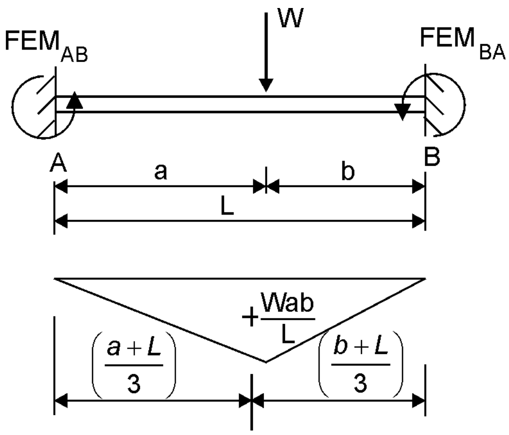
Slope Deflection Examples:

Fixed End Moments

For a member AB with a length L and any given load the fixed end moments are given by:

Where: g_B and g_A are the moments of the bending moment diagrammes of the statically determinate beam about B and A respectively.

Example: Determine the fixed end moments of a beam with a point load.

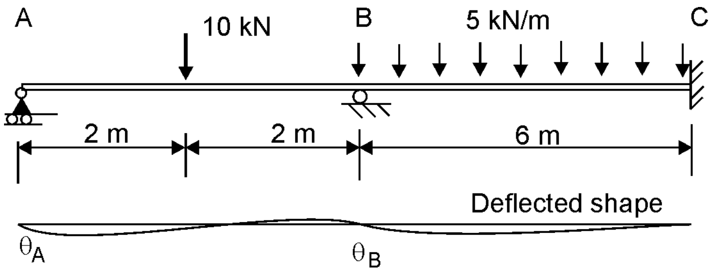


Simply supported beam with bending moment diagramme. Centroid in accordance with standard tables.

In a similar way FEM_{BA} may be calculated.

Use of slope-deflection equations:

Example 1: Determine the bending moment diagramme of the following statically indeterminate beam.



The unknowns are as follows: θ_A, θ_B, $\theta_C = 0$, $\Delta_{AB} = 0$, $\Delta_{BC} = 0$

We require two equations to solve the two unknown rotations:

$EI\,\theta_A + 0{,}5\,EI\,\theta_B + 5 = 0$ **(1)**

$M_{BA} + M_{BC} = 0$

$0{,}5\,EI\,\theta_A + 1{,}66667\,EI\,\theta_B + 10 = 0$ **(2)**

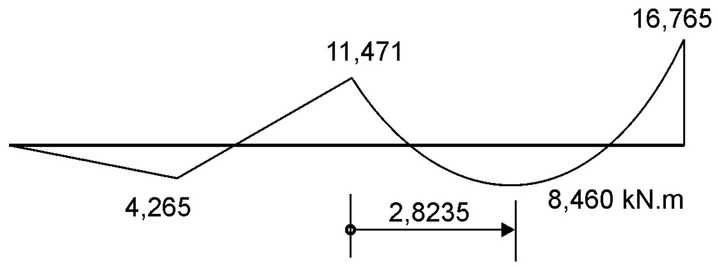
Solve the unknowns:

$\theta_A = -\,2{,}35294/EI$
$\theta_B = -\,5{,}29412/EI$

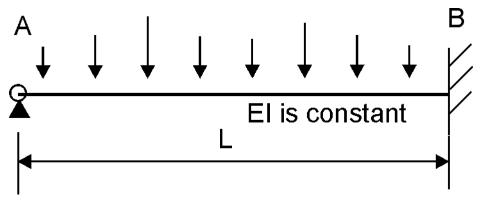
Calculate the values of the moments:
$M_{BA} = 0{,}5 \times -2{,}35294 - 5{,}29412 - 5 \quad = -11{,}471$ kN.m
$M_{BC} = 0{,}6667 \times -5{,}29412 + 15 \quad = +\,11{,}471$ kN.m

$M_{CB} = 0{,}3333 \times -5{,}29412 - 15 \quad = -16{,}675$ kN.m

Draw the bending moment diagramme.



