just a sophisticated way of rejecting logic. It just goes to show how logical the proof is, that the most intelligent atheists thought that they had to go that far, to avoid the conclusion that they did not believe in. It is some evidence, before we begin, that it is worth your while to continue reading this paradoxical proof that God exists. There is this proof, as well as the academic denial of that fact. A paradox is basically a lot of reasons to believe a fact, plus a very plausible denial of that fact. Cantor’s paradox is a remarkable proof minus the surprising theorem that it proves.

Nineteenth-century mathematics was paradoxical enough to have warranted a redefinition of “number” that remains, to this day, at the heart of modern mathematics. Now, the position of logic in academia is a bit like the position of mathematics in science, but there are important differences. The job that logic does in all of our thinking is very different to all the jobs done by mathematical models and methods. Logicians aiming to replace logical thinking, in academic work, with calculations done within mathematical models of logic is a very different kind of thing to mathematicians doing mathematics by way of a mathematical model of itself.

Redefining logic in order to make an undesirable proof go away is simply unscientific. Symbolic logic looks scientific, and it reliably delivers desired results. But such things are rhetoric, not logic. Still, it was a natural mistake for modern logicians to make. The idea goes back, through Russell, to Leibnitz (who thought that the Grandi series summed to ½). And what modern logicians have been doing is superficially similar to what modern mathematicians did, when they made the standard set theory the foundation of (standard) mathematics. There was nothing wrong with that, by the way. After all, ancient mathematicians did not just draw accurate pictures, they worked logically with symbolic descriptions of space; so why would they just calculate accurately with numbers? Furthermore, superficially similar redefinitions are also a perfectly natural part of how ordinary language works. When we chance upon inconsistent descriptions of reality, it is natural for us to see something like an equivocation.

Chapter 3 amounts to evidence against common sense, if the God hypothesis is rejected, precisely because chapter 4 is the only logical solution. Why would we reject the God hypothesis? Well, it can seem self-evident that there is no God. Biology can explain what we are. There are no miraculous events, requiring a non-biological explanation, no sign of God anywhere. The problem with that argument is that chapter 3 is arguably such a sign. Another problem is that those biological explanations are presumably going to be good explanations because they are logical, because they make sense (to people who care about truth and logic). What happens to those explanations, when we are entertaining doubts about common sense and the consistency of logic? What happens to the evidence that there is no God, in a world where evidence need not necessarily be judged sensibly or logically? (What happens to all the other important evidence? Not only does this doubt seem to undermine itself, insofar as it does not, it could threaten the status of science as being above and beyond politics.)

It really can seem that there is no God. And so, it can seem to make sense, that we should rule out chapter 4 as an explanation of chapter 3. But what if it seemed that there was a God? Could we take that seeming, and do the same with it, as we are considering doing with chapter 3? If that is silly, then it is silly—not sensible—to rule out chapter 4, as an explanation of chapter 3, on the grounds that it is clearly the case that God does not exist.

Biological explanations do not need to invoke God. But that says nothing about the existence of God. Astrophysical explanations do not need to invoke the artichoke. Perhaps that shows how much smaller than galaxies artichokes are. But it basically says nothing about the existence of artichokes. We do not see God around here. But nor do we see algebraic entities around here. The God that we are considering here, as an explanation of chapter 3, is not a being that would be likely to need a political party, or its own religion. Perhaps it does have a religious relationship, with religious people. But who is to say what that is? When we make a decision about who would know about that, that says something about us. Perhaps there are a few biological explanations that should invoke God. But modern biologists could always find other explanations. God would be unlikely to need to do something biological that could not, with hindsight, have been something more ordinary.

When it comes to questions of existence, it does not matter how much says nothing much, only whether anything says something. Suppose you are a detective. A banker has died, apparently of natural causes. You investigate the scene and find nothing suspicious. But you then notice that there were no fingerprints on the handle of the safe. You might have expected the banker's fingerprints to be on that handle, though. Something as slight as that absence of a few fingerprints makes all the difference between probable and improbable. The evident nonexistence of a murderer, which you saw as you glanced around the room, was nothing by comparison.

The reason usually given for why modern logic is so mathematical is that there are a lot of logical paradoxes—a lot of seemingly logical derivations of contradictions. The main ones are considered and solved in the next chapter. Since we are probably highly evolved primates, and it is easy enough to think of primates as not being perfectly logical, our natural logic has been blamed. Mathematical models of logic, which are not so accurate that they are also contradictory, can then seem preferable when precision is required, such as in scientific thought, which is very mathematical anyway. Such "logics" have to capture as much as possible of what we mean by logic, so they are mathematical models of logic. But because they are mathematical, they are judged primarily according to mathematical criteria, such as their elegance. And allowing some true contradictions—for our most abstract thinking—could give us an especially elegant symbolic logic.

Cantor's derivation of a contradiction was so logical, he took it to be showing that the largest infinity—his largest collection of numbers—was contradictory. And that is essentially what the standard set theory models. There are all the sets of that theory, in the theory—which is a mathematical model of collections—and there is only no set of them all because if there was such a set then there would be a contradiction.

Another popular resolution calls all such paradoxical statements nonsensical. That is perhaps most popular amongst those materialists who call all religious statements nonsensical. And it can always be compared with the fairly popular view that all experts spout nonsense. Of course, rejecting our natural way of reasoning was never a very reasonable option. While we can – and should – try to think more logically, we could hardly try to think more inhumanly. We could only ever play at that.

Perhaps it did make some sense, to try to work around a natural illogicality scientifically, when the contradictions seemed to result from perfectly logical arguments. Although even then, it might have made more sense to wait for the logical solution to such logical puzzles. However, to suppose such an illogicality, just to avoid the conclusion of an undesirable proof, was certainly not scientific. We would not then be seeing what a logical look at the evidence would show there to be. We would be seeing only what we wanted to see. And redefining "logical" in order to try to change that would not make it any better. If you and I are arguing about some matter – you find X nonsensical, whereas I find not-X absurd – and you finally find and present a perfectly logical argument why you are right, and I say that since not-X is absurd, you have shown me why "logical" needs to be redefined, what would you think?

Why do scientists believe that we are highly evolved primates? It is not because scientists like the idea of evolution: evolution is a terrible process that works by way of the slaughter of trillions of immature individuals. Nor is it because scientists could write up evidence for evolution in an artificial language, and calculate 'evolution' in that language by using a mathematical model of logic that has been tweaked so that its results are never too unpalatable for scientists (or those who fund them). If that was the reason, then religious denials of evolution would of course be perfectly reasonable. But it is not; and insofar as scientists implicitly act as though that is the reason, they are simply failing to understand their own thought processes. No, the reason why scientists believe that we are primates is – of course – because of a logical look at the evidence. So if there was something wrong with logic, there would be something wrong with our reason for believing that we are highly evolved primates. And there would be something wrong with all of our symbolic logics. Whereas, there may well be nothing wrong with logic. Indeed, logic is a prerequisite for thinking properly about anything.

Logic does give us logical paradoxes. But logic always tells us, of any contradiction that appears, that it is merely apparent. Logic makes us look for the cause of that appearance. While it is conceivable that there is no cause other than our own flawed logic – much as it is conceivable that you are now in a coma and dreaming all of this – the assumption of logic is the assumption that there is some other cause. And we do have to assume logic, in order to think properly.

Logic also gives us – by way of the science of biology – some reason to doubt our primate logic. And some would therefore have seized that opportunity to reject an otherwise very strong proof that there is, very probably, a God. It is understandable that they would have done that at the end of the nineteenth century, when academic science was still throwing off an old-fashioned metaphysics. It is also understandable that they would have done so during the twentieth century, when the culture of academic science had become atheist. But it would still have been unscientific.

Replacing logic with a calculating tool – a formal “logic” – that is not quite logical means that it is not exactly truth that is being pursued. Of course, an atheist would disagree, since an atheist would think that the truth of atheism was what was being pursued. But atheism ought to be defended logically. Why should other believers not put their beliefs ahead of the logical pursuit of truth? Because most scientists do not share those beliefs? Even so, taking an instrumentalist approach to the logic of arithmetic would have reminded most scientists of the Church’s approach to Galileo’s defence of heliocentrism (the view that put the sun in the centre, with the planets orbiting around it). The Church had famously observed that a heliocentric interpretation of the astronomical data was just one particularly neat mathematical structure that could assist with long calculations. It was a calculating instrument, not a picture of reality. Such neatness said nothing about the actual physical nature of the solar system, the Church had asserted. Whereas Galileo had famously insisted that the earth actually orbited the sun.

And an instrumentalist approach to logic is far worse than an instrumentalist approach to physics. Modern physics is a work in progress, and many physicists assume some sort of instrumentalism in order to do that work free from metaphysical assumptions (although in fact, most of them assume atheism). But what would instrumentalism about logic be freeing scientists from? Logical questions about their atheist beliefs? Once scientists have allowed themselves such latitude, what is to stop those employing them from putting simple economics ahead of logic? Or the political beliefs of the democratically elected representatives of the people? To see how unreasonable such a doubt about logic is, you need only reflect upon how such a doubt is never used anywhere else in science, to reject any other argument. Anywhere else and it would be obvious that it was simply unscientific. It would be like using a sceptical doubt, such as that we might be dreaming, in a court of law, to justify ignoring some inconvenient evidence, without ignoring any of the convenient evidence.

Or imagine an alternative historical timeline, in which the Copernican revolution (which replaced the earth being at the centre of creation with the sun being at the centre of the solar system) had never happened, and the Darwinian revolution had instead swept the mediaeval worldview away. Had the earliest evolutionary biologists still been using epicycles to predict astronomical events, then they would have been able to use such a doubt about primate logic against their Copernicus. They could have observed that it is only common sense that the earth is fixed. All humans can see the stars moving around it. If thinking in a very mathematical way, about very detailed observations of those stars, led humans to think that the earth was spinning around the sun, then that would have looked like a good example of primate logic being imperfect.

Those fictional scientists could have prided themselves on being able to do the relatively complicated mathematics of epicycles, which can always be fitted to observations. But would that have been a reasonable doubt, for them to have had about their Copernicus? It would have seemed reasonable to them—as reasonable as our scientists' similar doubts about this book's explanation of Cantor's paradox seem, to our scientists—until they think about it.

Given anything that looks like a proof of a contradiction, the most reasonable reaction would therefore be to take it to be a "proof" of a contradiction, a logical puzzle to be solved logically. Nevertheless, a lot of people—primarily atheists (but also classical theists, and others)—whose worldview is that we humans are not that rational do find it easier to consider the possibility of such a proof of a contradiction than to countenance a proof that there is (probably) a non-classical God. In effect, they promote their worldviews ahead of logic. Any inconsistencies that I could point out, with the rationality of that, in the case of the scientific worldview, they could put down to our being irrational. They could point to the empirical evidence for evolution, and to various logical puzzles, and to the irrational history of mankind, and ask me to explain all of that.

So, firstly, that empirical evidence, if it is considered logically enough, can easily enough be seen to be perfectly consistent with there being a logical kind of God. Secondly, the logical puzzles do have relatively easy solutions (as you will begin to see in the next chapter). Thirdly, history does indeed show the kind of thing that we might expect were we to throw the bathwater of reason out with this new proof (that there is probably a logical kind of God). Regarding the empirical evidence for evolution by natural selection, note that there was a lot of empirical evidence for Newtonian physics at the end of the nineteenth century. The only reason why the former appears to be incompatible with the existence of God is that serious thinkers have not been thinking much about such compatibility. It does not take much to get them thinking, as Einstein showed. One does not have to overturn all the previous observations. And there is certainly an incompatibility with a literal approach to Genesis.

I think that science has to be defined primarily in terms of its methods being logical, and only secondly in terms of its best current theories, although that is a question for philosophers of science; it is certainly tangential to the concerns of this book. This book is primarily an argument that, if you think about it, logic dictates that there is probably a God. It is not primarily an argument that you should care about logic. Nevertheless, you probably do care, especially if you are a scientist. And the particular doubt about that proof that I am addressing in this section is to some extent independent of such general questions as what science is, and should be; although a redefinition of "number" and "logic" and hence of "truth" has already been entertained as a result of the paradox that is at the heart of this proof.

This particular doubt puts the implicit atheism of major scientific worldviews ahead of the logic of the proof. It regards the argument of chapter 3 as—not flawed in any of the ways previously considered in this chapter, but therefore as—being an argument against the consistency of logic itself. This doubt takes that to be a reasonable possibility on the grounds that we are clearly just highly evolved biochemical structures. Those grounds are taken to be as solid as a massive amount of empirical evidence. However, evidence does have to be assessed rationally, if it is to amount to anything solid, and so this doubt does undermine itself, to the extent that it is not actually as reasonable a doubt as it does at first appear.

The problem with this doubt is not so much that the empirical evidence for evolution by natural selection is perfectly consistent with a wide range of theologies—which it is, although that is a very complicated (but fortunately tangential) thing to show—it is rather that it is akin to a Cartesian doubt. Consider how the night sky, with its masses and masses of stars in this one galaxy alone, could seem to show that intelligent alien civilisations are quite probable, somewhere in the vastness of space. You only have to ignore such counterintuitive details of modern physics as the universal speed limit (and all that complicated mathematics), or think that such an obscure limit might be surpassed somehow (say, via hyperspace), or simply consider how they could have already arrived here (having set out in the distant past), to think that secretive aliens are not that unlikely. Nevertheless, their effect on our knowledge—and especially on our knowledge of such things as that universal speed limit—can easily enough be seen to amount to a Cartesian doubt.

Still, blaming a result that you do not like on a failure of logic is not obviously a problem. It is not as though scientific proof will have thereby become a matter of what someone else likes. However, you would then be on a slippery slope. And the slipperiness of that slope might go to show how unscientific it is to blame a result that you do not like on a failure of logic. Consider, for example, some fictitious technicians, whose hard work has become threatened by some evidence that a certain product of that work is dangerous. Logic turns that evidence into a reason why that product is dangerous. They do not want to throw all that scientific-looking work away, but they observe that a lot of their scientific peers have little problem with logic itself being imperfectly logical. What could those peers say to their firm's lawyers when those lawyers invoke the flaw in logic that has been shown to exist by those peers' victory in the debate over the proof that they did not like? Would they fall back on that proof only having shown that there are very occasional true contradictions? Would they defend that symbolic logic? Would they finally admit that it was never scientific to doubt logic? But by then it might be too late. A lot of people might then be, either happy enough to do without perfectly logical logic, or else relatively powerless. By then, they might be drowning in dangerous products, of all sorts.

If some scientific atheists did want to carry on claiming that it was scientific to doubt logic, then they would not find it hard to find independent evidence that we cannot think logically about very remote matters, though. For example, while astrophysics gives us strong evidence that the temporal element of spacetime is not presentistic, atomic physics gives us some evidence that it is. The fact is, though, that despite our biological justification for doubting logic when it comes to the foundations of mathematics, there is no similar talk in physics about evolution not having fitted us to do astrophysics or atomic physics. What we find instead is a healthy debate about the nature of time. I think that if you look at this objectively, you will agree that the difference is that in the foundations of mathematics we find this proof that God exists, whereas there is no such proof (yet) in modern physics.

What would it mean, were science to become just another bunch of beliefs? Its adherents would maintain that it was more than that, but in that way too they would be just like all the other adherents of bunches of beliefs. For it to be more than that, it needs the scientific method to be logical. Logic – and each and every presupposition of logic, such as that there are things (like words and symbols, planets and particles, cats and dogs, people and their houses, with their numbers and rooms, with their tables and chairs) – is not something optional. Real scientists believe primarily in the scientific method, as a logical pursuit of truth, not in the truth (or probability) of any particular scientific theories – such as the theory that we evolved by natural selection (and are therefore unlikely to be perfectly logical) – and certainly not in the culture of science, or the image or lifestyle of the scientist (however much those things influenced their personal choice of career). Real scientists would not countenance the self-styled handmaidens of science redefining "truth" for them.

Committed atheists, by contrast, will see nothing solid in my vague talk of differentiating the whole numbers, not when they compare that to all the evidence for Darwinian evolution, in this physical universe. They will see no reason to try to understand such talk. They will simply throw my words in with all the religious wordage of much of our history, and many other cultures, and all that is written on the supernatural, and such like. A lot of academic scientists will therefore see no independent reason, beyond the intuitively logical argument of this book (if they grant that they are intuitively logical), to believe that there is an infinite person constructing whole numbers forever. The latter is simply fantastical, in their eyes.

Whereas there seems, to many academic scientists, to be a lot of hard evidence for there being an imperfection in primate logic. It is not only the way the masses move, not only the way the elites conduct their businesses, it is also a lot of academic work on a wide range of logical paradoxes, all of it looking like academic mathematics. Academics have already seen a need to formalise logic – much as intuitive Euclidean space has been replaced, in physics, by the complicated mathematics of general relativity – and so it is now quite simply a fact that academic logic is a lot of symbolic logics. And so even as they choose atheism over logic, they will think that they are choosing logic. They will, in that way, be behaving just like any other cultural group. But science was always supposed to be different. Scientists should not replace logic with symbolic logic. They should be wary of doing so by default, by deferring to the professional expertise of academic logicians.

A relatively modest claim for me to make would be that there are no scientific grounds for the claim that there is probably no God. A logical God does appear less fantastical the more you think about it, in academic rather than cultural terms. But of course, there is nothing to stop atheists from continuing to believe that the human mind evolved by natural selection, and that we are therefore not perfectly logical. Children will continue to learn about evolution in biology classes, and science will continue to be presented to them as logical, not as "logical". But it might be that fewer of the brightest children will go on to pursue careers in academic science. Of course, it depends on what one means by "brightest", and by "science". If science is defined to be what scientists do, and we already define scientists to be people with scientific jobs, then the brightest might already have been defined to be the best paid. Academic science might become even more full of atheists, at the expense of its more logical investigators. It might become duller (that is the kind of thing that cultural groups have been known to do).

Biologists, and most other scientists, tend, for obvious historical reasons, to be atheists. But all scientists defer to the professional expertise of other scientists. Professional logicians, faced with this proof, and unable to find an actual mistake in it, could simply remind us of their earlier replacement – which at the time was explicitly for scientific purposes – of logic with a wide range of symbolic logics. They would be wanting to give scientists the most appropriate symbolic logic for their needs. And they would themselves be implicitly deferring to the prevailing atheism of both science and academia. In their turn, academic biologists can then defer to the expertise of those logicians, as well as the standard mathematicians. There will not seem to have been any real need to choose between logic and atheism. Had they realised that there was such a choice, biologists would of course have chosen logic. But for historical reasons, the point was never made plain, and could still be very easily concealed.

Scientists tend to believe that human intelligence evolved because it facilitated social integration within highly competitive groups of a few hundred primates, in a purely physical universe. They tend to believe that human intelligence is akin to artificial intelligence. And in their work they use mathematical models to turn a lot of evidence into high probabilities of truth. Consequently they tend to find the idea of formal redefinitions of “truth” to be a rather scientific idea, in line with the familiar redefinitions of “space” in physics, “number” in mathematics and “logic” in logic. They would not take themselves to be changing the meaning of “truth”, so much as finding out more about it, as they explore more and more of the world. But “whole number” and “logic” are not like “space”, and nor is “truth” – even more obviously.

Imagine, for a too-simple example, to start with, an informal redefinition of “truth” as empirical adequacy. Empirical adequacy is very important in non-academic science. Epicycles were empirically adequate for astronomy, and then were not. And more recently, Newtonian astrophysics was empirically adequate for getting to the moon and back. Neither are empirically adequate in complete generality nowadays. And neither were ever simply true.

So, imagine that we have a theory or model that is empirically adequate. We are to imagine calling that theory “true”. The thing is, that “is” in “that is empirically adequate” is the “is” of classical truth (to say that what is the case is the case is to speak the truth). So we are still assuming classical truth, even with this informal redefinition.