The Modified Slope-Deflection Equation with a Hinge at A:



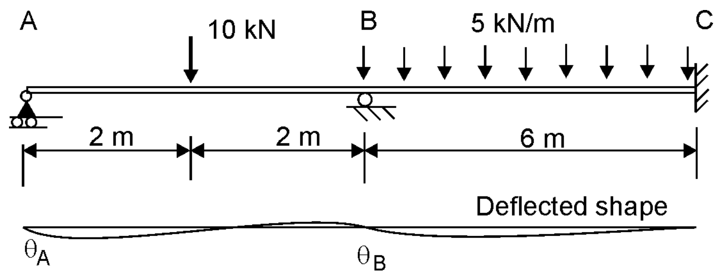
We would like to eliminate θ_A from the equation as we know that $M_{AB} = 0$.

Solve for θ_A.

Replace θ_A in this equation.

This equation may be used to reduce the number of unknown rotations.

Solve the previous problem using the modified slope-deflection equation.



The unknowns are as follows: θ_A use the modified slope-deflection equation, θ_B, $\theta_C = 0$, $\ominus_{AB} = 0$, $\ominus_{BC} = 0$
The total number of unknowns is reduced to θ_B

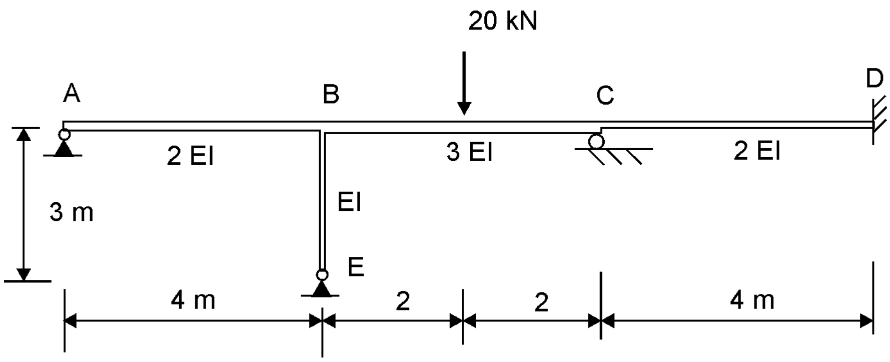
$M_{BA} + M_{BC} = 0$

1,41667 EI $\boldsymbol{\theta_B}$ ***+ 7,5 = 0*** ***(1)***

$\boldsymbol{\theta_B}$ ***= - 5,29412/EI***

Bending moments are as calculated previously.

Example 2: Determine the bending moment diagramme of the following structure.



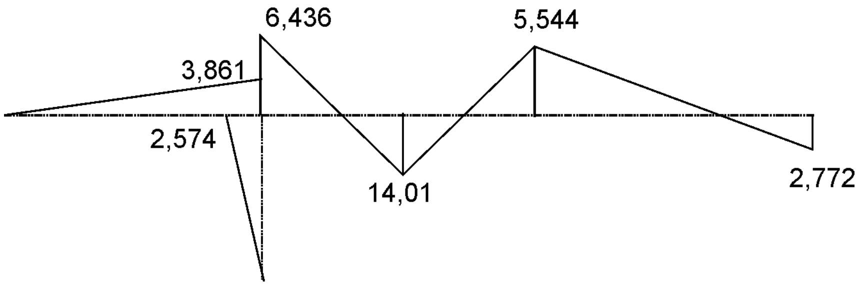
Unknowns: θ_A – use modified SD equation, θ_B, θ_C, θ_E – use modified SD equation, $\psi_{AB} = \psi_{BC} = \psi_{CD} = 0$

No Loads No FEM:

(1)

(2)

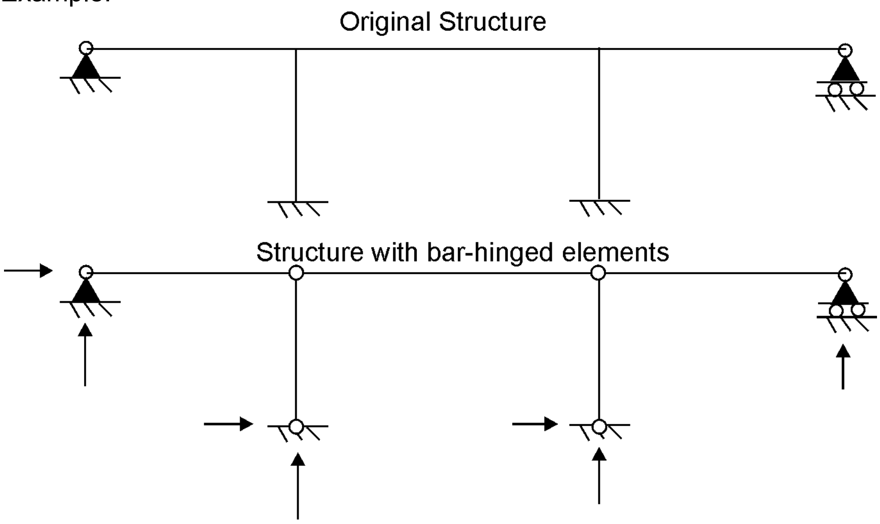
Solve for θ_A and θ_B.



Sway Structures

One of the ways in which can calculate whether a structure can sway and the number of independent sway mechanisms, is to convert the structural elements to bar hinged elements and to determine the degree of instability. The degree of instability will also be the number of independent sway mechanisms.

Example:



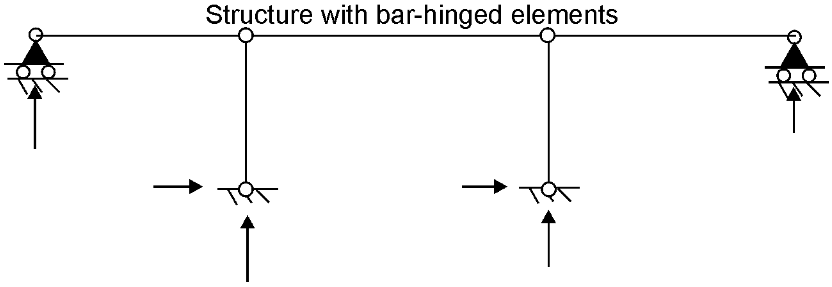
Structure with bar-hinged elements:

s = 5			
r = 7		s + r	= 12
n = 6		2n	= 12
2n – (s + r)	= 0	No independent sway mechanism	

Example:

Original Structure

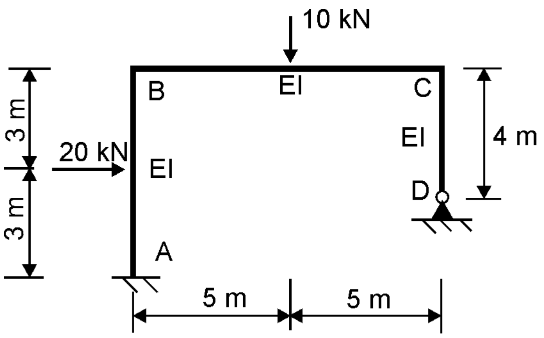
Structure with bar-hinged elements



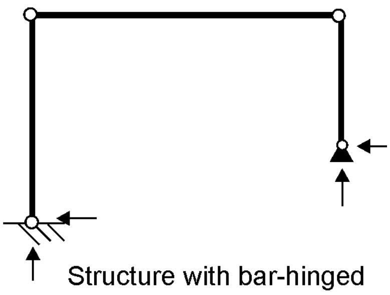
Structure with bar-hinged elements:

s = 5
r = 6   s + r = 11
n = 6   2n = 12
2n – (s + r) = 1   One independent sway mechanism

Example: Determine the bending moment diagramme of the following structure:

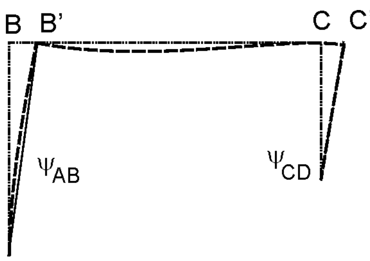


Original Structure



Structure with bar-hinged elements

s = 3
r = 4   s + r = 7
n = 4   2n = 8
2n – (s + r) = 1   One independent sway mechanism



Sway of the structure.