The alternative is that we call that theory “true” when it is empirically adequate to think of it as being empirically adequate; or rather, not so much “when it is” as “when it is empirically adequate”, with an infinite regress threatening. That regress might be benign, or it might turn empirical adequacy into something subtly but problematically different. And that alternative was not, in any case, what we had been assuming. Since empirical adequacy is not the same as truth, we can see the problem with such a regress very clearly by thinking of it as a kind of untruth. We have a theory that is empirically adequate, and – if we are not still using classical truth, then – we are concerned with the empirical adequacy of calling that theory “empirical adequate”. So it might not even be empirically adequate, and yet we shall be calling it “true”. And the difference between it being empirically adequate and it being called “true” – on this redefinition of “truth” as empirical adequacy – is going to be hidden within an infinite regress.

The thing about a formal redefinition is that there is a similar hiding of what is going on. Imagine a logician formalising troof, where to say that a proposition is troo – that it has troof – is to say that it is part of a coherent story that accounts for all the scientific facts and which scientists like the sound of. We may as well suppose that the logician has constructed a mathematical model of troof that is very neat. And if "true" is replaced by "troo" in any formal paradox of truth, then the model delivers no contradictions. Do we have, here, a pretty good formal theory of truth? Since the unwanted conclusions that are being avoided are not conclusions that the scientists think could possibly by true, it might not seem to be a different concept that is being defined and confusingly called by the same name as truth. Furthermore, it is now true (in this new scientific sense) that there is no God. Scientists really are concerned to have results that they like the sound of, so they are concerned with the truth of the matter (in this new scientific sense). Some of them wonder if this is really what "truth" means – of course – but their professional colleagues assure them that such is the truth of the matter. In other words, that this really is a coherent story that those logicians really do like the sound of. But of course, these scientists wanted to know the classical truth of the matter.

The best current scientific picture of ourselves presents us as being highly evolved primates, whose minds are essentially akin to computers. Why should it be possible for beings such as ourselves to be perfectly logical? In view of the above, we should take seriously the possible answer that: it is our logic. As individuals we often make mistakes, and groups of us do too; but if we work at it properly, we can hope to make few. After all, the argument from evolution – against natural arithmetic (and more generally, pre-symbolic logic) – is just another sceptical argument. It is very similar, the way in which we cannot rule out the possibility that a powerful alien has hypnotised us into thinking illogically, about matters of interest to the alien (and also when we are thinking about logic). Suppose that someone raised that sceptical scenario, in defence of ignoring an argument for something that person did not like. We would just laugh at that.

Do you want to say that the difference is that we are probably highly evolved primates, whereas there is probably no such alien? But to work out the latter probability you would have to assume that the world is as it appears to be. If it was the case that there was such an alien, then the world would not be as it appears to be (in just such respects). Because it is a sceptical scenario, the lack of evidence does not give us a low probability, it gives us an unknown probability that it would be unreasonable not to ignore. There are countless numbers of other sceptical scenarios, so a low probability of any one of them would be of little use anyway. And note that the illogicality that we might associate with being primates might crop up anywhere, for all those primates might know. Why raise it somewhere, and not elsewhere? And so on, and so forth; at the end of the day, why not think of primates as logical, in their own way? In particular, why not assume that, although we are probably highly evolved primates, we can be logical enough with our natural languages when we make the effort? The paradoxes certainly give us no reason not to assume that we are. And at the end of the day, questioning that assumption would not only apply to inconvenient proofs; and even when it did, it would not necessarily deliver atheism, rather than some other politically convenient belief.

Consider a fictional analogy; say, the evidence for climate change. Most of us cannot look at the actual evidence, and if we did we would not know how to check it for mistakes. We assume that the experts have done all that for us. But suppose someone asks: what if most experts are team players who are in it for the money? Can we check their sincerity? Even if the person asking such questions was clearly in it for the money, we might acquire some reasonable doubts about the evidence for climate change. Nevertheless, we work it all out, somehow, and come to have some reasonable beliefs about climate change. Now suppose that someone suggests that none of that matters, because we are highly evolved primates: we cannot trust our own logic, so we should not believe what reason has eventually told us. Let's all just fall back on what we originally felt. You cannot ask why not, because we cannot have a logical answer. In this real world context, you can just see how the latter sceptical suggestion differ from the former sceptical suggestion. The latter should quite clearly have absolutely no part to play in how we should think about climate change. The former might be annoying, but it would be wrong just to ignore the question. To ignore such questions as a rule would be dangerous (for obvious reasons). Now, why should things be so very different when it comes to the simplest properties of the real world, arithmetic?

And why should our being highly evolved primates, and therefore not perfectly logical, not mean that we would like any piece of rhetoric in support of our religious beliefs, even a proof? The alternative to logic is not necessarily an evidence-based atheism, it could as easily be an intuition-based religion. It might come down to a popularity contest, and in that contest atheism would have lost all of its most popular arguments! There would no longer be a logical argument for evolution. There would no longer be a logical argument for the improbability of the classical God. If we lose logic, what are we left with? Whatever it is, it is not science. I have already explained how it is no answer to say that we would have logic, a very scientific symbolic logic. So let's hope that it is not yet true that a logical argument – in the ordinary sense of 'logic' – has become of less professional significance to scientists than the general convenience of their professional disciplines.

Fortunately, the economic importance of science is entirely rooted in its slavish devotion to the pursuit of truth. We would expect academic biologists, for example, to be able to facilitate the regulation of the new biotechnological industries, in the best interests of the people as a whole. If the history of technology is anything to go by, such regulation will be needed, so it is a good thing that academic biologists would go with the logic, even if logic was not getting them what they wanted. Who would trust the regulation of such industries to scientists who would choose atheism over logic? What else would they prioritise over logic, and truth? It is not as if you could just ask that sort of scientist.

Of course, biologists are likely to say that biology plainly shows that there is probably not a God. They will suspect that the argument of chapter 4 was just bad mathematics. Mathematicians would agree, although they might refer us to the philosophy of mathematics. But the argument of chapter 4 was designed so that it made no assumptions regarding the philosophy of mathematics. It was all about things, logical possibilities, and collections. And it only made such assumptions about those as it would be far too odd not to have made. If academic biologists were to look across academia at the philosophers, to see what such assumptions would look like there, they might be surprised to see them busy redefining 'truth' for scientific purposes.

The relationship of truth to science is a bit like the relationship of goodness to God. In some ways it is natural to think that goodness derives from God, if there is a God: the good is whatever God likes. That thought – the divine preference view – goes naturally with God being the creator of the world. But in society, the good is not just whatever society likes, and so the divine preference view can seem wrong. And yet if goodness does not derive from God – if it is something that even God must choose, in order to be good – then there seems to be something bigger and better than God: goodness itself. So, there is something like a dilemma here. And similarly, because scientists are the experts on what is actually the case, it would be natural to think that they should have the best definition of 'truth'. But in general, people giving 'truth' a meaning of their own do not come across as trustworthy. And yet, who would know better than scientists, what science needs? So, there is something like a dilemma here too.

Space having been discovered to be non-Euclidean, shortly after Cantor discovered his paradox, it was not unreasonable to entertain the thought that the parts of logic that were closest to arithmetic might have been discovered to be in need of a similar modernisation. And while it is technically a logical impossibility for there to be something wrong with logic, that is just a technicality. The one thing that logical impossibility cannot rule out is there being something wrong with logic itself. If there is something wrong with logic, then the logical impossibility of that would not be what it would have been, had there been nothing wrong with logic. Nevertheless, the only way to replace logic with anything else is to make oneself more illogical.

The sciences result from looking logically at the facts, and the facts result from looking logically at the evidence. A huge amount is now known about particles, molecules, living beings, planets and galaxies. If logical paradoxes had showed that we did not evolve to be perfectly logical, that would have cast doubt over all of that scientific knowledge. Scientifically minded people should therefore be glad to discover that it is not, after all, likely that there is anything wrong with logic. You would have to be deeply into a subculture of atheism—or one of classical theism, similarly—for you to regard chapter 3 as evidence of a natural flaw in the logic of our primate brains—or, respectively, of our fallen, or merely finite, minds. If our natural logic was inherently flawed, we would only be able to think about that in flawed ways; and of course, maybe it is not flawed.

Suppose that someone you are deeply in love with is convicted of a crime. The evidence is very strong, but your beloved claims to be innocent. If the evidence is so overwhelming that the only way in which you can imagine your beloved being innocent is if your beloved had been set up by some undetectable aliens, then you might think that you had some evidence that there are such aliens. But of course, everyone else should consider your beloved to be guilty. After all, even if there were such aliens, there would be nothing that could be done about them. Anyone arguing that there were aliens would only be making trouble for themselves, whether or not there were aliens. And similarly, we should assume that our natural logic is not inherently flawed, whether it is or not. If it is, then our attempts to work around the flaws would themselves be flawed, and they would not even look logical.

It might be said that scientists assume the simplest hypothesis, and that it is simpler to think that evolution has given the most logical primates a logic that is at least infinitesimally flawed, than it would be to add any kind of God to science. But the former is only simpler for those well versed in the current scientific theories, and unversed in the philosophy of religion. A perfect being who is able to change is not necessarily a complicated thing. And in any case, scientists do not actually believe illogical theories just because they are simple. A very simple theory is that all propositions are true. No one believes that all propositions are true.

Much as evolution is a biological fact, the existence of God is a mathematical fact. We see neither directly, but we conclude that they exist by their effects. It would be too odd, were they not to exist. Such “facts” are very probable theories, so they might be superseded by future theories that rest on greater evidence. And note that evolution and God are more compatible than quantum mechanics and general relativity. Just because there is a proof that there is (very) probably a God—who is nothing like a Jove or Odin (who would hardly be able to individuate whole numbers)—it does not follow that our brains are not those of highly evolved primates. What follows is that it is less likely that we are identical with our brains (although the relationship between mind and brain will probably turn out to be at least as complicated as whatever future theory resolves the current inconsistency of quantum mechanics with general relativity).

The rituals and creeds of the religions of others naturally appear complicated. Scientists may well find it more plausible that evolution should give rise to a flawed logic, than that there is a God. But science is about much more than what the majority of scientists are currently finding plausible. Science is not just another part of our culture. Scientists themselves all think that the main thing is to take a long, hard look at the evidence, and see what it is actually saying. And if the evidence is not looked at logically, then it is looked at illogically. And science is of course not about seeing what the evidence says when it is looked at illogically.

Could it be that the logical flaws only arise in connection with this proof that there is (very) probably a God? Maybe, but that would be very convenient to the prevailing atheism of academic science; so convenient, there is surely an onus, on atheist academic scientists, to give a principled reason for such a significant distinction. Why is it that highly evolved primates thinking logically about hundreds of millions of years of fossils, together with huge amounts of data from molecular biology, which is just part of the evidence for our being highly evolved primates—why should they think that that is so different to highly evolved primates thinking logically about the elementary repetitions described in chapter 3? If we cannot do the latter, why would we be able to do the former? In the absence of such a principled reason, we should assume that the logical flaws would be affecting our thinking more widely, in ways that we would probably be unable to think logically about.

I have described a refutation of atheism. If atheism is true, then things like whole numbers, and things like these words, all sorts of things are contradictory. We cannot do science logically if we do not allow ourselves such things, or if we do not allow ourselves to think logically about them. Scientists use mathematics to move from facts to corroborated explanations. They cannot just make that mathematics up in order to get a desired explanation. Even if they do only respect the empirical evidence, and not such abstract patterns as numbers, they cannot just make the numbers up to get a desired explanation. It is precisely because the empirical evidence should be respected, that scientists should do something logical with it, that they cannot just make the mathematics up to avoid an undesirable result. And it really would not help the atheists to redefine "logic" too, not even if they did it in a way that looked scientific, and not even if that would be how computers would best be programmed.

It is true that the evolution by natural selection of primates such as ourselves might not have left us with the ability to do science in a perfectly logical way. But that is no reason not to try to do it perfectly logically. Why draw the line at a logical proof that God exists? Why not draw it at the empirical evidence that we do not survive death? Why not just believe that we do survive death, and that the logic that takes us from the empirical evidence to the contrary belief is not to be believed? Because that would obviously be wishful thinking, of course. So, when atheists draw the line at Cantor's paradox, that certainly looks like wishful thinking.

Atheist scientists think that they can draw the line at a logical proof that God exists, just as they do with arguments that we survive death, because they think that they know that God does not exist, just as they know that we do not survive death. They think that there is no empirical evidence that God exists, just as there is no empirical evidence for ghosts. However, this logical proof that there is probably a God of a certain kind is based on the empirical evidence that there are things, and on the empirical evidence that there is no solution to the puzzle of chapter 3 that is both logical and atheist. And it follows that it is more likely that we do survive death.

Will atheist scientists be happy to have their atheism as a cultural belief that is above and beyond the scientific method? Consider the scientific evidence for our evolution by natural selection. That evidence consists of observations of fossils, and the surrounding rocks – and also observations of current life, under various magnifications – all tied together with metaphysics, logic and mathematics. Now consider a scientist from a subculture whose foundation is the Bible. Her cultural beliefs mean that she is inclined towards creationism. But she is a scientist, and will change her beliefs in the light of sufficient evidence. However, she does find the idea of molecules randomly arranging themselves into living creatures, who then evolve into humans, absurd. She does not see that as a possible theory. Consequently she rejects all that evidence for it, on the grounds that there must be something wrong with it. Rather than question the scientists who made the observations, she questions the metaphysics, logic and mathematics.

There are various ways in which she could do that scientifically – ways that would correctly be called "scientific" if our atheist scientists were. She could simply replace logic with a symbolic logic in which the absurdity of our evolution is used to prove certain axioms. There would then be no evidence for evolution by natural selection – no more than there is any evidence for a God who changes, in the world as described with standard mathematics. Alternatively, she could do something more like our own replacement of natural collections with axiomatic sets, say by replacing the natural relation of being tied together in our thoughts with an axiomatic ribbon theory; or whatever she felt was most natural.

Such creationists could go on to develop their own biotech, which might be preferred to our atheist biotech by large slices of a multicultural market. Since biotech is all about unnatural selection, they should be able to do it just as well as atheist scientists. Those two groups of scientists would argue about who was scientific, but they would really be arguing, badly, over which subculture was correct. Both sorts of scientists would be likely to prioritise other cultural beliefs, over actually scientific conclusions, as they pursued their own version of biotech. If they got a problem doing it, they could simply paper over that problem, with more formalisations of previously unproblematic concepts – like Ptolemaic astronomers discovering yet another need for more epicycles. But as they built up their industrial power, they would just be making some problem that they could not paper over bigger and bigger. (Imagine if Ptolemaic astronomers had not lost to Copernicus, and had carried on adding epicycles, until they ended up in charge of navigation for the first heroic shot at the moon.)

Of course, we only have one of those sorts of scientists. But that does not make them any more scientific. Cultural distinctions within our predominantly atheist science will have more of an effect on our scientific understanding of the world than they should have, as a consequence. Even if we did have two sorts of science – and there may well be different sorts of science in different countries (some of them more classically theist than atheist) – there would not be a lot of short-term competition between them. But within each sort of science, there would be a lot of short-term competition, between scientists. There would politically left-leaning and politically right-leaning versions of that sort; there would be more libertarian and more authoritarian versions of each of those versions; and so on and so forth. This could affect metaphysics – through the importance of intuitions about free will, for example – and hence it could affect logic, and mathematics. Ultimately the deciding factors, on what was true, might be money or family. Once you allow atheism to do what we have allowed it to do, it is a slippery slope down to a very tangled web.

The possibilities are complex; so, let’s briefly glance at one simple fiction. A Catholic statue speaks, and says that you should have faith in Jesus. Catholic scientists investigate, and find it to be a miracle. Most people would suppose that non-scientific factors must have influenced that finding. They will react by thinking less of those scientists’ scientific integrity. But those scientists could say that they were just using a very logical kind of physical science. They could say that it is physics that has, in effect, prioritised a cultural belief over logic. And they could certainly use the opaque languages of modern physics and logic to make it impossible for anyone to prove them wrong. But that does not bother me. What bothers me is the possibility that physics is pursuing epicycles, instead of the truth.

You may think that by undermining the scientific status of our predominantly atheist scientists, I am helping those who have flown in the face of a scientific consensus by—for example—denying that climate change is largely man-made. But scientists involved in climate research will probably be as surprised as anyone, by how atheism trumped logic in the foundations of the language of their science. Like most scientists, technicians and engineers, they will have skipped the zeroth chapters of their mathematics textbooks. They would always have been applying actual mathematics, when they did their research. (Standard mathematics is a perfectly good model of actual mathematics.) Like everyone else who takes a rational interest in the world around them, and who is currently an atheist, such scientists will have to choose between atheism and logic. But even if they choose atheism, that will not necessarily mean that they are not excellent research scientists. Research scientists have all sorts of personal, political and religious beliefs, for all sorts of reasons.

The reason why atheism must now be regarded as unscientific is that no other logical way of avoiding such inconsistencies as that derived logically in chapter 4 has been found. Scientific atheists will think it logically possible that primates such as ourselves evolved to be incapable of understanding such unimaginably huge collections as those of chapter 4. But while it is of course logically possible for beings like us to be systematically irrational – in a variety of ways – if it was the case that those beings were ourselves, then that would not correctly be called by us "logically possible" because were we illogical, we would not give "logic" the meaning we would give it were we – as I assume we are – logical. And we do have to assume that we are logical, in order to be logical. We do not have to assume that we are infallible, of course; indeed, we should assume that we are fallible. But that is of little help to the scientific atheist after over a century of the cleverest academics and scientists searching for flaws in this sort of proof. The best that they have come up with are the symbolic logics that are mathematical, not logical, as previously noted.

The only logically possible cause of such paradoxical contradictions as that of chapter 4 is that we had incorrectly assumed that such logically possible selections – and similarly, the associated logical possibilities of that many things (or in other words, the existence of such whole numbers) – were timeless. That there is no other logical possibility is shown by how mathematicians and philosophers of mathematics have not found it, in over a hundred years of looking for it. And that it is indeed a logical possibility follows from it being logically possible for such logically possible selections (and associated whole numbers) to be differentiated by a creator, of the world out of nothing, who is able to change.

Cantor's paradox amounted, in such a way, to extraordinary evidence for an extraordinary claim. In order to see more clearly the logic of that, it may help to imagine what it would be like were the claim not so extraordinary. Without believing in such a creator, we can still imagine a perfectly ordinary culture whose God was such a creator. Like us, they would have assumed that numbers were timeless; so, they would naturally have seen Cantor's paradox as a simple proof that the most unimaginably huge whole numbers were not, after all, timeless. It would have been much like how the proof that the square root of two is not a rational length was seen by mathematicians who had assumed that all lengths were rational. Had our ancient mathematicians believed more strongly that all lengths were rational – perhaps because that belief was an integral part of their rational world view – then the school-mathematical proof of chapter 5 would have seemed as dangerously paradoxical as Cantor's paradox must seem to many of us. What makes the claim that God exists extraordinary is our culture.

Naturally, that culture has caused an extraordinary amount of evidence to be associated with Cantor's paradox, in the form of modern mathematics and philosophy of mathematics (this book would be far too long were I to do justice to all of that here). There are many reasonable doubts you might have about this logical proof that there is a God. You might have doubts that arise from your culture. Although many religious believers believe in a God who is less impersonal than an unchanging infinity, a lot of theological experts prefer that unchanging infinity. Or your culture might be an illogically atheist culture, into which you have to fit as best you can because it is your culture. Furthermore, the proof that I have been describing could – for all you know for sure – contain crucial errors. It could in time be trumped by an even more logical argument that there cannot be a God, for all I know for sure (although I have thought about this question a lot). So, this proof is not good enough to change your mind if you do not want it to. But it is a very good reason why scientists should keep more of an open mind about such metaphysical questions. It is certainly logical enough to counterbalance all conceivable evidence that there is not a God, in any logical assessment of the evidence.

A scientist can be an atheist, but it will have to be because of reasons other than a logical look at enough of the scientific evidence. Still, it is certainly not unreasonable to consider ourselves to be, as a species, illogical. The evidence is certainly there, in our history, and in the current state of the world. But scientists do have to pursue the truth logically, by amassing evidence systematically. That is just what "science" means. That pursuit cannot include redefining "truth", "logic" and/or "evidence" to suit the current cultural beliefs of scientists.