One assumes that the member BC does not deform as AE is so large that
If this is true, BB' must = CC'.

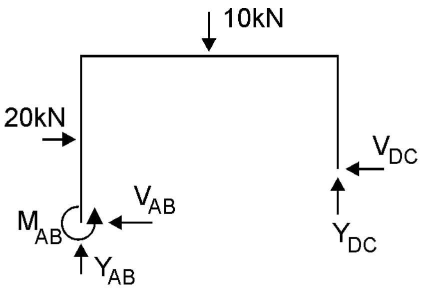
But BB' = 6 θ_{AB} therefore θ_{CD} = CC'/4 = 6 θ_{AB}/4 = 1,5 θ_{AB}.

Call θ_{AB}, - θ.

Unknown in this case: $\theta_A = 0$
θ_B ?
θ_C ?
θ_D use the modified slope deflection equation.
θ ?

We require three equations to solve the unknowns.

For the third equation, one must investigate all the external forces that are applied to the structure.

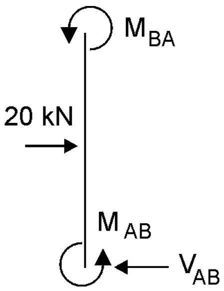


The axial forces Y_{AB} and Y_{DC} are usually difficult to determine, whereas the shear forces V_{AB} and V_{DC} can be calculated by taking moments about B of the member AB and C of the member CD respectively.

The third equation is obtained by:

(1)

(2)



Take moments about B.

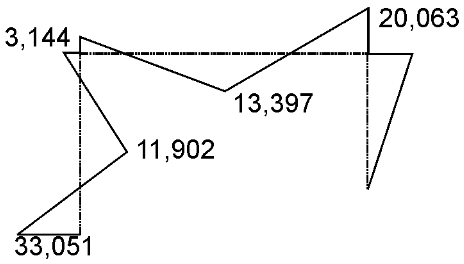
In a similar fashion one may calculate V_{DC} in terms of the unknowns:

(3)

Solve the unknowns:
θ_B = -18,582 / EI
θ_C = - 9,61572 / EI
Θ = 24,24397 / EI

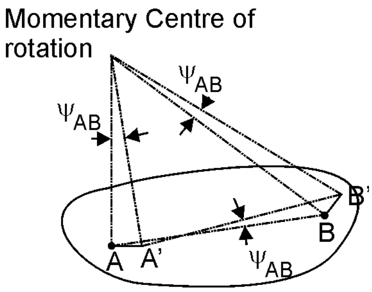
Substitute in the equations for the moments:

M_{AB} = 33,051 kN.m
M_{BA} = -3,144 kN.m
M_{BC} = 3,144 kN.m
M_{CB} = - 20,063 kN.m
M_{CD} = 20,063 kN.m