Where would such redefinitions take us? Where would they have taken us? The atheist worldview of modern science is one in which such redefinitions do make a lot of sense. But in such a world, what is the point of science? Is it a way of getting better and cheaper goods into shops that are easier for us to shop in? But what does "better" mean, in such a world? Science done a lot because it has been pursuing the truth logically. If science thinks that it can grow out of the pursuit of truth, then those goods might only be sold as "better", in a world that is really all about sales. The irony is the classical question of what such profits could actually buy the sellers, in such a world.

Imagine a future where the brightest students study convoluted and complicated symbolic logics within a worldview that shows them that they are imperfectly evolved primates, and another future where the brightest students learn how to think logically about complex problems. The difference between those two futures is not just that the former students will be better at formulating and defending “truths” but not so good at pursuing the truth. The former will be “bright” rather than bright. They will have been selected for having few problems pursing “truths” rather than the truth, fewer of the inspired insights that moved us out of the ancient world, and then the medieval world, and which we might now need. The two groups will tend to do different jobs, in the increasingly different economies and cultures of those two futures. Society is a package deal. In the former future, problems of all sorts get papered over, efficiently and effectively. A plausible story would be told of how the latter future was unrealistic; but basically, you can either have a lot of problems that you cannot think properly about—but be able to address what seems to be happening in a way that looks good—or else you can be able to think about how you would have had those problems, by choosing to do that instead of something more “useful” (like merchant banking or corporate law). It is the difference between scholastic, and academic; or war and peace.

The people who had the biggest problem with the Copernican revolution were not those who thought that the earth was flat. Ordinary people tended to be interested in exotic stories about the new world. Nor were they the astronomers who were proficient with epicycles. They could learn the new calculus easily enough. The people who lost most in the Copernican revolution were those who had become most committed, in strongly societal ways, to the old metaphysics and its timeless “truths”. Similarly, it is not so much ordinary people and academic scientists who will have big problems with this proof—that there is a God who is able to change—which does seem to imply that the physics of the future is likely to be theocentric.

Logic shows there to be a God; and of course, if there is indeed a God, then nothing is actually lost by preferring logic to symbolic logics. And note that even were there not really a God, we would be bound to make an illogical choice anyway, because there would still be this logical proof, which would then be a proof that there is something that there is not, a proof that logic is flawed. We might believe that we could isolate the illogic at infinity, but why would that not be an illogical choice? We might think that it was not an illogical choice because it made the problem go away. But what sort of pursuit of truth is that? It is the pursuit of truth at the end of an empire.

I have been introducing a choice – logic or atheism – but for most of us there is only the unanswerable question of whether my argument did contain a subtle mistake. We can always believe that there must be one somewhere. We can do that with any argument that we do not like. But that would be unprincipled. We would just be believing whatever we liked. If we applied the principle that we should not accept the conclusion of an argument if there might, just possibly, be a mistake in it – if we applied that principle to all arguments, then we would end up with no proofs at all. Whereas proofs are important.

Juries should convict criminals on the basis of the evidence, for example. Voters should vote for politicians who can assess rationally the evidence for, say, man-made climate change. Suppose we have a proof that someone is guilty of a crime, and the jury say that although they accept that proof, they do not see a criminal when they look at the defendant. Or suppose we finally have a proof that there is man-made climate change, and some politicians say that while they can now see that there is such a proof, there is no man-made climate change. That is what it would be like, were scientists to say that although there is a proof that there is a God, there is no God. So of course they would be very reluctant to say anything like that. Scientists are very logical, very realistic, and very honest people, as a rule. That is why when they do say that chapter 4 does not mean anything because we are only highly evolved primates, we can be sure that there are no mistakes in chapter 4, or in other versions of Cantor's paradox. Scientists would much rather point to an actual mistake in the reasoning of such paradoxes, if that was at all possible. Since they are very clever, very industrious people, as a rule, they would have found an actual mistake, if there was one to be found.

Given that there is no such mistake in chapter 4, the only question is whether there are any better explanations of the contradiction of chapter 4. The standard scientific explanation is out of the running at the start. Standard set theory is simply a way of avoiding the contradiction in the mathematical sciences. Another common response from scientists is the denial of the power of non-empirical proof, based on our being primates. And yet they do respect empirical proof; and what is that, but a thinking rationally, about some evidence? The argument from our being primates is an argument against rational argument, so not only is it circular, it would affect a lot of the empirical proofs in modern science, just as much as it would this proof that God exists, were it not too circular. It would also severely limit our legitimate political activity.

The fact that such responses are so common is some evidence that there is no better explanation, than that there is a God. You might not like that explanation, because you feel sure that there is no God. You might, for example, associate 'God' with religions that you would not wish to be associated with. But a jury, faced with a clear proof of guilt, and returning a 'not guilty' verdict on the basis of how charming the criminal has appeared to them – perhaps pointing out, in their own defence, that they are only human – is not a good jury. Of course, anyone might be unsure about the absence of important mistakes in an argument (especially in light of Curry's paradox). But we have to try, nonetheless. We do expect our juries to have common sense. And our logicians should be able to tell us where the mistake in the informal version of Curry's paradox is. But our logicians say that the mistake was to be informal. They will say that the best explanation of the contradiction in chapter 4 is that chapter 4 was too informal. There is no contradiction in modern mathematics, they will point out.

The logical argument in chapter 4 takes us from an assumption about things – basically, that their higher arithmetic is timeless – to a contradiction. That assumption was thereby shown to be false. We made that assumption in the first place because its contrary is so strange that it would stand in need of some explanation. Having found the explanation – there is a God who is able to change – we can see that we should never have made that assumption in the first place. We should instead have assumed that such a God was possible. But we did not. The original assumption in academia was that if not classical theism, then atheism was the case. It is a bit like how we naturally assume that space is Euclidean. When we find out that spacetime is non-Euclidean, it shows that we really were right to have added non-Euclidean axioms to geometry. Logically, we should never have assumed only Euclidean geometry, although it was of course only natural that we did so assume, to begin with.

So, we have a proof that there is a God. Plenty of people do not like that, of course, and so some scientists and academics – whose opinion naturally matters – suggest that logic cannot prove so much, not the logic of primates, like ourselves. But we primates will of course follow our own logic, as best we can, and so we will either believe in God, or else we will be illogical. Atheists can say that we should of course follow our own logic, but point out that nowadays our logic is symbolic logic. However, that is only because of a successful attempt – following Russell – to hide the proof of chapters 4, 5 and 6 underneath a lot of superficially similar logical puzzles. Such atheists might say that it is actually nothing to do with those puzzles, but rather such mathematical terms as 'set'. But most modern mathematicians do defer to logicians when it comes to the ins and outs of such formal languages as axiomatic set theory. And logicians have connected the set theoretic paradoxes and the semantic paradoxes, in a variety of ways.

Cantor's paradoxical contradiction does seem to arise from our taking every collection of things to be yet another thing: that collection. And it does seem to be false, that many things are one thing. Many things are many things. A collection of things is just those things, being referred to collectively; but in the absence of such reference, there are just those things. There is not really that other extra thing. We cannot even assume that there is that extra thing without getting a contradiction. The contradiction proves – such atheists could add – what we should have already known: that many things are not another extra thing.

We do often want to refer to some number of things collectively, though. And in science we want to do that properly. And so we have axiomatic set theories – these atheists conclude – which allow us to make such references in a consistent way. We can have any axioms we want – we want something that is like a collection, but not productive of contradictions – because it is a fiction that there is this other thing. The problem with that story is that the paradoxical contradiction of chapter 4 actually arose from our taking a subcollection and a different subcollection to be two different subcollections. We were not adding a new thing; we were having different ways of having fewer things.

There is not really anything complicated about collections. A collection of things is just those things, being referred to collectively (as we just did with that 'those things'). Objectively, there are just those things. We do not really need axioms for collections, no more than we need them for things per se. Now, logicians can try to describe, as precisely as they like, what we mean by 'thing' or 'object' and so forth. They might create a range of axiomatic theories of 'thing', based on those descriptions (axioms are simple descriptions). But we begin by knowing what is meant by 'thing'. We do not need an axiomatic theory in order to know that the following sentence has four words in it. These are four words.

Suppose that biologists defined 'individual' so that we became individual human beings in the tenth week after conception. The motivation behind such a definition would clearly be political. But definitions are a normal part of biology. For example, biologists have to define species. And biology is an empirical science. Having replaced logic with a choice of empirically adequate symbolic logics, why should biologists not create such a definition? It might be what their employers wanted from them. You might have lots of problems with such a definition, or you might not. But the fundamental problem with it – the logical problem – is that that is simply not what 'individual' really means.

When we go through all the ways in which the contradiction of chapter 4 can be avoided, we find nothing very plausible. All the alternatives to there being a changing God are paradoxical. The only way in which they are better than just keeping the contradiction – accepting true contradictions as part of the fabric of reality – is that they are less obviously absurd.

What makes some scientific sense is talk of our human logic being unable to cope – due to our being highly evolved primates – with very large infinities. But while the contradiction appeared at the end of chapter 4, it followed logically from what went before. What we were unable to think logically about, without getting that contradiction, were the smaller infinities, and what came before them – things and possibilities. But if scientists took seriously the idea that we humans cannot rely on our most logical thoughts about infinity, then standard mathematics would be constructive mathematics. And why draw the line at infinity? Why would we expect even highly evolved primates to be able to think perfectly well about such complicated mathematics as that of the evidence for, say, man-made climate change?

Imagine someone who does not believe in man-made climate change, who says that what we actually have – with all this talk of evidence for it – is further evidence against logic. This person – say, Mr E – says that it shows that the well known human problem with thinking logically about infinity can also manifest more generally, with very complicated mathematics. There is no way of dealing properly with all that evidence that does not involve a lot of complicated mathematics. And for primates, there is likely to be no difference between very big finite numbers and infinity. Even saying that there is really a difference involves talking about something that we certainly do have paradoxes thinking about, Mr E observes.

My immediate reaction to such remarks would be to point out the obvious absurdity of rejecting mathematics itself just because of a dislike of the result of a calculation. Of course, Mr E would take offence at my use of the word "dislike" and point out that it is not a matter of liking or disliking the conclusion, it is a matter of the conclusion being absurd. Mr E really does find the very idea absurd. He will describe the awesomeness of nature, and recommend that I get out more, into the countryside and beyond, into the wild places, and see for myself just how vast they are. Mr E recommends that I climb some mountains, which I politely decline to do, at which point Mr E shrugs, as if to say "there you go then, you have lost the argument." At that point it all depends upon who Mr E was, and what my relationship to him was. Have I failed to get a job? Has he?

I could try to point out that the idea of man-made climate change is far from absurd, but of course, all my arguments would come from the scientific literature, and would involve a lot of complicated mathematics in their justifications. You might try to point out that Cantor's paradox is a well-known paradox of the very large infinite, which throws no doubt over ordinary mathematics. But Mr E could say that that is just a matter of not liking such a shadow. It is precisely the way in which ordinary mathematics leads logically to the very large infinite, Mr E could add, which makes the large infinities paradoxical. And such paradoxes as Russell's do intimately link the paradoxes of the very large with other logical paradoxes.

What do scientists think of redefining "truth"? If they take us to have discovered that truth is more complicated than we thought (much as we discovered that space is more complicated than we thought), then they will presumably think that they should aim for a redefined "truth" when doing their most important thinking (much as physicists assume relativistic equations when doing high energy physics). But such a redefinition of "truth" will have been produced by people who chose atheism over logic. That is because there is this logical resolution of Cantor's paradox, and because there are even simpler solutions of the other apposite paradoxes, as you will see in the next chapter. Following Cantor, various logical puzzles were taken to be clues as to the true nature of logic and truth. But that word “true” must have its natural meaning there, to avoid vacuous circularity. And surely logic—in the natural sense—has to be assumed in order to work out what clues mean. And since logic and truth are prerequisites, doubts about them are akin to Cartesian doubts.

If we had evidence that our logic was flawed, it would be different. But it would have to be very strong evidence (a glitch in the Matrix is just evidence that we have been overworking, even if it is a glitch in the Matrix; an alien artefact is just evidence that people make strange things and sometimes leave them lying around, even if it is an alien artefact; et cetera). Looking at this question from within a predominantly atheist culture, there can seem to be such evidence, but there is actually no such evidence logically. There would only have been a choice of atheism over logic, if my proof is indeed logical. And that could easily be very bad for the public perception of science. When a scientist tells the world about some discovery, and it is inconvenient to some, why should they not regard it as a “discovery” that is really the product of those scientists choosing some cultural or political assumption of their own? Such things happen too often already. Scientists will not want to make matters worse. Consequently, they will not think much of redefining “truth”.

The most familiar numbers are the ones we use for counting: 1, 2, 3 and so on. But there are also numbers used for measuring – such as ½ – and negative numbers, and complex numbers – which generalise positive and negative into a whole plane of directions – and plenty of more complicated things that are also called “numbers” by mathematicians. And for thousands of years, mathematicians have also worked with geometrical shapes, and with more abstract patterns such as algebra. But it is the counting numbers that will mostly concern us.

Collections and counting numbers are clearly closely related, and so a problem in the logic of arithmetic would, eventually, have pointed towards the concept of a collection, once every other conceivable possibility had been exhausted. That concept would not have been the first place to look for a solution, though, because there is almost nothing to it. It is such a simple concept – a collection of some things being just those things being referred to collectively (as we just did, with that “those things”) – there is obviously nothing in it that could have been the cause of the problem. Still, that is why the axiomatisation of that concept could, relatively easily, facilitate the avoidance of the problem. The axioms of standard set theory are self-evidently true of collections, so the bulk of mathematics is unaffected by the standard axiomatisation; and the final paradoxical contradiction is simply used to exclude that collection from standard set theory.

It would only be natural to assume that mathematicians have been discovering what numbers really are. But this sentence has six words. There is nothing more to the meaning of “six” for mathematicians to discover, than that it is the number of things in such collections of six things. The standard mathematical definition of the counting number 6 gives the name “6” to something that exists in standard set theory – exists in a sense, much like the sense in which Sherlock Holmes exists. That entity has many structural properties in common with the actual number 6. Similarly, Sherlock Holmes is defined, in those stories, with descriptions that could also apply to real people. And that entity has some properties that simply enable standard sets to avoid Cantor’s paradox. You could say that Cantor’s paradox shows that the actual number 6 has those properties, although that would be quite a controversial claim. There are a lot of non-standard set theories, and several of them have been preferred, by serious mathematicians, to the standard theory. They all avoid Cantor’s paradox, in one way or another. And of course, the paradox can also be avoided without invoking any set theories at all, as I am doing in this book.

The idea that standard mathematicians have been discovering what numbers really are is therefore controversial, to say the least. And such controversies, in the philosophy of mathematics, do not affect the main argument of this book. The aforementioned controversies are of some small tangential interest, because of the role that Russell and other philosophers played in the standard reaction to Cantor’s paradox, which naturally influenced the development of the logicians’ “logics”. But our primary concern here is the logic of collections of things. It does not really matter to us here what “6” is a name for. If six is pretty much what we all thought it was when we learnt English – naturally enough, I have been calling that “the actual number 6” – then this book is about arithmetic. But even if six is really something else – such as the standard set that is so-named (which it also very naturally looks like it is, because mathematics is what mathematicians do) – then that would only mean that here we are not really interested in mathematics, only in logic. Note that even then, it would still be legitimate to use such counting numbers to talk about such aspects of the logic of collections of things as equinumerosity. We are simply using them as names for those aspects – a perfectly natural use for them.

But the idea that collections of things are really so complex, we should think of them as standard sets, is essentially the same as the idea that we could be wrong about anything, and everything. It is only then that it makes any sense at all. (A puzzle that we have yet to solve is of course not a good enough reason to have such an idea.) And what could we do about such Cartesian doubts? If we are not so wrong, then we need do nothing; but what could we do were we so wrong? We could hardly think very logically about that idea. To see this, suppose that we are indeed no more than highly evolved primates. Then we could only do what such animals could do. We could only assume our natural logic – at best (we could always fail to be that logical). How could we do anything else? Some of us might think that they were doing something else – something better – but they would have to be fooling themselves.

In this book I am arguing that if we accept logic – deductive and explanatory – then we should reject atheism; and I am also arguing, because of the logical strength of that first argument, that academic scientists should accept logic. The argument of chapters 4, 5 and 6 is the most straightforward part of that argument, with the argument of chapter 4 being the most straightforward, being essentially deductive. It shows that what we already had, in our background assumptions, was contradictory.

Were the whole of the argument of chapters 4, 5 and 6 deductive, I should have had to assume the untruth of atheism in my philosophy of mathematics. You can only get out what you put in, with deductive logic. That is how it can so reliably take truths only ever to truths. But the argument of chapters 4, 5 and 6 does not assume any philosophy of mathematics, or any other complicated metaphysics. It assumes only the most basic facts about explanatory logic. It is the defenders of atheism who have had to introduce complicated intellectual convolutions, in order to make atheism appear more logically possible than it really is.

My argument does not need any of those complexities. I have to address some of them in broad terms, because they are there in the standard approaches to such basic entities as numbers. But to address directly all that has been constructed in the twentieth century, in defence of atheism, would be to bury my own argument. The atheism of most academic scientists has changed little since the days when it was a reasonable rejection of some old-fashioned religious metaphysics. So most of them will think, in one way or another, of the number 4 as being something like 1 + 1 + 1 + 1.

The most relevant scientific terminology—set theory—appears to have been designed to obscure my main argument. I must therefore use ordinary language to describe that argument. An atheist would therefore find it easy to say that a more scientific approach would show that that argument is not really much of a problem for atheism. A lot of this book is therefore about why that atheist would be wrong. But the logic of my main argument is relatively straightforward. And it began simply enough. We began with a lot of scientists saying that there is probably not a God, and it is precisely because they say that—and not that God's existence is impossible—that we can safely assume that there might, just possibly, be a God.

Were God's existence impossible, more scientists would presumably say that it was. Consider how few scientists would say that 2 + 2 is probably not equal to 5. Most would simply say that it is impossible for 2 + 2 to be 5. I say "few scientists" instead of "no scientists" because of the standard mathematical definition of the counting number 2. There is no proof that standard set theory is consistent. Mathematicians are generally confident that it is consistent, but mathematics is all about proof. In the absence of a proof, 2 + 2 could be part of an inconsistent whole. It might equal 5 as well as 4. But of course, there can of course be no reasonable doubt that 2 + 2 equals 4 and not something else. The standard definitions are—as definitions (not just models)—flying in the face of logic. How could it have been thought logical, to define something as basic as the counting numbers in terms of something much more obscure, at the cost of a loss of certainty? Mathematicians are logical, so there must have been some good reason for the standard redefinition of the counting numbers. The standard definitions enabled mathematicians to isolate a problem—Cantor's paradox—that they were unable to solve. They did that by creating a mathematical model of the counting numbers. To regard the standard definitions as new improved definitions of our actual counting numbers is to assume that there is no logical solution to Cantor's paradox.

We should begin by assuming that there is a logical solution to that problem. And chapter 4 described one. But those scientists who are sure that there is no God will regard that as no solution. And they will take my arguments for there being a proof that there is (very probably) a God to be, at best, arguments for there being no logical solution. Many of them will say that the concept of God is illogical—contradictory (a contradiction describes an impossibility) or nonsensical (nonsense describes nothing). But such arguments fail for the God of chapter 4. So, we are left with our original assumption that God is logically possible. There might, just possibly, be a God, of some consistent sort or other. So, if it could be shown that atheism comes with a contradiction that some consistent theisms do not come with, then atheism would have been refuted. Most people would accept the logic of that. But of course, atheists would examine such a refutation very closely, looking for anything resembling a weakness. And ordinarily, when something is shown to be the cause of something else, we will not have ruled out all the other possibilities. We will just have shown that all the obvious alternatives could not nearly so easily have been the cause. When the alternatives are not that unlikely, we might talk about probability, rather than proof. In science, we try to put numbers to those probabilities. But even when we do talk of proof, we do not rule out all the other logical possibilities.

Consider the hypothesis that all swans are white. That hypothesis might be used in logical deductions: all swans are white, and this bird is not white, so it is not a swan. If it was a dirty swan, then that would be a bad argument. You would then have been giving two senses to the word "white". Now, suppose that you find a black swan. That refutes our hypothesis, presumably. However, did we think to rule out the possibility that it was a white swan that had been dyed black? Perhaps, but even then, it remains the case that our hypothesis could be used, perfectly logically, to show that our strange bird was not a swan – that "swan" was not the right word for it. In effect, we could take "all swans are white" to be part of the definition of "swan". That is the kind of thing that mathematicians have been doing, with their axiomatic redefinitions.

And furthermore, we will not have ruled out a whole range of sceptical scenarios (such as give rise to Cartesian doubts). Perhaps we simply imagined the black swan, for example. Or perhaps we were hypnotised by powerful aliens. Or by demons with power over us caused by our own failings, such as our vanity. And of course, it is logically possible that the black swan was sent by God to drive us away from logic, toward theology. The possibility of God is a little like a sceptical scenario. Now, the problem with sceptical scenarios is that while they do not seem to be impossible, they are designed so that we cannot know anything about their likelihoods, even though we naturally treat them as though they were collectively highly improbable. It is a puzzle, even nowadays. But it is tangential to the concerns of this book. Still, the possibility of God is a little like a sceptical scenario.

Most people would think that if there was such a contradiction, as chapter 3, then scientific atheists would have to reject atheism. Science is all about thinking logically, after all. It is precisely because scientific atheists would, then, have to reject atheism, that scientific atheists begin by being confident that there is no such a contradiction. Had science already led them to believe that they were not really very logical, they would not be having such serious doubts about the existence of such a logical refutation.

So, when it becomes clear that there is such a contradiction, not only will that be surprising, it should be even more surprising if a lot of scientific atheists then decide to reject the logic, instead of their atheism. They will have left it too late for such a discovery, about human logic, to be very plausible. You are probably well aware of how scientific atheists usually want people to take a logical look at the evidence, and then let logic dictate what the evidence is evidence for. In this century, scientific atheists have made a great show of reasons why there is probably no God.

And in chapter 4 you will see how there is such a contradiction. It is not just atheism that is in that way contradictory – some theisms make the same inconsistent assumptions. So the contradiction is almost a logical consequence of nothing – almost – and if that had been what the contradiction was, then rejecting some logic would have made a kind of sense (although rejecting logic could never have made much sense). But as you will see in chapter 4, there is a logically possible God whose existence would mean that there was no such contradiction. So the contradiction is not actually a consequence of nothing. It is a consequence of assuming that there is no such God. The contradiction is a reason to reject atheism – and also some theisms. But it is not necessarily much of a reason to reject those theisms, because theological thinking is quite different to logical thinking. Most religious believers take their God to be above and beyond logic. Religions are all about faith, hope and charity – whereas atheism is strongly associated with logic, science and reason.

We begin with a lot of evidence that we evolved by natural selection. Such selections very probably shaped everything about us, including our common sense. Those who were predisposed to thinking less logically about the world around them tended to be less successful at reproducing their predispositions. However, that process might, just possibly, have left us with some marginal flaws, in even our best reasoning, about more remote matters. A scientific atheist faced with a proof that there is a God could therefore regard it as evidence of such a flaw. However, the more logical and elementary the proof, the more that such a flaw would undermine other logical arguments. And the argument of chapter 4 is very logical, and is as elementary as the logic upon which arithmetic is based.

Can you imagine such a proof appearing within the mathematical sciences, just as science was in the process of replacing religion at the top of academia? Atheists would have seen an obvious threat to the scientific pursuit of truth. Leading scientists might have defended that pursuit by engineering a new definition of the word “proof” as it appears in scientific work: the problematic proof would no longer have been a proof in the new scientific sense, while other proofs remained proofs. Would that have solved the problem described at the end of the previous paragraph? Well, the problem could have been evaded in that way. It would have been buried, along with the problematic proof, under the new terminology.

Something similar does seem to have happened, because I reverse engineered chapter 4 from a part of modern mathematics that had originally been a mathematical paradox. Cantor had been looking at infinity, as part of his work on the calculus, when he derived a contradiction, apparently logically. He took his contradiction to be symbolic of the mysterious ineffability of the God who had, he presumed, inspired him. And quite generally, religious believers need not feel threatened by his contradiction. However, his less religious peers may well have regarded it as some evidence of a flaw in human logic, probably due to our having evolved by natural selection. Darwin’s theories were the talk of university towns at the time. And there would have seemed to be supporting evidence – for

Cantor’s paradox being a flaw in human logic – in the form of other logical paradoxes, some of which had puzzled logicians for centuries, as well as some new ones. Those other paradoxes are still cited, quite frequently, by academics concerned with the definitions of such words as ‘proof’, ‘logic’ and ‘truth’. But you will see in chapter 7 that they have – and have always had – resolutions that are not too odd. Unless you were looking for evidence of such a flaw, you would not have thought of them as significantly paradoxical.

Most people assume that science embraces logic. And it does. But scientists are only human; they naturally have cultural beliefs. For scientists who are atheists – which is a lot of them – atheism is part of doing science properly. They may well not think about how they regard thinking about such metaphysical questions as a bit unscientific. And regarding numbers in particular, they will naturally defer professionally to mathematicians, who have redefined the word ‘number’ in such a way as to hide the real choice between atheism and logic. And less explicitly, they will also defer to logicians – when it comes to logic – who, in related ways, have redefined ‘logic’. When they do think about it, atheists may well think of logic as being like spatial awareness – a natural way of structuring information – and as being just as capable of being scientifically improved. They may liken axiomatic logics to Einstein’s theories of relativity. They are superficially very similar – lots of counter-intuitive mathematics – and from an atheist perspective, they are very similar.

Because atheism is effectively a cultural belief of many scientists, a lot of scientists believe that by choosing it over pre-axiomatic logic, they are pursuing the truth scientifically. However, not only is atheism refutable, it has been refuted (as you will see). And it is pre-axiomatic logic that takes truths to truths (in a sense of “truth” that is not similarly redefined). Axiomatic logic is essentially mathematical, and could at best be applied only to a mathematical model of truth.

The English word “logic” can be described in many ways, but those ways are invariably such that logic takes truths to truths. I think that we should assume that thinking logically is not going to cause any inconsistencies in our thoughts. Any inconsistencies that do appear, as a result of our thinking logically about things, are going to have been in our conceptions of those things, or in our background assumptions. They will have been revealed, not caused, by our thinking logically. To see why we should assume that, we need only suppose that the assumption was false. We could not think about that supposition very well, if it was correct.

If we were inherently illogical, we might find some contradiction in our thoughts about reality. Of course, our evolution would never have caused us to believe that 2 + 2 = 5. But suppose that an actual inconsistency was found in the arithmetic of some very large numbers. Mathematicians would check those calculations and, finding no mistake, check them again, and again. They might wonder if their computers had been got at. Having checked everything that they could check, they might assume that the problem was above their pay grade. They might stop using computers, until the problem was solved. But they would probably do something less rational – we are supposing that there is an actual inconsistency, and hence that their thinking will be inconsistent. They might blame a foreign power that they already disliked, for example, taking themselves to have firm evidence of that power's meddling. Or they might change the definition of the counting numbers.

They might wonder – rather rationally – what could have caused an actual inconsistency in what is essentially nothing but a lot of very logical thinking. According to the best current scientific theories, our evolution might have resulted in such an inconsistency. But the evidence for evolution by natural selection would not be as strong as we think it is, were there such an inconsistency: these mathematicians would not be able to think as logically as we assume that we can. And furthermore, there are plenty of other conceivable causes of such an inconsistency. Aliens might have hypnotised them, or demons might have bewitched them, or they might be diseased brains in malfunctioning vats (not unlike the Matrix), and so forth. All of these are well known sceptical hypotheses, much discussed by philosophers (following Descartes). But the worse human logic was, in this scenario, the worse these mathematicians would be able to think about such conceivable causes.

So, while it seems to make sense, to suppose that our natural evolution from more primitive primates might have left us inherently illogical, that supposition is not really a sensible one. It is essentially the same as the supposition that we have just been considering.

There is a well-known situation where we find something that is clearly black being exactly the same as something that is clearly white; it is Edward Adelson's checker shadow illusion (which is online at http://persci.mit.edu/gallery/checkershadow). When we look at Adelson's picture, we never conclude—indeed, we do not even entertain the possibility—that black is sometimes white, and that we must therefore be inherently illogical, presumably because of our evolution. We see how in one part of the picture, the grey rhombus represents a black square, and that in another part—of the same picture—it represents a white one. (You can hardly believe your eyes, when you look at that picture.) This is, of course, no evidence at all against logic; which raises the question: Why were the logical paradoxes taken to be evidence against the consistency of (natural) logic?

Even if we could not find a logical explanation, we could always suppose that we might, if we tried harder. There would never be a good time to decide that we had actually found evidence of our inherent illogicality. We should always assume that there must be some logical explanation. That could only be wrong if we really were inherently illogical, and even then, why should we not assume that we were not? In that case, much of our thinking would be wrong anyway.

We are in a similar situation with Cantor's paradox, where we seem to have a contradiction for extremely large whole numbers. Mathematicians could always have assumed that there must be some logical explanation. And to some extent they did just that: they now use a mathematical model of whole numbers that enables them to sidestep the paradox. They can say that they actually say nothing at all about the paradox, or its logical explanation. It is only when atheists say that we can ignore the proof that there is a God, which is based on Cantor's paradox, on the grounds that that paradox does not arise in modern mathematics, that mathematicians seem to be saying something else.

Even if there was no known logical explanation of Cantor's paradox, we would not have any evidence that we are inherently illogical. We would have to assume that there was some logical explanation, even if we had not found it. Now, our evolution as inherently illogical primates can seem to be a logical explanation. Biology is certainly supposed to be logical. But this only looks like a logical explanation. Evidence that we evolved has to be evidence that logical thought could have arisen naturally. It cannot be evidence that we are not capable of properly logical thought.

You may well think that it can be. Biology paints a scientific picture, of our evolution from primitive primates, which could of course have left us inherently illogical. Indeed, the picture makes that look quite likely. But that possibility is akin to a sceptical scenario: we could have been brainwashed by aliens, we could be dreaming, and so on.

Biology has to assume logic. We have to assume that there is some logical explanation. And when we do have a logical explanation, then it is all the more absurd to think that it is our inherent illogicality that is being revealed by the paradox. The fact that the explanation is one that a lot of scientists do not like explains why this has happened, in the twentieth century. That fact does not make those scientists less unscientific. It would be easier to see this were this not our own culture, of course. So, imagine that the scientists are those astronomers who had trained in the use of epicycles, before Galileo. Faced with an argument that there are not really any epicycles in the sky, they say that there is an argument that shows how the minds of men are, since the Fall, somewhat illogical. They are of course not being scientific. It really does not matter how comfortable they are with their own culture's stories, and how competent they are with their science. What matters is the logic of the argument against the reality of epicycles.

One reason why you might not be sure about that is that the hypothesis of a God is a lot like such sceptical hypotheses as aliens, demons, brains in vats and so forth. If you came upon a burning bush that spoke to you, and which was not consumed by the fire, you might suppose that the fire was a holographic projection. If you knew about such things, and if what you were seeing was far in advance of anything that you had heard of, then you might suppose that it was secret technology. You would not have much evidence for aliens. You would have less evidence for demons. And to suppose that it was God would not be very logical at all. That would be like concluding that you were a brain in a vat, experiencing a virtual reality, only even less logical.

The hypothesis that there is a God can explain anything, and it can do it too easily. To say that God did something with a miracle is like saying that an alien did it with superior technology, or that a demon did it by magic, or that the supercomputer controlling your vat made it look like that to you. It is not much of an explanation. As much as the theoretical possibility of brains in vats threatens our scientific knowledge, that is how much of an explanation it is.

However, a particular God hypothesis – such as that in chapter 4 – can have particular explanatory value, even though the general God hypothesis is strongly associated with religious belief, and is a lot like a sceptical hypothesis. That is because it is not so much the case that God miraculously solves the problem of chapter 4, somehow; rather, it is that there is only one logical way to avoid the contradiction of chapter 4 – basically, there are ever more whole numbers, without limit – and that way happens to require the particular power of a certain sort of God. Aliens, demons and supercomputers might be able to make something like that burning bush, but not even they could make ever more whole numbers.

If you saw such a burning bush, you would have little evidence for aliens; but if you saw sunlight reflecting off their spaceships as they arrived, then you would have quite a bit of evidence, either that you were hallucinating, or that there were aliens. Talking to the people around you might convince you that you were not hallucinating. It would not then matter that aliens are often associated with conspiracy theories. And of course, their arrival would not prove that they had been abducting anyone before then. Similarly, there being a logical proof that there is probably a God does not of itself say much about any religious beliefs.

And the clarity with which the earth is perceived to be far from heavenly does not logically say anything about the God hypothesis. Logically, it follows only that either there is no God, or our perceptions of such things are not very good, or we have heavenly lives after our deaths (so that these lives are more like time spent in a character-building school), and so forth.

There is a similar clarity to the earth appearing to be flat. But the earth being spherical (or an oblate spheroid) had been known since ancient times, at least to philosophers. Ancient Greek mathematicians had even calculated (not too inaccurately) the diameter of the earth. And there is a similar clarity to ourselves not seeming to be biochemical robots, of course. So, the major metaphysical views of academic atheists—such as naturalism, physicalism, materialism and so forth (all of which are what is ordinarily meant by materialism)—are not a lot less mysterious than the hypothesis that there is a God.

It is of course nice to ignore such philosophical speculations and concentrate on hard science, or its useful application, and equally nice to approach God by way of religious belief, especially if one was inherited. But this book is about a logical proof, and so it all rests upon the reasonableness of the possible hypotheses. It is important to be as objective as possible. It is not so much about what seems reasonable – which will naturally be whatever you are already assuming in the background of your worldview – as it is about what stands up to a very well-balanced scrutiny.

I will not be assuming much about the metaphysics of a logical theism. What little I end up with will be whatever the proof shows to be necessary. As for the metaphysics of atheism, that has always tended to be materialistic because of well-known difficulties hypothesising anything else in the absence of a God. Were there a God, the laws of nature would be laid down in accordance with the purpose of creation. There would be no particular difficulty in having spiritual beings inhabiting physical bodies. The laws governing the interactions of the two types of substance would be laid down in accordance with the purpose of such a creation.

But in the absence of a God, the ubiquity (omnipresence) of the physical in the observable world makes it very difficult to explain how there could be a separate substance that does all the feeling and has all the moral responsibility. There is no need for all those difficulties to be gone over in any detail here. If you are interested, look up "Cartesian dualism". Balance is important, but here I am simply countering a common prejudice against any God hypothesis: the feeling that such things are just too odd. God is strongly associated with religious belief and the supernatural. But common sense says that a future medical science might be able to alter the human brain in various ways, while keeping the subject alive. If your brain was changed gradually into the brain of an elephant, common sense says that you would still be you, but you would be an elephant. Reincarnation is not too far from common sense.

Most adults find it hard to recall much about being a baby, and so we are too far from possible earlier lives for the general lack of any memory of them to be any evidence against their possibility. Indeed, we might wake from these lives much as we wake from dreams. In our dreams, we do not recall who we really are; and then, when we wake up, we generally forget most of the details of our dreams.

What we know about the world we usually know by way of observations. We observe various natural laws (such as those of chemistry). Whereas, what we know of ourselves by direct acquaintance is that we have feelings and make decisions, we care about things, and so forth. It is therefore quite hard for us to think of ourselves as purely physical (or biochemical). In a nutshell, this is the problem of how a robot could have feelings. Molecules are basically very mechanical things, for all that the mechanics is quantum mechanics. How can such mechanical stuff have – in certain arrangements (for all that they are very complicated arrangements) – feelings? Basically, we do not know; and two things follow immediately:

Firstly, if atoms are such that, in some arrangements, they can have feelings – and there certainly are feelings – then what else might atoms be capable of? It is hard to know where to draw a logical line. If feelings, then why not ghosts? Why not magic? There is an absence of evidence for ghosts and magic; and yet, where is the evidence for feelings? For that, we have to rely on testimony. And there are accounts of ghosts, and of magic. Such accounts are not usually very scientific, but all the scientific evidence is for the mechanical nature of the physical world. In short, the obvious metaphysics of atheism seems to be the friend of superstition.

But of course, atheists regard themselves as being opposed to superstition. So, atheists who have thought about this have tended to see a real need to know more about this hypothetical line. And yet, it is very hard to find out anything about it. Now, not knowing more about it does not make atheist metaphysics less likely. All atheists would agree to that. But consequently – and this is my second point – atheists should not suggest that theist mysteries make theism less likely. Atheists will be saying that the God hypothesis of chapter 4 is too vague, that too much of that hypothesis is too mysterious, for this to be a proof that there is probably a God. But they will be wrong to say that.

There is not, of course, any inconsistency in arithmetic. Arithmetic is basically a lot of certainties, such as 2 + 2 = 4. What differences there are, between such small numbers and the unimaginably large numbers that are the bulk of the subject matter of arithmetic, are clearly not such as could possibly result in an inconsistency.

Still, it is not as though there have not been problems with the logic of arithmetic. At the very start of our scientific age, Galileo found his paradox. There are fewer even numbers than counting numbers; but secondly, there are as many even numbers as counting numbers. Both descriptions are true. But that second true description does seem to be contradicted by the first true description. So we seem to have a true contradiction. And while true contradictions are logically impossible – because each description rules the other out, in any pair of contradictory descriptions – that means that we seem to have some evidence against the consistency of logic.

We have already seen several reasons why our logic might not be completely consistent. As well as evolution by natural selection, there are countless sceptical scenarios. In Galileo's day, they might have observed that infinity is an attribute of God, who is above and beyond logic. Did Galileo's paradox provide any evidence for true contradictions? Of course not. True contradictions are logically impossible. What follows logically from the apparent contradiction is that some word or words in the paradox must have had two different meanings there. To see what those meanings are, we need only look at the reasons why each of the true descriptions was true.

Cantor developed the concept of a whole number in a perfectly logical way, clarifying the sense in which there are not fewer but just as many even numbers as counting numbers. I suppose that some people might prefer to reject the idea that there are as many even numbers as counting numbers, on the rather logical grounds that there are clearly fewer of the former. We do get the former by taking away a lot of the latter. And our intuitions about the meaning of "as many" do come from our experiences with finite collections, not infinite collections. Galileo's paradox could be turned into an argument that we should not extent the concept of a whole number to infinite collections. Furthermore, Cantor went on to get his own paradox. According to his own account of that paradox, the concept of a whole number results in actual inconsistencies, when it is freely applied to infinite collections. And it is not just some people who think that a concept that causes inconsistencies should be rejected.

Now, the actual cause of the Cantorian inconsistencies – which were of a kind with the contradiction that I will be constructing in chapter 4 – was not the concept of a whole number being freely applied to all collections of things, but was a more subtle assumption, as you will see. But I do not want the soundness of my construction to depend upon equinumerosity being a perfectly good sense of "as many" when there are infinitely many things. There are very reasonable doubts about that. So, the logic of my argument will be based upon the underlying logical possibilities for pairing up the things in various collections. Still, it does not really matter what we call that. And it may well be easier for you to follow my argument if I use the words "as many", it is such a neat and intuitive words. So, I shall use that words in chapter 4, to refer to the logical possibility of such a pairing.

After all, equinumerosity does seem to be the concept upon which arithmetic is based. To count five apples is to pair them all up with 1, 2, 3, 4 and 5. We originally learnt the meanings of such number words as 'two' by associating such words with that many things of the same kind: two apples, two pears, two chairs. An apple and another apple, a pear and another pear, a chair and another chair – those are all instances of the same pattern: 1 + 1. Counting numbers are most naturally defined to be the products of repeatedly adding 1. There are infinitely many counting numbers. The number of those numbers would be an infinite number, were there such things as infinite numbers. And since the counting numbers and the even numbers are equinumerous, there is such a thing as an infinite whole number.

Cantor had been developing the arithmetic of such infinite numbers – as part of his work on the calculus, as it was being developed in the second half of the nineteenth century – when he found an inconsistency, between two descriptions of the collection of all such numbers. A similar inconsistency will be described in detail in chapter 4.