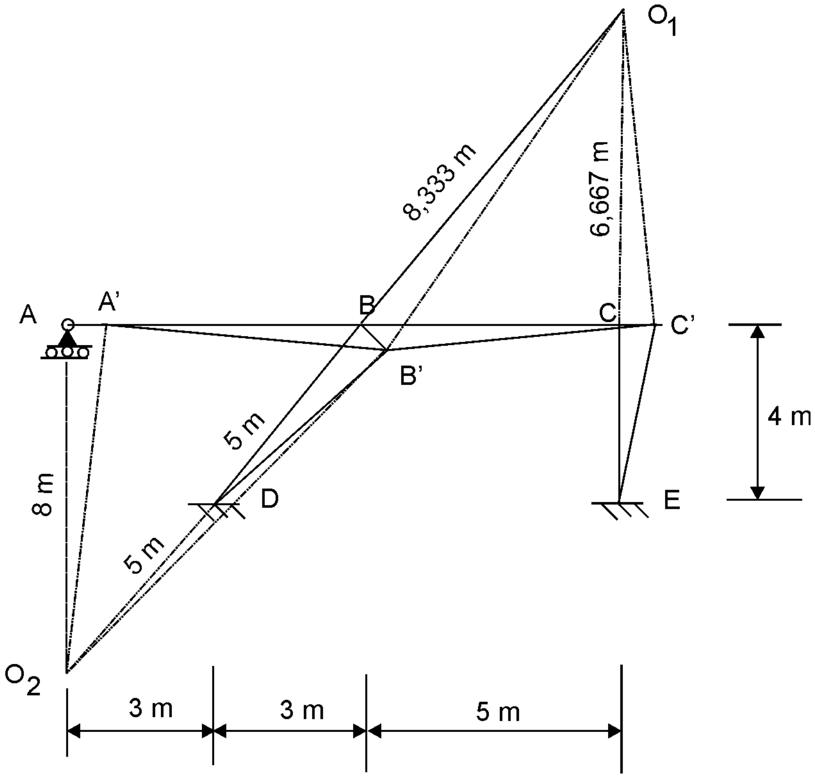


Momentary Centre of Rotation

When two points on a rigid body undergo a small displacement, the body rotates about a momentary centre of rotation and the following angles are equal:



Example: Determine the sway angles of the following structure in terms of the sway angle $\ominus_{DB}$ of the following structure.



D is a fixed point so that the point B may only move vertical to the member BD. B moves from B to B'. In a similar way E is a fixed point and C can only move vertically to the member to C'. A may move horizontally. If one looks at the member AB both ends may move so we will find a momentary centre of rotation O_2 vertical to the direction of movement. Both ends of member BC can move so we will find a momentary centre of rotation, O_1 vertical to the direction of movement of B and C.

Because movements are small relative to the length of the member, the tan $\ominus$ = the angle $\ominus$.

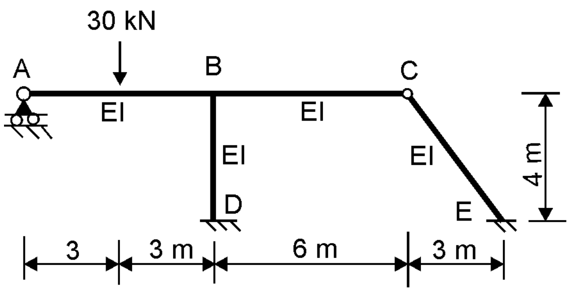
BB' = 5 x $\ominus_{BD}$

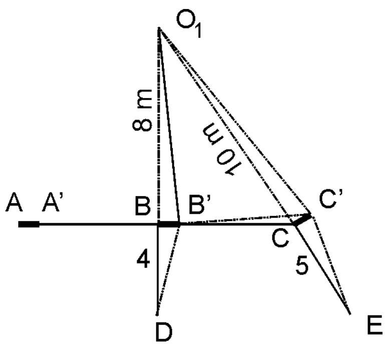
CC' = 6,667 x $\ominus_{BC}$ = 4 x $\ominus_{BD}$

The direction of the angle is important. If it is clock-wise it is negative. $\ominus_{BD}$ as shown is negative so $\ominus_{BC}$ will be positive.

Example:

Determine the bending moment diagramme of the following sway structure.





Structure with sway mechanism.

Determine the number of independent sway mechanisms:
s = 4   r = 5   s + r = 9
n = 5   2n = 10
2n –(s + r) = 1
1 independent sway mechanism!

Unknowns	θ_A	use modified slope-deflection equation
	θ_B	?
	θ_C	use modified slope-deflection equation
	θ_B	0
	θ_E	0
	Θ	?