Since infinite numbers had long been thought to be paradoxical, this was not especially surprising. Cantor had presumably made a mistake, either in his derivation of this contradiction, or else by developing the arithmetic of such infinite numbers in the first place. The problem was that no mistake could be found.

Note that when Galileo's paradox was thought to be a contradiction, rather than two compatible descriptions, people did not think to question the consistency of arithmetic. They assumed that infinity had been shown to be beyond human comprehension. But the paradox was clearly not saying anything about arithmetic. And similarly, Cantor's paradox did not have to threaten the consistency of his infinite numbers. Those numbers – such as the whole number of all the counting numbers – seemed to many mathematicians to have the same right to be called 'numbers' as the counting numbers did. The standard response to Cantor's paradox has therefore been to keep all of those numbers, without having the infinitude of all of them. That might sound strange, and it is. It is strange because to have all of those numbers is to have their collection. A collection of things is just those things, being referred to collectively (by saying "those things" for example). Modern mathematics made this work by introducing a scientific theory of collections. Ironically, this meant replacing arithmetic with something else that was called "arithmetic".

I shall briefly explain how that works – although it is tangential to the argument of chapters 4, 5 and 6 – but let's first remind ourselves what scientific theories are. A scientific theory is a description of reality. It will be a good description, for various purposes. That is why it exists. Within the theories themselves, entities are completely defined by their descriptions. The point-masses of Newtonian mechanics, for example, are the size of a point. They model actual objects quite well, because an object's centre of mass is a point. But in the theory, each object is the size of a point by definition. Those objects exist (so to speak) at points in an imaginary Euclidean space, and slavishly obey Newton's laws of motion. Newtonian mechanics is a good theory of actual objects – it was used to get people to the moon, and back – but actual objects obey quantum mechanics and relativistic mechanics.

Entities defined by a scientific theory exist in the same sort of way as fictional entities. Philosophers of physics may wonder what exactly matter is – and what time is, and so on (and also what theories are) – but physicists just develop better and better descriptions of how matter moves (as and when they need to). Now, it would clearly not have been scientific, for physicists schooled in nineteeth-century physics, to ignore the evidence for quantum mechanics and relativistic mechanics, on the grounds that matter had been defined, in their nineteenth-century school books, to be such that it obeys Newton's laws of motion. That is not how scientific theories work.

Cantor's paradox was originally a problem with whole numbers. But whole numbers are basically properties of collections. So the problem in chapter 4 will be about collections. A collection of things has a number of things in it, much as a coloured object has a colour. Following Cantor various theories of collections were developed. The collections defined by those theories were called 'sets'. The numbers of standard mathematics are defined to be certain sets in a standard set theory, called 'standard set theory'. The contradiction of Cantor's paradox is avoided, within that theory, by there being no standard set of all of those numbers. (Rather neatly, the contradiction is used to prove that there is no such set, as you will see.)

Philosophers have been debating what exactly numbers really are ever since. But all of that is a red herring. We can ignore that huge debate here, as well as all of that set theory. We will only be interested in collections. Only in what they are really doing. And really, there is nothing very complicated about collections. They can be perfectly accurately described as follows: some things, being referred to collectively (as we just did, with that "some things"). And it would just not be very scientific to take the paradox to be showing that we have to ignore real collections.

Mathematical theories are a little unlike other scientific theories. Mathematics is like a language, in which other scientific theories are written; and that is especially true when it is modern mathematics – a mathematical model of mathematics. You could write up a scientific theory in a variety of languages, perfectly legitimately. But you cannot just pick a made-up logic, to get the answers that you want from a scientific theory, and still be doing science.

In standard mathematics, there is no number of all the numbers. So there are no inconsistencies in standard arithmetic (so far as we know). But, nor are there such equations as 2 + 2 = 4, not in the sense in which we naturally read them. It is a highly artificial reading of '2 + 2 = 4' that is to be found in the standard textbooks. The whole number 2 is usually defined to be the set whose two members are the empty set and a set whose only member is the empty set. And those sets are defined by the axioms of standard set theory.

Since those axioms cannot be shown to be consistent – in twentieth-century mathematics, this was an important result – some mathematicians are unsure of the consistency of standard arithmetic. Whereas arithmetic itself clearly is consistent. Once we have learnt what arithmetic is, we can see that it is going to be consistent. But standard sets are much more complicated. They are the product of an attempt to avoid a contradiction without addressing what that contradiction means. It has been a very clever attempt. So, mathematicians do think that the standard axioms are very probably consistent. But, were standard arithmetic not consistent, we could derive a contradiction like 0 = 1 logically from the standard axioms. Scientists taking their colleagues in the mathematics department at their apparent word, and taking arithmetic to be standard arithmetic, might therefore say that 2 + 2 is probably not equal to 5, rather than that it cannot possibly equal 5.

Now, Cantor's original justification for having the whole number of all the counting numbers does apply just as well to the totality of his infinite numbers. The counting numbers can be paired up with the even numbers, and Cantor's infinite numbers can be similarly paired up with various subcollections of those numbers. And equinumerosity is a very logical concept. That is how arithmetic can be based upon it. It will be logic – not mathematics – that will get us the contradiction of chapter 4. And the standard redefinition of mathematics will not be able to solve the puzzle of chapter 4, as you will see in chapter 5.

Still, the standard redefinition is very probably consistent. It was designed to be perfectly adequate, at least for the immediate purposes of science and engineering (which use mathematical models that can be numerical, or set theoretic, or digital, it makes no difference in the application). And the standard redefinition looks reassuringly scientific. That is how it effectively hides this proof. But it cannot undermine it. It remains the case that 2 + 2 = 4 in the sense in which we naturally read that equation. And the generalisation of such equations that gives rise to the Cantorian inconsistency – given natural background assumptions – remains logical, despite that inconsistency. Had it not been logical, there would have been no need to redefine mathematics: the generalisation could have simply not been made. But the generalisation of counting number to whole number was perfectly logical.

You might be thinking that we no longer have arithmetic, if we include infinite numbers. But arithmetic can be extended beyond the counting numbers, in various ways. For one other example, the ancient Chinese had extended it by including negative numbers. Positive and negative numbers have signs (or directions), as well as magnitudes, and together with zero – which we inherited from Indian mathematics – they comprise the integers. Imagine money going into a bank account, or going out of it. If you have nothing in and you keep taking money out, then you go into debt. Debt is like negative money. When you pay money in again, it fills in that hole of debt before it begins to pile up again. Negatives are like depths of holes, when ordinary (or positive) amounts are like piles above ground. Negative amounts are going in the opposite direction to ordinary amounts, if and when those ordinary amounts can be thought of as having a direction.

Cantor's generalisation of arithmetic has at least as much right to be called 'arithmetic' as integer arithmetic does. All whole numbers – finite and infinite – are essentially properties of collections. And the counting numbers themselves give us an infinite collection. Still, it does not matter whether we call it 'arithmetic' or not. The main thing is that there was clearly nothing illogical about it. Had there been anything illogical about it, there would have been no need to translate mathematics into standard set theory. Cantor's paradox could have been avoided much more simply by just not including his infinite numbers, had there been anything illogical about them.

Cantor's infinite numbers were the product of a perfectly logical generalisation of the numbers of ordinary arithmetic, then; but furthermore, there was no obvious mistake in his derivation of a contradiction. Had there been, it would have been noticed by now, and the standard redefinitions would have been shown to be unnecessary. You will be able to see for yourselves how logically a contradiction can follow from the natural presuppositions of ordinary arithmetic, in chapter 4. There are numbers of things in the real world, and such numbers naturally have whatever arithmetical properties they have here. And in chapter 4 you will see how those real-world properties need not be contradictory if there is a God who is changing. Either such a God changes with us, in time – as, for example, process theologians and open theists believe – or else those divine changes occur above and beyond our created spacetime. This proof that God exists says nothing about which of those is the case. The nature of time is a tangential matter.

Now, a creator who actually changes could conceivably – as you will see in chapter 4 – be starting with some very general possibilities, and be distinguishing more specific kinds, within those general ones, with an authority so authorial that the effect is that – from our creaturely point of view – objective possibilities are becoming more numerous. That is how the hypothesis of chapter 4 will reveal the contradiction of chapter 3 to have been illusory. In chapter 3 we will be assuming, naturally enough, that if something is ever going to be possible – so that that specific thing was always possible – then there was always that specific possibility. Whereas there might, just possibly, have been only a more general possibility to begin with.

The standard mathematical response to Cantor's paradox replaces counting numbers (of things) with standard axiomatic sets (of things). So the standard view is that numbers are effectively human creations. And the next most important response – mathematical constructivism – also treats numbers as human creations. The latter is less comfortable with infinity; but they are equally implausible, when it comes to the corresponding collections of things. It is hard to imagine that Tom, Dick and Harry, for example, being able to pair up with the three primary colours – red, yellow and blue – is not an objective capability. That is why it only really makes sense for a God to be constructing numbers. It does not make sense for the numbers to be changing all by themselves. How can a collection of things change when the things do not change? And Cantor's paradox shows that it does not make sense for them to not be changing.

Of course, if you are happy to throw logic away, whenever it is getting in the way, of what you believe, then things can seem much simpler. And Russell led the philosophy of mathematics into a golden age, following Cantor's contradiction. But such philosophising is just another tangential matter for us here. Cantor's contradiction was originally about whole numbers, in the pre-set-theoretical sense, so it was caused by the corresponding logical property of collections. That is why scientific collections were replaced by axiomatic sets in the twentieth century. In chapter 4 we get a contradiction akin to Cantor's by considering some obvious properties of collections of things. The natural relationship between those properties and number will be assumed, whenever it is convenient to do so (it is a natural use of number words), but the philosophical defence of taking such a natural view of what has become, post Cantor, controversial is a tangential matter for philosophers of mathematics. Related tangential matters are arithmetic itself, and mathematics generally, and set theory – and even logic, in the modern sense. The following proof that God exists is logical, but symbolic logic – which is to logic as axiomatic set theory is to mathematics – is another tangential matter.

There is no point in not assuming that logic is consistent. An atheist might think that the point would show itself were there a logical proof that atheism was false. But a flat-earther could say the same thing about a scientific proof that the earth is not flat. Much as atheists see no sign of God in the world, our flat-earther could say that you can see that the earth is basically horizontal – you only have to go to the seaside, or a desert, to see how flat it really is. Now, you could reply that at the seaside clouds and boats appear over the horizon. But in response our flat-earther could talk about mirages. And if you start to discuss mirages at this point, then you will have been side-tracked.

We could instead point to some scientific proofs that the earth is not flat. But our flat-earther could observe that those scientific arguments are clearly very logical, and add that by giving such logical arguments for the earth not being flat – giving them after failing to counter the evidence of our own eyes that the earth is flat – all that had actually been shown was that logic is not consistent. What would we say to that, if our response to the proof of this book had been to say that logic is not consistent (and therefore to redefine "logic")? We could say that the logic of the scientific proofs is different, but the flat-earther could muddy those waters too. Our flat-earther could say that straight lines are not circles, as a matter of mathematical fact. But if we travelled in a straight line on the surface of a globe then we would be travelling in a circle. That contradiction, it could be said, proves logically that the earth is not a globe.

Of course, it does no such thing really, but if we have already played around with our own definitions of mathematical objects, then we will again be on weak ground. And academics do tend to move from evidence that there are infinite numbers, to wondering whether there are any such things as numbers. What sort of thing would a number be, if it existed, they ask. Do numbers exist? In particular, if the number 2 does exist, where is it? It is not a physical thing, so it does not exist in this physical universe, so how do we know so much about it? Such questions count as philosophical in our academia.

I think that the existence of 2 amounts to the logical possibility of there being a thing and another thing. Similarly, the existence of a shape amounts to the logical possibility of there being things with that shape. Counting numbers seem to be properties of collections, much as shapes are properties of extended objects. But whether this is how we should think of numbers, or not, is actually tangential to chapter 4. I will there be assuming that some ordinary objects exist – although chapter 4 could have started with any things at all (the letters of the alphabet, for example, or subatomic particles) – and then that some logical possibilities exist. When I talk about those logical possibilities I will naturally use numbers. But if you think that numbers are really something else, then you should read my talk of numbers as talk of those logical possibilities. The philosophy of mathematics is therefore tangential to this proof. The proof is logical rather than mathematical (although it was inspired by a mathematical paradox).

As for those logical possibilities, they certainly exist in the way that colours exist, or names exist, or jokes exist and so forth. We can have three logical possibilities, just as we can have three colours, three names, three jokes, and so forth. It does not matter that you can wonder what a joke is. It does not matter that there are different theories of what a name is. It does not matter that the colours shade imperceptibly into one another. We can pair up red, yellow and blue with an apple, a banana and a carrot, which is all we need for this proof to work.

In academic science, "2" usually refers to something like a fictional bag that contains two bags, one of them empty and the other containing nothing but a bag that contains an empty bag. The standard foundation of mathematics is the algebraic definition of those fictional bags, called "standard sets". Standard set theory was designed so that the new arithmetic would be a lot like the arithmetic that you will have originally learned, plus Cantor's extensions of it, except that it would not be contradictory.

I should point out that arithmetic is of course not contradictory. It was thought to be contradictory because of how logical Cantor's extension of it beyond the finite counting numbers was, even though that extension led to a contradiction – Cantor's paradox – and because of an argument that we should never have assumed that it was not. That argument is basically that we are highly evolved primates, and that we should never have assumed that primate arithmetic would be consistent. This argument makes a lot of sense if we think of ourselves as brains, rather than minds, and of brains as being akin to computers. Even the simple perception of two or three things would then be a complicated computation. And while such computers would have been naturally selected to be empirically adequate at such computations, we should not have assumed that they would not be full of all sorts of errors, which might show up with Cantor's extension.

The first problem I have with that argument is that we should surely not assume that such errors would not show up with computations corresponding to anything alien to an evolving primate. That would include the whole of modern science. But when we get a contradiction anywhere else in science, we try to solve it properly, we do not redefine the scientific terminology around it. The fact that Cantor's paradox is part of a proof that God exists seems to be what makes it special. And my second problem relates to that fact. This argument assumes a materialist world-view, but if God exists then we are quite likely to be souls, primarily, only currently incarnating within these physical bodies. We should begin by making very few assumptions – neither atheism nor theism. And our immediate perception of two or three things, as two or three things, appears to be a very simple perception. The foundation of arithmetic is essentially just logic. And we should of course assume that logic is consistent.

The foundation of arithmetic is the counting numbers, and primarily their addition. Multiplication is introduced as repeated addition; and subtraction and division are just those two in reverse (for example, it is because 2 + 2 = 4 that 4 – 2 is 2). Clearly addition cannot cause a contradiction, even with very big counting numbers; and everything else is introduced in ways that are by definition consistent: were they not consistent, they would not be introduced. And similarly, Cantor's new numbers were only introduced into arithmetic because they were as clearly consistent as the old ones. If we are not sure about the consistency of some introduction, then we entertain it as just one logical possibility. That is just common sense. But in any case, the following proof does not require Cantor's new numbers to be consistent. Chapter 4 works with logical possibilities, and where there is some reasonable doubt about their existence, such doubts do not obstruct the proof. On the contrary, they deliver it more immediately.

We have the counting numbers: 1, 2, 3, and so on, ad infinitum – and so we have an infinite number of them. Such infinite numbers were once thought to be contradictory (as you will see), but Cantor showed how they were not. Their introduction can be seen to be consistent (much as the addition of very big counting numbers can be). The details of that introduction are largely tangential to the following; but clearly, standard mathematicians must have had good reason to accept the introduction of such infinite numbers. Those reasons were clearly strong enough to withstand Cantor's contradiction, which could have been expected to have shown the new arithmetic to be inconsistent. And most importantly, if there is not an infinite number of the counting numbers, then since there are all of those counting numbers, there must be something very strange going on with them. And the only conceivable possibility is one that delivers the proof anyway, as you will see.

An argument that God exists is an argument that probably has a mistake in it. It would be surprising if it did not have one. The most surprising thing about it, though, would be how little it would matter, even if it did not have one. Part of that lack of significance is unsurprising: it would be difficult to know for sure that it had no mistake in it. What is surprising, though, is how easily such an argument could be ignored even if it was obvious that it had none. Still, to be completely justified in being absolutely certain that nothing had been overlooked – not even something extraordinarily subtle – in such an argument would, of course, be very difficult. Because of that difficulty, we tend to defer to expert opinion in such matters. And the most influential experts in such matters tend to be well placed in the major religions and scientific disciplines. Of course, religious believers already believe in a God (or something like a God). While they might not believe in the kind of God that such an argument showed there to be, it would not matter much if they did not (even if it was obvious that the argument was free of mistakes). Religions are about faith, not proof. Believers could easily take their God to be above and beyond our logic. The surprising thing is how easily scientists could ignore a proof of something that they were quite sure was false.

Scientists arguing that, for example, the scientific evidence shows that we evolved by natural selection – or rather, that we very probably did – certainly come across as logical. But what if there was a proof that God exists? As I write, most scientists are quite sure that there could not possibly be any such thing. But suppose there was such a thing. For some scientists such a thing would just remind them that they should never have expected primates – not even highly evolved primates – to be perfectly logical anyway. Such scientists could easily take such a proof to be the fatally flawed product of an inherently imperfect logic. They could point to other logical puzzles (such as those of chapter 7) and invite us to see further evidence of the imperfection of our logic. They could invite us to see that, but of course, they could not so easily argue – except imperfectly – that our logic really was imperfect.

Such scientists could also highlight the theoretical possibility of mistakes so subtle that they had gone unnoticed, by such imperfect beings as ourselves. Although they would want to downplay that possibility in the case of the evidence for evolution, and other scientific theories. So they might find doing that too duplicitous. There is something very honest about science. Scientists gather evidence in order to get to the truth. They argue logically in order to get to the truth. It is only because of a logical look at the evidence that scientists believe that we are (very probably) highly evolved primates. And it is only logical to think that: if there is something that would not have been logically possible, had there not been a God, then there is a God.

We should never rush to fill gaps in our current knowledge by invoking the power of a God. Indeed, not even a miracle would prove that God existed, except to those who already want to believe. To prove beyond reasonable doubt that God exists is certainly looking like a tall order. But notice how the existing arguments that God exists all failed for specific reasons. They did not fail because we humans are not perfectly logical.

Nor because of the mere possibility of some unnoticed error (or because of any Cartesian doubt, of course). They did fail, though; and given such failures, it would be reasonable to suppose that there is unlikely to be a proof that God exists. Nevertheless, you will see in chapter 4 that were there no God, things in general would be contradictory – would be logically impossible – whereas there are numbers of things. Chapter 4 begins with some arbitrary things. They could be any things at all. About those things we assume only what we would never ordinarily think to question. We then take a few logical steps – primarily Cantor's diagonal argument – and arrive at two contradictory statements, describing a logical impossibility. But logical steps only ever take us to truths – to descriptions of actuality and possibility – from truths. We must therefore have started with something false. One of our perfectly natural assumptions must have been surprisingly false.

And as you will see, in chapter 4, there is one of them that could, just possibly, be false if God exists. So, if we find that it would indeed be far too odd for any of our other assumptions to be false – and far too odd were this particular assumption false and there not a God – then we will have a surprising but very logical proof that God exists. Things that are as Cantor's diagonal argument shows them to be are only possible if there is a God of a certain kind. Note that while this proof is essentially logical, it is also vaguely empirical: there are of course numbers of things in the world (there are these words, for example). Such things are of course as logic shows them to be, and so there is the kind of God that can block the derivation of the contradiction of the next chapter.

The logical argument of chapter 3 is either showing us that there is something wrong with human logic, or else it is showing us that logical possibilities of particular numbers of things become differentiated from the more general logical possibility of things, for all that it is inconceivable how anyone less than a God could possibly do that. The final chapter undermines the belief that a lot of other logical puzzles do amount to a lot of evidence that there is something wrong with our natural logic, by describing solutions to those puzzles.