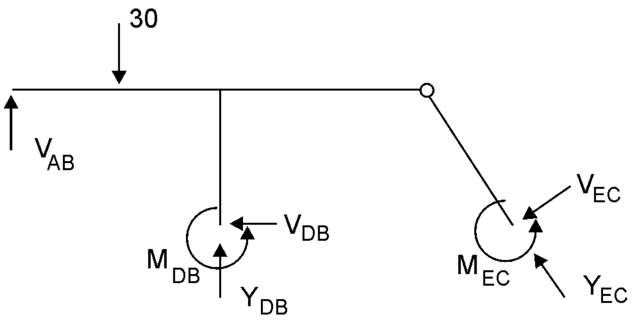
2 unknown – we require 2 equations

Set $\Theta_{DB} = -\Theta$
BB' = 4 Θ

CC' = 10 Θ_{BC} = 5 Θ

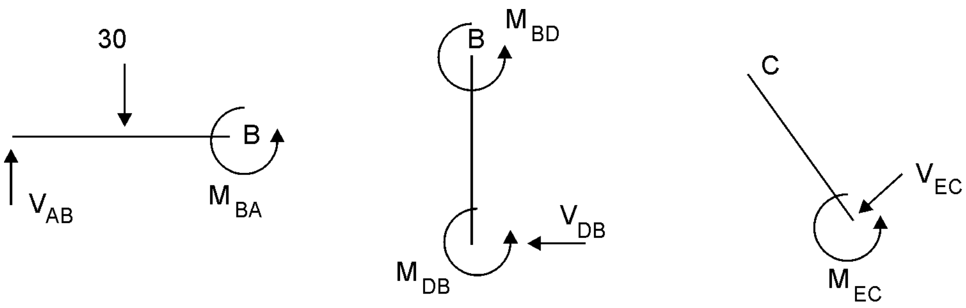
(1)

To determine the second equation one must view all the external forces on the structure:



As it is difficult to determine YDB and YEC we will take moments about a point where their moment is known to be 0. The momentary centre of rotation, O_1, is such a point.

Determine the unknown forces in terms of the unknown rotations and translational angles.



Member AB

Member BD

Member CE

Take moments about the momentary centre of rotation:

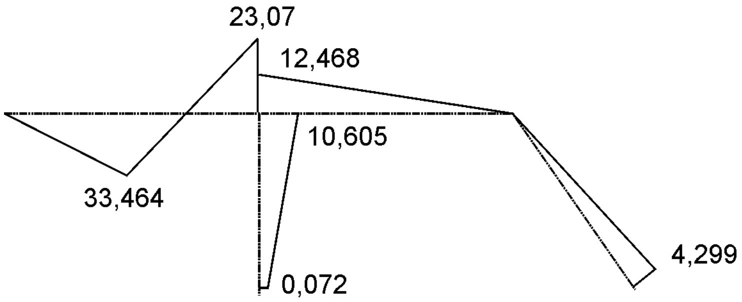
$V_{AB} \times 6 + V_{DB} \times 12 + V_{EC} \times 15 - 30 \times 3 - M_{DB} - M_{EC} = 0$

(2)

Solve the unknowns:

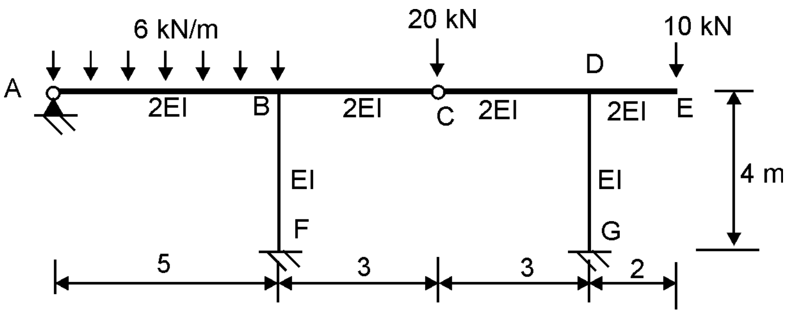
$M_{BA} = -23{,}073$ kN.m
$M_{BC} = +12{,}468$ kN.m
$M_{BD} = +10{,}605$ kN.m

M_{DB} = -0,072 kN.m
M_{EC} = -4,299 kN.m

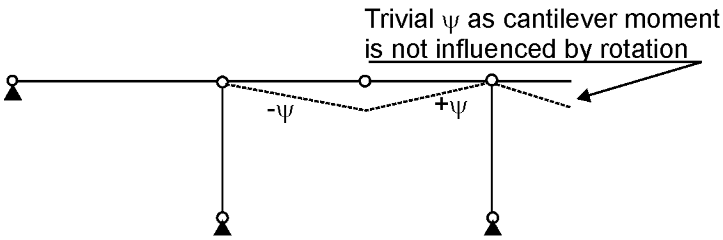


Bending moment diagramme

Calculate the bending moments and draw the bending moment diagramme of the following structure.