Logical puzzles are puzzling because they are showing us where our ability to reason logically needs to be improved. Those puzzles could of course not be showing logic itself to be unreliable. Even if logicians had tried everything to solve a very puzzling puzzle, how could they know that they had? And even if they could show—beyond a reasonable doubt—that they had, how could that have been a reliable demonstration, if reason itself was unreliable? But such questions do not even arise, because as you will see in the next chapter, there are reasonable solutions to all the logical puzzles that have been cited by modern logicians as evidence that logic should be replaced, for academic scientific purposes, with a symbolic logic, a mathematical model of logic.

Symbolic-logical versions of those puzzles will not be considered, in the next chapter, because they are not so much paradoxical as mathematical. Insofar as they are a problem, the problem is how to program a computer to interact with human language as comprehensively and as logically as possible. Logicians have been concentrating their efforts on such versions of the puzzles. But the solutions of the next chapter are good news for academic science, because the status of academic science would be fatally undermined by the belief that our natural logic is inconsistent. Anyone would be able to say, of any perfectly logical argument for a conclusion that they found inconvenient, that it was merely logical. And there is already too much of that kind of thinking.

Appendix

I Descriptions

Howard Uno is planning to use lots of individual grains of sand, during the making of an artwork called The End of the Society of the Spectacle. For that purpose, he has already got himself a lot of sand, all piled up into a heap.

After Howard has used the first grain of sand, he still has a heap of sand—of course (one grain of sand here or there makes no difference to a heap of sand being a heap). This means that when he uses another grain, he is again taking a grain from a heap. So, he is again left with a heap.

When he takes a third grain away, he is again left with a heap. And so forth: Each time he uses a grain, he takes it from a heap, and so a heap remains.

Except that that is not true. It is true for hundreds of thousands of grains, but one day Howard will notice that he has only got a small pile of sand left.

Of course, there is nothing very mysterious about that. If we have a lot of things, and we remove just a few of them, we will have a lot left. But of course, if we remove a lot of them, we might not be left with very many.

Nevertheless, have an apparently logical argument for an untruth, a logical paradox. And like the liar paradox, with which this book began, this logical puzzle was known to the ancient Greeks. It is called the sorites, which is ancient Greek for heap. Unlike the liar, this puzzle has a fairly obvious solution. What may surprise us nowadays is that that solution leads very logically, and in quite a straightforward way, to a logical solution of the liar paradox, and of the related paradox that we met in chapter 2. This chapter, and hence this book, ends with that solution to Curry's paradox.

But to begin with, consider how different it would have been, if we had instead been talking about, say, a mill of sand, where a mill is defined to be a million or more grains. Instead of saying "each time she uses a grain, she takes it from a heap, and so a heap remains," we could then have said "when she uses grains, she begins by taking them from a mill, one at a time, and until that mill is exactly a million grains, a mill remains after each removal."

There is nothing puzzling about how a mill stops being a mill. A heap is a similarly large amount, but it is not any particular amount. It is a vague, not a definite amount. A heap is a lot of stuff that has all been piled up—according to my dictionary; another dictionary might say that a heap is a large pile, and that a pile is an amount of something that has been heaped up. Dictionary definitions tend to be a bit vague and circular, because they are only descriptions of the meanings of words.

We learn those meanings by coming across various uses of the words. We abstract their meanings from what we know by acquaintance with those uses. So, the way in which our words have acquired their meanings is a rather rough-and-ready process. Our words are defined well enough for their ordinary uses. But in this scenario, the use of an ordinary word seems to defy logical analysis:

For each way that things could be, each particular thing either is that way, or else, if that is not the case, it is not that way. That is so obviously true, logic itself seems to be dictating that it is true. And in particular, something is either a heap or else, if that is not the case, it is not a heap. When it is not the case that it is a heap, then it is the case that it is not a heap. So, logic seems to dictate that Howard's sand must stop being a heap suddenly, with the removal of a single grain. But of course, a heap does not stop being a heap with the removal of a single grain. One grain here or there makes no difference to a heap of sand being a heap.

To look at this another way, suppose that each time Howard took a grain of sand away, he asked himself "is this pile of sand a heap?" The answer is at first "yes," and then, several months later, it is "no." And a pile of sand cannot stop being a heap with the removal of a single grain. So, this pile of sand must stop being a heap gradually. As it gradually stops being a heap, "yes" must gradually become incorrect.

"Yes" must at times be in between correct and incorrect. But how could it be neither correct nor incorrect? If "yes" was not the correct answer to that question, then it would be incorrect.

Our words are defined well enough for their ordinary uses. But when we put them to such extraordinary uses—or misuses—as the question "is this pile of sand a heap?" asked hundreds of thousands of times, it is hardly surprising that we get problems. It is a tricky kind of question. It is not a nonsensical question, so it ought to have some logical answer. But we should expect that answer to be more cumbersome than a good answer to a straightforward question would be.

Analogously, the trick question "have you stopped beating your children?" looks like a question with a "yes" or "no" answer. But "no" seems to imply that you are still beating them, while "yes" seems to imply that you used to beat them. A better answer is therefore "no, I have never beaten them," although that does look like an answer to a different question, such as "have you ever beaten your children?" To object to that answer on the grounds that it answered a different question would be to ignore the fact that it answered a trick question.

When answering a tricky question, we need to remind ourselves of what we know, and see how well—or how badly—the question describes that. And if we are faced with a pile of sand that is not obviously a heap, and not obviously not a heap, then some of us would say that it was a heap, and some of us would say that it was not a heap, but a lot of us would want to say only that it was a pile. We should, I think, call it a pile.

But if we had to use the word heap to describe it, why not call it a pile that was about as much a heap as not? That description is rather vague; but then, a heap is a vague amount, not a definite amount. And asked of such a pile, the question “is this pile of sand a heap?” would then be answered correctly by “about as much as not.” So, it would make sense that by “yes” it would be answered about as correctly as not.

“Yes” would then be, not so much not correct, as only about as correct as not. And it would be, not so much not incorrect, as only about as incorrect as not.

Those descriptions are a bit clumsy, and a bit vague, but only because it is not natural for us to use the word heap in the way that the heap puzzle demands we do. Ordinarily, if a pile was neither obviously a heap nor obviously not a heap, then we would just use the word pile. We could then apply the ordinary logical rules. One of the ordinary logical rules—something is either a heap or else, if that is not the case, it is not a heap—gradually stopped applying in the middle of the heap puzzle, and then gradually started to apply again. But when that law did not apply with the word heap, it did apply with the word pile. The pile that is about as much a heap as not is more simply a pile. It is simply not the case that it is not a pile.

The heap puzzle is simply showing us a very natural limitation to those ordinary logical rules. It is only natural for us to think like this: When it is not the case that something is a heap, then it is the case that it is not a heap. But when we are going to be using the word heap to describe something that is about as much a heap as not, this would be more accurate: Insofar as it is not the case that something is a heap, it is the case that it is not a heap.

The reason why it is regarded as a law of logic that either things are a certain way, or else they are not that way, is that it is logical to use words as they are supposed to be used. When we are not misusing them, the ordinary logical rules do apply. And that particular rule of ordinary logic goes back to Aristotle. It is known as the law of excluded middle, and it applies where it is not the case that things are about as much that way as not. It is essentially a law of logic, because we could not do much logical thinking without it. (In particular, all proofs by reductio ad absurdum use it.) Nevertheless, we can think logically about the heap puzzle, if only by using such cumbersome wordss as “not so much not incorrect, as only about as incorrect as not.” So, this law of logic is more accurately called a rule of ordinary logic.

Our words developed their meanings in the context of our use of them. Naturally, their development was heavily influenced by their use in helping us to make decisions. When we have a problem, and a possible solution, we have to decide whether to act on that possibility or not—that is why logic is based on there being two alternatives. Are things a certain way, or not? Our words are defined well enough, in their ordinary uses, for the ordinary logical rules to apply. But when we use words in such extraordinary contexts as the heap puzzle, we find that things are certain ways about as much as not. To think logically about that perfectly natural boundary to the applicability of ordinary logic, we have to use such cumbersome wordss as “about as much as not.”

Now, scientists investigate by making observations, and by making mathematical models of their observations. And classical symbolic logic is a pretty good mathematical model of ordinary logic. And in classical logic, the law of excluded middle is modelled by an axiom, called LEM. And this is a very natural axiom for any symbolic logic to have, because the terms of the artificial languages to which symbolic logics apply are defined with mathematical precision. However, the reasoning of the heap puzzle takes us step by logical step into the ordinarily excluded middle ground, and then out again, into an untruth. When logicians investigate a puzzle like this by modelling it mathematically, there is therefore something—this axiom—that is both appropriate and inappropriate.

Many logicians have therefore solved the problem of the symbolic-logical “paradox” that corresponds to such logical paradoxes as this heap puzzle by using a non-classical symbolic logic. And it does make sense, in the context of applications to computer science, to introduce a middle, or third option. It is like adding a category of “other,” to the bottom of any list that is not—or might not (for all we know)—be exhaustive. But such mathematical modelling does not help us to solve the original logical puzzle. If something was neither the case nor not the case, then it would be both not the case and the case—which is simply contradictory.

A scientific theory, of something, is a mathematical model, of it. And for many people, science is the epitome of good reasoning. Consequently, these puzzles can seem to amount to a good reason for doubting reason itself. However, while the mathematical precision of a symbolic logic can be appropriate when logicians are studying ordinary logical reasoning, it is simply inappropriate for these puzzles, which arise because the meanings of our words are naturally a little imprecise.

Ordinary reasoning involves choosing between various cases. And it is not always perfectly logical. One of the common mistakes (described by Johnson-Laird in his “How We Reason”) can be illustrated by the following question. Suppose we know that either John is sitting on the sofa reading a book, or else Jane is in the garden mowing the lawn; and we know that John is sitting on the sofa. Is he reading a book?

Most people think that he is. We know that there are two cases, one of which is true, and that only one of them is true (that is what “either” and “or else” mean). But if Jane is in the garden mowing the lawn, then John could be sitting on the sofa watching TV. The mistake arises because of how we naturally reason about cases. So, we should not be too surprised to find subtly false dichotomies giving rise to such paradoxes as the liar and the heap.

As I say, when something is about as much a heap as not, we would usually just call it a pile—but would it be wrong to say that such a pile was not a heap? Well, it is not a fact that it is not a heap—the fact would be that it is only about as much a heap as not. But to say that it was not a heap would be, not so much wrong, as only about as right as not. You might even, in some contexts, be able to express a truth well enough by saying that it was not a heap. The next section addresses truth; here, the main thing is that the words "about as much as not" enables us to use poor terminology consistently.

As a rule, it would be better to improve our terminology—using the word pile, instead of heap, for example. We would then be able to choose between various logical possibilities, one of which is correct, the others incorrect. However, when we are solving puzzles like the heap puzzle, we have to use the terminology of the puzzle. That is why these puzzles are so puzzling.

The heap puzzle is not just about heaps, of course. So, let's consider again the apple from chapter 5, which had all of its molecules removed one by one. That object began as an apple, and became very gradually smaller and smaller, until eventually it was just part of an apple, and then just a few molecules, counting down to zero molecules. In between being an apple and not being an apple, it will have been about as much an apple as not.

What about when it was in between being an apple and being no more an apple than not? Well, it might then be described, not too badly, either as an apple, or as about as much an apple as not. If we had problems regarding it as an apple, then those problems would just show that it was better described as being about as much an apple as not. If we had no such problems, then we could of course regard it as an apple.

If we needed to describe what was happening in more detail, at such times, then we would have a need to improve our terminology. We might talk in terms of trillions of molecules, for example, instead of using the word apple. We could think of it as being about as much an apple as not, and a bit more apple than not. But there would be no fact of the matter, of whether it was an apple, or only about as much an apple as not. So, to think of it like that would be quite cumbersome, insofar as our thinking was very logical. And by using better terminology, we would be able to apply the ordinary logical rules.

This is not fundamentally about heaps, grains and molecules; this is a matter of logic. There is naturally a slight vagueness to the meanings of (almost) all our words, and so there is this natural boundary to the applicability of the ordinary logical rules. To see this more clearly, imagine a man staggering through a desert and seeing a mirage, which he takes to be a pool.

Coincidentally, there really is a pool where he thinks there is, but it is obscured from his view by the mirage. As he staggers on toward the pool, he is obsessively thinking "that pool looks cool." Over and over, he thinks that thought, as the image of the real pool gradually replaces the mirage. Gradually, the thing referred to by his "that pool" changes, from the illusory pool to the real pool, without his noticing. Presumably he would, at some point, have referred to the pool with that words only about as much as not. And then the referent of "that pool" would have been about as much as not the pool.

As this pool gradually dries out, there is eventually a body of water that actually is about as much a pool as it is not. And the water in this body of water might be a blue-grey colour. Some people would see it as a very dull shade of blue, others as a vaguely bluish grey. All our eyes are slightly different, and some might see it as neither blue nor grey, while others see it both as blue and as grey. We might say that this water was, in short, about as blue as not. And furthermore, even those who find it blue would find some slightly greyer colour to be about as blue as not. It is easy enough to see that there must, of logical necessity, be such colours:

If, on the contrary, all colours were either blue or else not blue, then the blue part of a rainbow would have two sharp edges. Colours within those edges would be blue, and colours outside them would not be blue. But of course, the colours on either side of those sharp edges would be very similar. Indeed, they would be indistinguishable to any human eye. Even though one would be blue, and the other would not be blue. Which is absurd because the word blue is defined in terms of observable distinctions.

What separates the blue colours from the others must therefore be, not so much like a mathematical line (whose width is zero), more like the line that a sharp pencil would draw—a thin line, with a non-zero width, but not a precise width. The edges of this thin borderline are fuzzy.

One fuzzy edge of the borderline is a bluish indigo, the other a bluish turquoise. There is no fact of the matter of whether the colours within that borderline are blue or not. And there is, moreover, no fact of the matter of whether some colours are in that borderline or not. There are always facts, of course; but the fact of this matter is that the colours within that borderline are about as blue as not.

As a rule, the “law” of excluded middle applies when we are talking about the colours of things. Most things with one colour are either blue, or else not blue. So it is easy to overlook this fact. But it is easy enough to imagine a student considering a turquoise object in a white light, against a range of coloured backgrounds. She thinks that it is a very greenish shade of blue and a very bluish shade of green. In pursuit of objectivity, she asks some other students what they think, and it is easy enough to imagine that they would be divided in their opinions. Since there is nothing else to colours, other than how they seem to us, she decides that it is not so much blue, nor not blue; it is vaguely, but only vaguely blue.

These logical puzzles do not show that our natural logic is inconsistent—not even marginally inconsistent—they show only that it has natural boundaries, not mathematical boundaries. How we describe those boundaries is to some extent a matter of taste. But the main thing here is that these puzzles do not give us any reason to doubt the consistency of logic. And the words “about as much as not” can also help us to make sense of Russell’s paradox. To be a bit more precise, it solves the predicate version of that paradox. So, I will begin the next section by explaining what a predicate is.

II Russell

Assertions or descriptions, like “that pool is blue,” have two basic parts. There is a referring words, like “that pool,” which tells us what the description is about. Descriptions are about the subject of the description. And there is a descriptive words, like “is blue,” which tells us something about the subject. This part is called the predicate, by grammarians and logicians.

Russell came to think of the paradox that bears his name by thinking about Cantor’s paradox, which was about sets. And the set version of Russell’s paradox—which I shall shortly describe—had a considerable influence upon the standard response to Cantor’s paradox. But Russell had relatively little to say about the predicate version of his paradox, which is usually called the Grelling-Nelson paradox, after two German philosophers (who were also mathematicians) who did have something to say about it—Kurt Grelling (1886–1942) and Leonard Nelson (1882–1927).

The predicate version of Russell’s paradox is a simple logical puzzle about predicates like “does not describe itself.” Descriptions tend not to be descriptions of themselves. But some, such as “this sentence contains five words,” do describe themselves. And predicates can, similarly, describe themselves. For example, “is a predicate” is a predicate, and “has three words” has three words. Some predicates do not describe themselves. For example, “has five words” has three words. The problem with “does not describe itself” is that if it does not, in fact, describe itself, then clearly it does—which the meaning of the word not rules out—and conversely, if it does, in fact, describe itself, then clearly it does not—which the meaning of not also rules out.

Those accustomed to applying the usual logical rules, when thinking logically, may well have found that paradoxical. But if “does not describe itself” describes itself about as much as not, then there is no problem, and so this puzzle can be solved easily enough, if predicates can describe themselves about as much as not. And it is fairly obvious that they can. If the three words of “this is blue” had been printed in the right shade of turquoise, for example, then they could have described themselves about as much as not (some shades of turquoise being about as blue as not, in some contexts). And there are lots of other examples.

This is a boring description—or is it? On the face of it, that self-description is not very interesting. But it is a matter of opinion. And the more you think about it, the more interesting it must have been. Furthermore, either it is already obvious that predicates can describe themselves about as much as not, or else the validity of this solution will depend upon the validity of some such example. It is therefore quite plausible that, as someone wondered about that self-description, it was at times described about as well by the words “boring, but not very boring” as by the words “not exactly boring, but certainly not very interesting.” And it is similarly plausible that “is boring” could sometimes describe itself about as well as not.

For another example, the word pretty does not have an especially pretty shape, but it makes me think of prettiness. I therefore find “is pretty” to be about as pretty as not. It is a matter of opinion, but that does mean that I could hardly be wrong about that. And it may well be that as many people agree with me, that “is pretty” is a pretty predicate, as think that it is not. If so, then “is pretty” could be said with some justification to be about as pretty as not, which would mean that “is pretty” described itself about as well as not.

For yet another example, “is long” is certainly not very long, but “is rather long for a predicate in a book that is not a textbook” is rather long. So, some words with a length in between those two lengths and a similar meaning may well be described by itself about as well as not. In short, the predicate version of Russell’s paradox is not very puzzling—in natural languages. But Russell, following in the footsteps of such British mathematicians as George Boole (1815–1864) and John Venn (1834–1923), had been taking a more mathematical approach to logic.

Consider, for example, the logic of the deduction with which chapter 2 began: All humans are animals, and the Queen is a human, so the Queen is an animal. Instead of the predicate “is an animal," Russell would have used the set of all the animals, in the sense of an imaginary bag containing all the animals. Because humans are animals, the set of all the humans is a subset of the set of all the animals. (Subsets of a given set are sets whose members do not include anything that is not in the original set.) The logic of the deduction is that because the Queen is in the subset—because she is one of the humans—she is in the original set—she is one of the animals.

Is there a set of all the animals, though? There is certainly no collection of all the humans that there have ever been, because there was no first human. And as we have seen, most of our words are like that. Instead of the predicate "is blue," for example, Russell would have been working with a set of all logically possible blue things. But some logically possible things (such as "this is blue" printed in turquoise) are about as blue as not. Are they in that set, or not? They seem to be in it about as much as not.

Axiomatic sets can have things in them about as much as not. Axiomatic sets are howsoever their axioms say they are. But logic is not symbolic logic. And another problem with sets had arisen, as Russell was starting out: Cantor's paradox. The things in Cantor's sets were mathematical objects—like numbers, and other sets—so they were all definite things. None was like a pile of sand that is as much as heap as not, which would be as much in as out of the hypothetical totality of all and only the heaps.

Let's use the acronym "Cos" to name the set of all of Cantor's other sets. Cantor's paradox was that a diagonal argument would show that the set of all the subsets of Cos was bigger than Cos—which is clearly inconsistent with Cos containing all the sets other than Cos, because subsets are sets. Cantor's imaginary bags were eventually replaced with the more precisely defined sets of modern mathematics—but all that was to come. Cantor was thinking about the set of all of his other sets because it makes little sense to think of such a bag as being one of the things that it contains. You can only put things that you have already got, into a bag. But you can then put those bags into bigger bags, and so on.

If you build up sets in that way, your sets are said to be well founded. Cantor's sets were well founded, Russell's were not. The sort of sets used by Russell, in place of predicates, could contain themselves. The set of all the sets corresponds to the predicate "is a set," much as the set of all the animals corresponds to the predicate "is an animal." The set of all of Russell's sets was itself a set, much as "is a predicate" is a predicate. So, the set of all of his sets was a member of itself. And so it was perfectly natural for Russell, as he thought about Cantor's paradox, to think about the set of all the sets that were not members of themselves.

Such a set would be a member of itself if it was not, but would not be a member of itself if it was. That is the set version of Russell's paradox. And as Russell observed, it is a bit like the following puzzle. Imagine a barber who shaves all the men who do not shave themselves—and of course, none of those who do shave themselves. Does this barber shave himself, or not? If he does shave himself, then because he shaves only those who do not shave themselves, he does not shave himself. But if he does not shave himself, then because he shaves all those who do not shave themselves, he does shave himself.

It is only natural for us to think of the barber as shaving himself and all those who do not shave themselves. So, it seems to be the description of the puzzle that is at fault here. My description of the barber should presumably not have included the words "none of those who do shave themselves." Does that help us to solve the set version of Russell's paradox? Could the solution involve a set of all the other sets that do not contain themselves?

That set—let's call it Other—would not be a member of itself. Russell could not have used Other, instead of the predicate "is not a member of itself," because Other would not have been in that set. Russell needed a set of all the sets that did not contain themselves. Similarly, all the animals are in the set of all the animals—that is why that set could replace the predicate "is an animal" in Russell's mathematical approach to logic. Could Russell have used a set containing all the members of Other and also Other? But then that set would not have been a member of itself, and so it should have been.

This was therefore a problem for Russell's mathematical approach to logic. Those who thought that natural languages were not good enough for logic had to develop more sophisticated symbolic logics. But Cantor, by considering a set of all his other sets, had never had this problem to solve. Cantor may have talked about "a set of all the sets," but he only ever filled sets with pre-existing things, and that biggest set was implicitly excluded from the range of that word all.

Russell concluded that his paradox showed there to be different types of predicates. And there undoubtedly are different kinds of predicates. There are predicates of ordinary objects and predicates of descriptions. Although we could also distinguish between predicates of numbers and predicates of shapes, and of real people and fictional characters, and perceptions and ideas, etc. Such distinctions are not in themselves very profound.

Still, objects can be fluffy and round, so we have predicates (of objects) like "is fluffy" and "is round." And "is a predicate" does seem like a different kind of thing, so let's call it a quasi-predicate (of such a predicate). We could then say that "is a quasi-predicate" is neither a predicate (of an object) nor, similarly, a quasi-predicate (of a basic predicate), but a pseudo-predicate. And so on, endlessly. And the corresponding sets would then be well founded. Nevertheless, my quasi-predicate and my pseudo-predicate do seem like the same kind of thing. And Russell's hierarchy of predicates did seem to be similarly artificial.

The following paradox is another puzzle that Russell found, as he was thinking about sets, although it was even less puzzling. It is called the Burali-Forti paradox, after the Italian mathematician Cesare Burali-Forti (1861–1931) who wrote the mathematics paper that inspired Russell to think of it. But Cantor was aware of this puzzle and had, like Burali-Forti, not found it very puzzling.

The Burali-Forti paradox concerns the totality of all the ordinal numbers. The counting numbers are what most of us think of as numbers, and when we use the counting numbers to count things, we count one of those things first, then another, then a third—if more than two things are being counted—and so on. That ordering of things as a first thing, a second, a third and so forth, is called a sequence; and that use of one, two, three and so forth is what makes one, two, three and so forth ordinal numbers. For one and two, this is clearer when symbols are used: 1^{st} for first and 2^{nd} for second.

That connection with counting does make that use of one, two, three and so forth the natural choice. However, anything could have been used; for example, we could have used letters. We would then have found it perfectly natural to say an ayth thing, a beeth thing, a seeth thing and so forth. Beyond the zeeth (the twenty-sixth) thing, we would presumably have been using AA for the AA^{th} (or twenty-seventh) thing, and then AB for the AB^{th} (or twenty-eighth) thing, and so on. But as I say, it is perfectly natural for us to have been using numbers. Still, if we had we used letters, instead of numbers, then the Burali-Forti paradox would have concerned the totality of all the ordinal letters. So, the Burali-Forti paradox is not fundamentally a paradox about numbers.

Still, it is a paradox of infinity, intimately connected with Cantor's paradox. Cantor introduced infinite ordinal numbers before discovering the infinite whole numbers that this book has been about. Infinite ordinal numbers extend such sequences as a first thing, a second, a third and so forth beyond the endlessness of the counting numbers.

Ordinal numbers could also be extended the other way, by adding a zeroth thing in front of the first thing, and other things in front of that zeroth thing. That would be like the move from the counting numbers—1, 2, 3 and so forth—to the integers. (Integers go from a negative endlessness up to –3, –2, –1, 0, +1, +2, +3 and so forth.) Ordinal numbers are not normally extended into the negative, but they do normally begin with the ordinal number 0. For various reasons, the first ordinal number is normally 0, in academic mathematics. Academic mathematicians do not say “zeroth” where we say “first,” but that is the logic of having zero as the first ordinal number.

Anyway, early in his career, Cantor was working on the calculus, and in the calculus, differentiating a function generates another function. (For example, differentiating the distance we have travelled, as a function of time elapsed, gives us our speed, as a function of time elapsed; and differentiating speed gives us acceleration. Speed is the first derivative of distance, acceleration the second.) Cantor had been differentiating a wide range of functions again and again, and considering the limits of those differentiations as their number tended to infinity. Then he differentiated those limit functions, giving him infinitieth derivatives. Differentiating those gave him infinity-plus-first derivatives, and so on.

Cantor needed a new kind of symbol, for those infinite amounts, and those symbols were the first symbols for infinite ordinal numbers.

Suppose we have these things: 1, 2, 3, …, A, B. If we say that the ordinal number of the counting numbers, when they are in that natural order, is infinity, then the ordinal number of 1, 2, 3, …, A, B would be infinity-plus-two.

Moving B to the front of the sequence gives us the following sequence: B, 1, 2, 3, …, A. The ordinal number of that sequence is infinity-plus-one.

If we relied on counting to tell us how many things we had, when we had infinitely many things, then these would be two different numbers of things. Whereas, they are obviously the same number of things: They are exactly the same things. So, infinite ordinal numbers are not the infinite whole numbers that measure how many things there are. The latter is such a basic concept in mathematics, Cantor found that he had to define infinite cardinal numbers as well as infinite ordinal numbers. Two collections have the same cardinal number of things in them when they are equinumerous (when the things in one collection can all be paired up with all the things in the other). Cardinal numbers are simply the whole numbers that we have been considering throughout this book.

The ordering A, B, 1, 2, 3, …, shows how the things in that particular sequence are equinumerous with the counting numbers.

Ordinal numbers are usually defined to be the sequence of all the preceding ordinal numbers, starting with 0. For example, the ordinal number 2 is (0, 1). And 3 is (0, 1, 2). The ordinal number 1—the second ordinal number—is written in symbols as (0), a sequence with only one element (so not really a sequence), and so 2 can also be written (0, (0)). That may look familiar, because it is similar to the set-theoretical cardinal number 2 (which we saw in chapter 1). Set theory is neater if the cardinal numbers are defined in terms of the ordinal numbers. And defining ordinal numbers in terms of preceding ordinal numbers is also very neat. It means that they can all be constructed out of pre-existing things, starting with the empty set, which accords with the idea of sets being well founded. For such technical reasons, the first ordinal number is 0, not 1.

Anyway, if we consider the sequence of all the ordinal numbers, then we are considering something with exactly the same basic pattern as those ordinal numbers. And yet, it cannot be a sequence of preceding ordinal numbers—it cannot be an ordinal number—because all the ordinal numbers are in that sequence. And that would be a set-theoretical paradox, had ordinal numbers been defined to be sequences of ordinal numbers. Which is precisely why mathematicians like Cantor and Burali-Forti did not define them in exactly that way.

Ordinal numbers are defined to be such sequences with the exception of the sequence of all of them. That biggest sequence cannot have 1 added to it, precisely because it is the biggest, and so it is not an ordinal number. This is very similar to the natural solution of the barber puzzle, and it is very mathematical solution. There is no mathematical justification for taking the sequence of all the ordinals to be an ordinal. If it was ever the case that mathematicians did have a need to do something more times than all the ordinals in a set theory, then they would simply have a need for a set theory with more ordinals in it. They could always get one by adding an axiom to the effect that the sequence of all the old ordinals was one of the new ordinals—which models rather well the idea of there never being all of the ordinals.

Could the solution to Cantor's paradox be like this solution to the Burali-Forti paradox? Well, supposing that there are all of the things that such steps as we were taking, in chapter 3, could possibly get to, then you could of course not get more of them by taking just such a step—that is just what it means to have all of them. So, there cannot be such a step. However, that thought does nothing to solve the problem of chapter 3. To avoid the contradiction, we need to find a logical mistake in our reason for thinking that there would be such a step—in the thought that, since there are all of those things, there would also be all the different ways of having just some of them—or we need there to not be all of those things, somehow.

The paradoxical totality of logically possible selections (from similar collections)—which was called X in chapter 3—is a collection of things that such steps as we were taking could possibly have taken us to. So, all of them were there already. The paradoxical contradiction shows that not all of those possible selections do exist, as distinct possible selections. But by itself that thought does nothing to undermine the contrary thought that, given that totality of things that such steps as we were taking could take us to, each and every subcollection of those things would exist. That contrary thought is strongly supported by the thought that, since all of our things are already there, hence so are just some of them. So, this does nothing to eliminate the contradiction. On the contrary, this thought is merely part of the contradiction. We need an explanation of how it could possibly be that not all of those logically possible selections are distinct possibilities.

Note that even if the Burali-Forti paradox had been more puzzling, that would only have given mathematicians a reason to change the definition of the ordinals. It would not have shown there to be anything wrong with the idea of infinite ordinals. And furthermore, even if there had been something wrong with that idea, there would have been no threat to the consistency of logic. Mathematicians would just have limited their use of infinite ordinal numbers accordingly (much as they limit their use of infinitesimal quantities). Russell's "paradoxes" meant something to Russell because he was constructing a symbolic-logical foundation for scientific thought. But none of them threatened the consistency of logic.

Do they give us any reason to think that the natural concept of a collection leads to inconsistencies, when used logically? Could it be the case that the most complicated of the logically possible selections of chapter 4 were logically possible only about as much as not? Having introduced that words, this is the sort of question that does now need to be asked. When we were considering the collection of all of those logically possible selections, might we have taken ourselves too close to such a borderline? Was that the lesson of Russell's paradoxes?

One conceivable way would be if the possible selectors got less and less real, the more complicated the selections became. However, even if they did become less real, somehow, the underlying things would remain just as distinct from each other. Another conceivable way in which the possible selections might be there only about as much as not would be by their being there originally in an undifferentiated way—although that is essentially the solution of chapter 4.

If I am right about Cantor's paradox, then the standard axioms do describe realistic properties of numbers. So, academic mathematics is fine for the mathematical modelling of reality. But we should, primarily, think about reality logically. Two cats and two dogs make, in total, four animals, and we cannot make them five by redefining "five" to mean four. No one can redefine "five" with authority. Arithmetic—in the sense of school mathematics (not academic mathematics)—is discovered, not invented. And it is when we describe reality well enough that our descriptions are true.

III Truth

Truth was addressed indirectly in the first section of this chapter. To say that the correct answer to the question “is this pile a heap?” is “yes” is just to say that the pile is a heap. “Yes” means “yes, that pile is a heap.” And to say that the words “that pile is a heap” are true is similarly to say that the pile is a heap.

We have seen how a pile might be about as much a heap as not. And we have seen how “yes” would then answer the question “is this pile a heap?” about as correctly as not. So, it makes sense that “this pile is a heap” would then be about as true as not.

To tell the truth is to tell it how it is. The classical definitions of true and untrue—which tradition ascribes to Aristotle—are therefore:

(T) To say, of what is the case, that it is the case—or to say, of what is not the case, that it is not the case—that is to say something that is true.

(U) To say, of what is the case, that it is not the case—or to say, of what is not the case, that it is the case—that is to say something that is untrue.

Because there is a marginal vagueness or fuzziness in the definitions of most if not all of the words of our languages, those two have always implicitly come with something like the following.

(V) To say, of what is vaguely the case, that it is the case, or that it is not the case—or to say, of what is the case, or of what is not the case, that it is vaguely the case—that is to say something that is vaguely true.

A description that is much truer than not will be true enough to count as true. That is just what that “much” means. And a description that is not much truer than not will be only vaguely true, or about as true as not. That is what that “about” means. If we needed to make sharper distinctions than that, we would have a need to improve our terminology, so that we could use ordinary logic, and so (V) is an adequate addition to the classical descriptions.

Furthermore, there is no need to make any changes to the ordinary logical rules. We would not normally be doing much in the way of logical thinking, were (V) applying, except trying to improve our terminology. The heap paradox was puzzling precisely because it was asking about a deliberately poor use of words. If we needed to do a lot of reasoning about a pile, when it was about as much a heap as not, then we would have a need to use a better predicate than is a heap. We would not have any need for a non-classical symbolic logic.

Ordinary logic with appropriate predicates can—and should—be assumed when reasoning logically about anything. Logicians are of course free to model (V) mathematically. For that purpose, they might use three truth-values, such as 1 or T, for truth, 0 or F, for falsity or untruth, and ½ or U for undecided or borderline cases. (Such symbols are used in such fields as artificial intelligence.) Most non-classical symbolic logics can be regarded as mathematical models of (V). But however useful such models might prove to be, they could never give us a sensible replacement for (V), were (V) telling us things that we would rather not believe.

Vagueness can be found in the misuse of most if not all of our words, and another example that ancient Greek philosophers discussed is baldness. You were probably born bald; and if you were, then “you are bald” would originally have been a true description of you. You would soon have had a head of hair, though. And of course, no one went from being a bald baby to not being bald by growing just a few hairs. “Bald” just does not have so precise a definition. Nor did they suddenly grow a lot of hair, of course. No, they only gradually stopped being bald. So, in between the times when “you are bald” was true, and the times when it was not true, it gradually stopped being true. There must be, at such times, a mixture of true and not true, a mixture that changes gradually, from true enough to count as true, to not true enough to count as true.

We might say that insofar as you had only a bit of hair, you were bald, and insofar as you had quite a bit of hair, you were not bald. It would be like a true contradiction. But this is just because our words are defined only as precisely as our purposes have required them to be. The marginal vagueness that remains is a legacy of the flexibility that natural languages needed in order to have developed in the first place.

Vagueness is not the enemy of ordinary logic. On the contrary, the underlying flexibility means that we can always make our words more precise when we have to. For example, when “you are bald” is becoming problematic, logically, “you are getting hairy” will be more straightforwardly true. Note that it is not that “is getting hairy” is less vague than “is bald.” It is rather that the bounds of its usefulness in logical arguments are in a different place.

Another textbook example is colour (which was given as an example in the second section of chapter 1). To begin with, the concept of truth certainly does apply to colours. It can be true to say that snow is white, for example. Indeed, a textbook example of the meaning of “is true” is this:

(iff) “Snow is white” is true if, and only if, snow is white.

When snow is white, “snow is white” is true. Snow can sparkle with all the colours of the rainbow, even when it is white; but snow may also become very discoloured by mud, so that it is brown. And when snow is not white, “snow is white” is not true. But what about when it is only vaguely discoloured? What about when snow is about as white as not?

If a description did apply about as much as not—and that does seem to be a realistic possibility—then it would make sense for it to be about as true as not. That textbook example did not allow for that possibility. That is because proofs, such as you might find in a textbook, use ordinary logic. For ordinary logic to apply, we have to be using words well enough—and in particular, well enough away from the marginal vagueness in their meanings that is shown to exist by the "paradoxes" of the previous section.

It is not so much the case that even students of logic find such "paradoxes" puzzling—with the implication that they are indeed paradoxical—as that they are especially puzzling for those working from the classical definition of truth, (T). The classical definitions, (T) and (U), are definitions in the way that dictionary definitions are definitions—they are descriptions of the meanings of the words in question. They naturally focus on the most significant aspects of those meanings. But in the logical puzzles of this chapter, a different bit of the meaning of truth is being emphasised. All competent speakers know what truth is, though, and all three of (T), (U) and (V) do seem to follow from what we know.

For academic purposes, a better example of the meaning of "is true" would therefore be this:

(isfa) "Snow is white" is true insofar as snow is white.

The earlier textbook example exemplifies the rule for ordinary logical thought, but this exemplifies the underlying logical law. Academics pursue the most general truths, of various matters. They use ordinary logic—which was abstracted, over many centuries, from the best of human reasoning—in order to pursue it. But that does mean that they should be modifying their terminology as appropriate (which is the good reason for academic writing not being an easy read).

To want the truth of any matter is to want things to be made clear—it is to want to eliminate vagueness. But to insist on the use of the ordinary logical rules even when one's terminology does not support them—that is just not very logical. Ordinarily, coming across such contradictions as we find in these puzzles would indicate a need to eliminate the vagueness by improving one's terminology. But with such puzzles, the poor terminology is the puzzle. To change the terminology would be to stop looking at the puzzle, not to solve it. To solve these puzzles, we need to realise that (iff) is essentially the most useful part of (isfa).

Less abstractly, suppose that we have a turquoise toy that is about as blue as not. There is no fact of the matter of whether this toy is blue or not. Or rather, the fact of this matter is that it is only vaguely blue. If we took a poll of people's opinions on this matter, they would probably be divided. Some would say that this toy was blue—and they would not be wrong (although you would not be wrong to say that they were wrong). It is a matter of opinion (and everyone is entitled to theirs). Some would say that it was not blue—and they would not be wrong. Should there be a fact of the matter of whether “this toy is blue” is true or not?

Because I am introducing (V), I am tempted to say that “this toy is blue” is not true, and that “it is not blue” is also not true. That is because I am thinking of how it would be better to use a words like is about as true as not. But “this toy is blue” is, not so much not true, as only about as true as not. Now, I want to avoid saying that “this toy is blue” is neither true nor untrue, because that would be to say that it was not true and yet was true, and I want to avoid asserting a contradiction. However, asserting this contradiction would be, not so much wrong, as only about as right as not. And insofar as it expressed the fact that “this toy is blue” is about as true as not, it would be right.

Naturally, (T) and (U) meet in (V). A variety of logical puzzles involving the truth predicate can therefore be solved by way of (V), including the version of the liar paradox that we met in chapter 1, and the version of Curry’s paradox that we met in chapter 2. The difficulty of solving such puzzles without (V) amounts to further evidence that (V) is a realistic addition to the classical definitions, although I shall not be critiquing such attempts here. The main thing here is that all of these logical puzzles do have logical solutions. And we can work up to the solution of our puzzling version of Curry’s paradox gradually, through a sequence of increasingly complicated puzzles. The most trivial one is me saying the following thirteen words.

This description, of the thirteen words that I am currently speaking, is true.

That description is not saying much. It says that what it says is the case—but that is all that it is saying. It is consistent for that self-description to be true, but it is also consistent for it to be untrue. If it was untrue, then it would not be true that what it says is the case, so what it says would not be the case: it would not be true. Given how vacuous it is, we cannot do much with it, logically. So, it would make sense for us to think of it as being about as true as not.

Would it be better to think of those thirteen words as being exactly as true as not? But people do have different views on what those words are. Some people think that they are true. And all of us have a natural tendency to interpret words that are ambiguous so that they are true (as when we hear “pigs are not pigs” as true). But some people think that those thirteen words are untrue. And there is common tendency to view vacuous words as nonsensical, and nonsensical words as not true. And such tendencies could hardly be expected to add up to something exact.

Does it make sense for us to think of those thirteen words as nonsensical, in view of how vacuous they are? I do not think so, because it was so obvious what they meant. Consider how the following question has an obvious meaning. “What am I, with only these ten words, talking about?” The answer is “nothing,” of course, but the fact that there is that obvious answer does show that a perfectly sensible question was being asked. Still, even if it does make sense to think of those thirteen words as nonsensical, it also makes some sense to think of them as true. And thinking of them as about as true as not is a way of including all such senses.

A more complicated puzzle is associated with this question: Is the answer to this question “no”? Although the answer to that question could be “it is not,” without there being any actual contradiction, there is no such easy answer to the following version:

Is it correct to answer this question in the negative?

To answer that question in the negative is to say that it is not correct to answer that question in the negative. So, if it is correct to answer that question in the negative, then it is correct to say that it is not correct to answer in the negative, presumably because it is not the case that it is correct to answer in the negative.

So, if it is the case that it is correct to answer that question in the negative, then it is not the case that it is correct to answer in the negative. So if we assume that either it is the case, or else it is not the case, then because either way it is not the case, it would follow logically that it is not correct to answer that question in the negative. But saying that it is not correct to answer that question in the negative would then be correct. In other words, it would be correct to answer that question in the negative. We have another contradiction, and now there is no obvious assumption for that contradiction to be showing to be untrue.

Could it be that there is no correct answer to that question? But then it would, in particular, not be correct to answer that question in the negative. And we have just seen how that pans out. So, we do need some correct answer. It cannot be “no,” or words to such a negative effect. But also, it cannot be the case that “no” is not correct. We have also seen how that pans out.

An assumption that the contradiction might be showing to be untrue is the law of excluded middle. It is not an obvious assumption, because it appears to be part and parcel of logic itself, rather than an assumption. By reasoning logically, we were naturally assuming that law. That is why it can seem to be logic itself that is generating the inconsistencies. And that is how these logical paradoxes can seem to give credence to the idea that Cantor’s paradox has no logical solution. One could certainly be forgiven for thinking that there could not possibly be any logical solution to this particular puzzle. But of course, not seeing how to solve a puzzle is not quite the same as seeing that it cannot be solved.

Lateral thinking is the way to get out of such tight logical corners. Note that answering in the negative was appearing to be incorrect because it was being taken to be correct. And then it was appearing to be correct because it was being taken to be incorrect. That seemed to put us in an impossible situation—but many actual situations have this basic form. Consider, for example, the ancient Greek story of the hare and the tortoise. A hare is so confident of winning a race against a tortoise, he has a nap, half way through the race—which the tortoise therefore wins. The hare lost because he had a very good reason to think that he would win.

For another example, if you go to a concert expecting to have a great time, then you are more likely to be disappointed, compared to going without that expectation. You are going because you like the music, so you have high expectations. But if you know that this increases your chances of feeling disappointed, then your expectations are naturally a bit lower. And that decreases your chances of feeling disappointed.

Could we say that answering in the negative is incorrect insofar as it is correct, and that it is correct insofar as it is incorrect? We have already seen, with the heap paradox, that the answer "yes" to the question "is this pile of sand a heap?" varies gradually from being correct to being incorrect. So, correctness does seem to be something that can—in such exceptional circumstances—come in degrees.

Our reasoning about "is it correct to answer this question in the negative?" should therefore conclude, not that if answering in the negative is correct, then it is incorrect, and if incorrect then correct, but that it is correct insofar as it is incorrect, and incorrect insofar as it is correct. The greater the incorrectness, the greater the correctness; and the greater the correctness, the greater the incorrectness—which would not have seemed so impossible.

The obvious solution is then that answering in the negative is about as correct as not.

It could not be very correct to answer in the negative, as it would then be very incorrect (and it cannot be both very correct and very incorrect). And similarly, answering in the negative could not be very incorrect, as then it would be very correct. But answering in the negative could consistently be about as correct as not, as it would then be about as incorrect as not. If the answer "no" was about as correct as not, then it would not so much be that it was not correct, as that it was only about as correct as not; and it would not so much be that it was not incorrect, as that it was only about as incorrect as not. That way of putting it is admittedly cumbersome, but it is perfectly consistent.

If there was indeed no other logical solution to this puzzle—as its appearing to have no logical solution would seem to suggest—then this puzzle would, all by itself, amount to some evidence that there were indeed degrees of correctness. And there is a lot of other evidence, as we have seen. And there being degrees of correctness, or truth, gives us a simple solution to the liar paradox.

“These words are not true,” Howard had said. And if he had been telling us the truth, then he would not—by (T)—have been doing what he said he was not doing. What he said would not have been true. Whether he was telling us the truth or not, what he said would not have been true. He would not have been telling us the truth. But paradoxically, because that was what he had said he was not doing, he would—by (T)—have been telling us the truth. Which is, of course, impossible: Telling us the truth is precisely what his not telling us the truth rules out.

In short, his words seemed nonsensical. But with (V), sense can be made of them. The problem was that Howard was telling us the truth because he was not telling us the truth. If we express that by saying that he was telling us the truth insofar as he was not telling us the truth, then it would clearly make sense for him to be telling us the truth about as much as not.

The big difference between the previous paradoxical question and the liar paradox is that when one says that one’s words are not true, one is saying, not only that they are not true, but also that that is not true—that is, that they are true. Simultaneously, two contradictory statements seem to be made.

The use of symbols might make this additional oddity clearer.

Consider any claim that that very claim is not true. For example, in the claim “this claim is not true,” the words “this claim” refers to that very claim.

If we call such a claim Dint—the word Dint is made from the initial letters of the words in “Dint is not true”—we can express its paradoxical nature like this:

Dint = Dint is not true.

Replacing the second “Dint” with the first results in this:

Dint = “Dint is not true” is not true.

As a rule, a description being contradictory means that the thing being described—such as a round square, or in this case, the description itself—is an impossible thing. Whereas, Howard’s description clearly did exist. It was paradoxical, not nonsensical. And there is a perfectly logical explanation of how there could be two such different ways of expressing the same meaning: They are two equally poor expressions of the same meaning.

Such a big difference would usually correspond to two different meanings. Contradictions usually describe logical impossibilities. They describe them very well. But some extraordinary possibilities are described not too badly by them.

We have seen how the answer “yes” to the question “is this pile a heap?” was at times—and to some extent—neither incorrect nor correct. At such times, it was not so much not incorrect, as only about as incorrect as not, and it was not so much not correct, as only about as correct as not. At such times, the contradiction “it is correct and it is incorrect” might not have been too bad a description of that answer.

“Yes and no” might, similarly, not be too bad an answer to the question “Is it correct to answer this question in the negative?” Insofar as we say “Yes,” we are saying that yes, “No” is a good answer. And insofar as we say “No,” we are saying that no, “No” is not a good answer. This reply is not as good—not as true—as “It is as correct as not.” But it does express an important element of that truth. And it expresses it in an appropriate way for normal discourse, because it does not introduce any new terms.

When we receive “Yes and no” as the reply to a simple question, we do not normally assume that the obvious contradiction is meant. Why would anyone assert such an obvious untruth? No, we usually assume that it means something like “Yes, in one sense, but in another sense no,” which may well be true.

Suppose that I am sitting at my desk, and I suddenly say “I am lying.” Perhaps I am saying that I am lying down, and simultaneously saying that I am lying about that. Such ambiguity is common enough in puns and poetry. A good answer to “Was I lying?” would then be “Yes, I was lying, because no, I was not lying but sitting,” or “Yes and no” for short.

And in response to a half-truth, “Yes and no” is not a bad thing to say. It acknowledges the fact that the half-truth contains an element of truth and an element of untruth. If it is a contradiction, then it won’t be true. But could it be as true as not? Contradictions are certainly not very good descriptions of logical possibilities. But are they always very bad descriptions of them?

A contradiction—such as “it is blue and it is not blue”—is a pair of contraries. And a claim and its contrary cannot both be true. If it is the case that something is not blue, then it is not the case that it is blue. But if something is only vaguely blue, then it could be that “it is blue” overstates its blueness, while because it is vaguely blue, “it is not blue” understates it. In such a use, those two contraries would not be in conflict, but would be complementing each other. It could therefore make sense that, if “it is blue” was about as true as not, so that “it is not blue” was also about as true as not, then “it is blue and it is not blue” would also be about as true as not, perhaps a little less true than not, in light of it being a contradiction.

But perhaps a bit truer than not, given how they complement each other. Ordinarily, a contradiction like “it is blue and it is not blue” would simply not be true—in the common sense that it would therefore be untrue (or false)—so it is strange to think of any contradiction as being, not so much untrue, as only about as true as not. But this is an unusual situation. We would usually call vaguely blue objects by the name of their colour—indigo perhaps, or turquoise. Nevertheless, “it is blue and it is not blue” could be used to convey the truth that such an object was about as blue as not, and so it would not be too odd to think of such a contradiction—in such an exceptional use—as about as true as not.

If Howard’s liar-paradoxical words were, as he spoke them, about as true as not, then his description of them as not true—the obvious meaning of those words—would have been, not so much false, as only about as true as not, as would my implicit description of them as true. Why should such a poor description not be its own contrary? We seem to have seen, not only that it is its own contrary, but how it could have been. It is basically because Dint = Dint is not true.

Howard’s words being about as true as not therefore seems to be a logical possibility, for all that it is not an ordinary-logical possibility. Would it be better to think of those words as being exactly as true as not? But the words true and not are naturally associated with a rather fuzzy borderline. That is just how natural languages are. About as true as not is a natural kind of thing for words to be.

Would contradictions not necessarily being false be another possible way of resolving Cantor’s paradox? No, because the things considered in chapter 3—all those logically possible selections from collections of similar things, starting with three distinct things—were all as distinct from each other as those three things were. The logic of chapter 3 was able to use the ordinary logical rules, which we tend to become very familiar with.

Another version of the liar has two statements, instead of one. Let’s call whatever you are thinking One, and let’s call the assertion that One is not true Two. If you were thinking “black is white,” for example, then that would be One. The assertion that it is not true that black is white—which we would then be calling Two—would then be true.

If you were thinking “Two is true,” then that would similarly be One. And Two would then be the assertion that it is not true that Two is true. So, Two would be asserting that Two is not true. And so, Two would also be asserting that it is not true that Two is not true.

So, Two would be its own contrary. And similarly, One would be its own contrary—but that might be clearer when these statements are written like this:

(1) (2) is true

(2) (1) is not true

Those two lines mean that (1) says that (2) is true, and that (2) says that (1) is not true.

By replacing "(2) is true" (in the first line) with (2), and then replacing that with "(1) is not true" (as the second line says we can), we can see quite graphically that (1) says that it is true that (1) is not true. Perhaps it is also easier to see how (2) says that it is not true that (2) is true, so that it says that (2) is not true. (Symbols, laid out more vertically, can make the underlying logic clearer.)

There is no explicit self-reference in this version of the liar. Some philosophers have therefore argued that self-reference is not essential to the liar. However, the self-reference is still there implicitly. When you were thinking "Two is true," you would have been thinking about Two—which says that "Two is true" is not true (whether you were thinking that it did, or not). Late in the twentieth century, the American philosopher Stephen Yablo observed in 1985 that you can also get a version of the liar with infinitely many statements, each referring only to those that follow it, in that infinitely long sequence of statements. Arguably, there is no self-reference at all in Yablo's paradox; although, perhaps it is just more subtle.

We cannot delve into that question here, and we do not need to. We cannot, because there are lots of other paradoxes involving infinitely long sequences. But that does not matter. In chapter 3, it did not matter whether it was M or X that was endlessly growing in size. And the main thing here is that all the statements of Yablo's paradox could be about as true as not—as could the statements One and Two. Insofar as what you were thinking was true, it would not have been true. And insofar as it was not true, it would have been true. Consequently, it would make sense for it to have been about as true as not.

All other versions and variants of the liar can be solved similarly. An interesting version is the following self-description, which I am going to call R (for revenge). When problematic versions of the liar are obtained by incorporating the terminology of a proposed solution into the wording of the new version, those versions are usually called revenge problems, in the academic literature.

(R) R is not at all true

In terms of the words “about as much as not,” that self-description, (R), says that (R) is not even about as true as not. And since (R) also says that it is not at all true that (R) is not at all true, it also says that (R) is at least as true as not.

Now, if we assume that (R) is about as true as not, then what (R) states can seem to be both untrue and true. It appears to be untrue because (R) states that (R) is not even about as true as not. It appears to be true because (R) also states that it is at least as true as not. And of course, two different ways of expressing the same statement should be as true as each other.

Still, statements that are actually about as true as not can easily seem to be both untrue and true (as we have already seen several times in this chapter). Suppose that (R) is about as true as not. It is not so much that (R) is untrue, as that it is only about as true as not. And it is not so much that (R) is true, as that it is only about as untrue as not. So, as that words “not so much” implied, it is not completely wrong to say that (R) is untrue, nor to say that it is true. That could easily account for (R) appearing to be both untrue and true.

Does it? Well, suppose that a more precise description of (R) is that (R) is about as true as not and, moreover, a bit less true than untrue. The words “about as true as not” does implicitly cover descriptions that are only a bit less true than untrue. And although such precision is not normally appropriate for borderline cases—which is why (1) and (2) are best described as about as true as not (rather than as true as not, with its implication of precision)—it is not entirely inappropriate for this version.

If we suppose that (R) is about as true as not, and a bit less true than untrue, then to say that (R) was not at all true could be, not so much untrue, as about as untrue as not. Adding “a bit less true than untrue” positions (R) closer to not at all true. And similarly, to say that (R) was at least as true as not could be, not so much true, as about as true as not. These descriptions are very cumbersome, but that is to be expected for this version.

Another reason why it makes sense for (R) to be about as true as not is that to say that something is not at all true is not that different to saying that it is not true. Were the ordinary logical rules applying, it would be to say the same thing. It therefore makes sense that we should have the same solution to this revenge version as we did to the ordinary version of the liar.

A more puzzling variant of the liar was introduced at the beginning of chapter 2, where we saw a logical (as opposed to the original symbolic-logical) version of Curry’s paradox, which can be written like this:

(C) If (C) is true, there is a round square.

In chapter 2, the paradoxical reasoning was essentially as follows. If (C) is true, then what (C) describes is the case. So, if (C) is true, then if (C) is true—which it would then be—there is a round square. So, if (C) is true, then there is a round square. That being the case, (C) is true—which means that there is a round square. There being no round squares, we must have done something illogical. The puzzle is to find out what it was.

And we have one clue to begin with: (C) effectively says that (C) is not true. So, (C) may well be about as true as not. Now, we did not use a false dichotomy explicitly, in this paradoxical reasoning. But in the second version of the liar paradox, at the very start of chapter 1, we used a *reductio ad absurdum* instead of an explicit dichotomy—when we moved from something like Howard telling the truth and pigs flying, to Howard not having told the truth. And similarly, we may here have been applying an ordinary logical rule that is only a rule when there is a sharp distinction between descriptions being true or else—if that is not the case—not true.

To find out which rule it was, it may help if we spread out the logical steps in that paradoxical reasoning. And to begin with, (C) is a claim of the form *if P then Q*, where the letters P and Q name, or stand for, any two claims.

For example, the assertion "all humans are animals," which means that if someone is a human, then that person is an animal, is of this general kind.

What the words *if* and *then* mean, in such contexts, can be spelt out like this:

Given that P, and that if P then Q, we also have Q.

That is a logical axiom. It goes by the Latin name of *modus ponens*, but despite the fancy name, it is essentially the very elementary logic with which chapter 2 began.

Under the supposition that (C) is true, the following two statements, (i) and (ii), are going to be true (the pretty obvious reasons are spelt out after those two statements).

(i) (C) is true

(ii) If (C) is true then there is a round square

Given that (C) is true, we clearly have (i). And much as "it is raining" being true means that it is raining, (ii) is what it means for (C) to be true, as you can see by looking at the definition of (C). Statement (i) is trivially true, but the point of explicitly writing it down is that if (C) is true, then ordinary logic—and in particular, modus ponens—should apply, to (i) and (ii). We therefore have, given that—or under the supposition that—(C) is true:

(iii) There is a round square

Because (iii) is the assertion that there is a round square given that—or under the supposition that—(C) is true, it is the same as:

(II) If (C) is true then there is a round square

I am now using capital roman numerals for names, to emphasise that these claims are not under any supposition; and note that there will be a (I) shortly. Just as we could get to “it is raining” being true, from the fact that it was raining, were it raining, so from (II) it follows that:

(I) (C) is true

Again, there is no hidden supposition. And again, from (I) and (II) it follows by modus ponens that:

(III) There is a round square

Since there are no round squares, (III) is false.

That has given us a more formally laid out “proof” of an untruth. The ordinary logical steps have been spread out, and made a little neater. There are a few, very simple symbols—but this is certainly not formal logic, in the sense of symbolic logic. This “proof” is not mathematical. So, it is just as paradoxical as the version in chapter 2. But it should be easier to see where we have been going wrong.

Thinking outside the box—thinking laterally, such as by using analogies—is the way to solve any paradoxical puzzle. And suppose, in particular, the truth of the following two statements, under the supposition that modus ponens is invalid under a self-referential supposition.

Firstly, any statement P.

Secondly, that P implies Q (for any suitable statement Q).

It would of course be absurd to use modus ponens to get Q from those two statements under that supposition. Under that supposition, such a use of modus ponens would be invalid. Since that would be absurd, modus ponens is clearly not always valid under self-referential suppositions. And since modus ponens is not always valid under self-referential suppositions, we should not have used modus ponens under the self-referential supposition that (C) is true.

It was indeed easier to find a place where we might have gone wrong, when the logical steps in the paradoxical reasoning were spread out. The corresponding step in the original reasoning was the step from this, “if (C) is true, then if (C) is true—which it would then be—there is a round square,” to this, “if (C) is true, then there is a round square.”

When an assertion is made, it falls upon ears and eyes that naturally assume that it might be true. Simply in order to understand the assertion—to work out what it means—we will suppose that it is true and think, in the ordinary logical way, about what that would mean. So, there was nothing wrong with supposing that (C) was true, and then proceeding by following ordinary logical rules. The resulting absurdity simply showed that not all of those rules apply to (C). Getting that absurdity was not a failure of logic, it was part of a perfectly logical reductio ad absurdum of the assumption that the ordinary logical rules applied.

There is then the question of why such rules did not apply. And when a claim is about as true as not, what would the words “if that claim is true” mean? Its meaning would surely be expressed more clearly by the words “insofar as that claim is true.” And insofar as the paradoxical claim is true, it seems to be untrue, because to say that if something is true then there is a round square is normally to say that it is not true. It does therefore seem to be the case that, insofar as the claim is true, it is not true. It follows that this claim is about as true as not. It is therefore a poor description of itself. Ordinary logical rules bring out its descriptive paucity in the form of paradoxical reasoning. So, ordinary logic is not valid here, even though it would be valid were the claim true, or untrue.

The Curry-paradoxical argument that God exists would be very similar if God did not exist; and if God does exist, it would be even less of a puzzle. In that case, the claim would be like the following claim. If this claim is true, then these are ten words. Which is unproblematically true. The Curry-paradoxical argument does not seem to depend upon whether its conclusion is true or not; but it is a “proof,” not a proof, and it does so depend.

Not all logical puzzles can be completely solved by use of the non-classically logical words “about as much as not.” Cantor’s paradox is essentially a logical paradox, and it cannot. And there may well be harder logical puzzles out there. But it should at least be clear by now that we should never take any inability to solve a logical puzzle to be any evidence against the reliability of natural logic.

We should always hope to find a logical solution to any puzzle, however hard it seems. Logic is not something optional. Logic is more than a very convincing kind of rhetoric, because logically is how we should think. We should certainly not be thinking that there is something wrong with logic, just because the only logical solution to a puzzle is something that we do not already believe, and because what we do believe is telling us that there may well be something wrong with logic. The thought that there could be something wrong with logic is akin to a Cartesian doubt.

And we saw in this chapter how a fairly obvious solution to the heap paradox actually leads fairly straightforwardly to a fairly simple explanation of the liar paradox (and of such variants of that paradox as the English version of Curry's symbolic-logical puzzle that we met in chapter 2). Both those paradoxes were known to the ancient Greeks. So the fact that scientific philosophers tended, in the twentieth century, to regard them as showing there to be something wrong with our natural intuitions about logic probably says something about those philosophers.

What it says may never be very clear, though. As the twentieth century began, scientific philosophers were trying to comprehend Einstein's innovations, as well as Cantor's paradox, when the first World War struck. Then—as the mathematicians working around Cantor's paradox were faced with Gödel's discovery, and the physicists getting their heads around Einstein were having to think about quantum mechanics—the second World War struck. And in the subsequent "cold war," the increasing importance of computing justified a lot of their symbolic logic. So it may well be too late now, for us to work out what those philosophers were really thinking.