Change the nodes to hinges and calculate the number of independent sway mechanisms.
s = 6
r = 6
(s + r) = 12
n = 7
2 n = 14 therefore 2 n – (s+r) = 2 with two independent sway mechanisms



We have three unknowns, namely θ_B, θ_D and Δ. We require three equations to solve these unknowns.

M_{BA} = 1,2 EI θ_B – 18,75

M_{BC} = 2 EI θ_B + 2 EI $\ominus$

M_{BF} = 1 EI θ_B

(1)

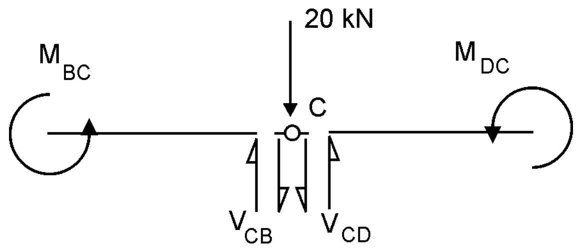
M_{DC} = 2 EI θ_D - 2 EI $\ominus$

M_{DE} = + 10 x 2 = + 20

M_{DG} = EI θ_D

(2)

Third equation may be obtained from the vertical equilibrium of node C



$-V_{CB} - V_{CD} - 20 = 0$

$-V_{CB} - V_{CD} - 20 = 0$

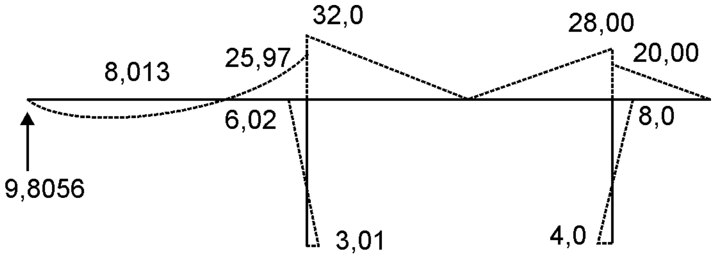
+ 2 EI θ_B - 2 EI θ_D - 4 EI $\ominus$ = 60 (3)

Solve the three simultaneous equations:

θ_B	-6.0185/EI
θ_D	8.0093/EI
$\ominus$	22.0139/EI

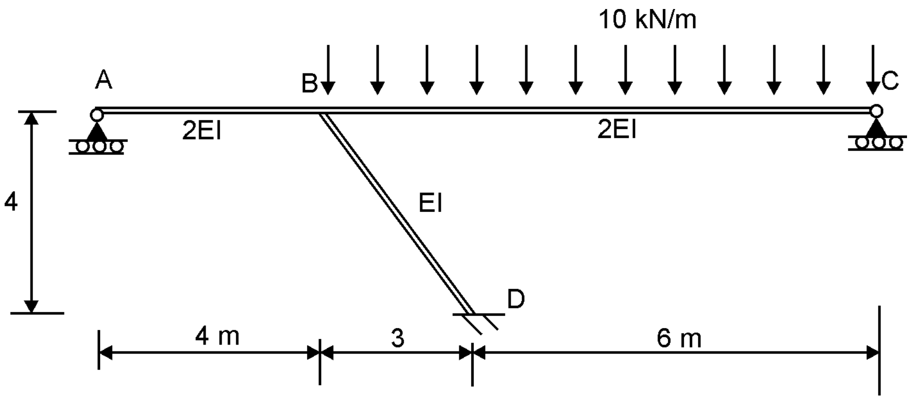
M_{BA} = - 25,972 kN.m
M_{BC} = + 31,991 kN.m

M_{BF} = - 6,0185 kN.m
M_{FB} = - 3,009 kN.m
M_{DC} = - 28,009 kN.m
M_{DG} = + 8,009 kN.m
M_{GD} = + 4,005 kN.m



Example 2:

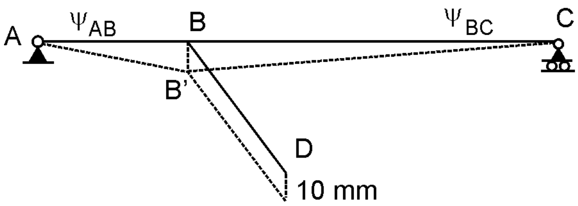
The support D of the structure undergoes the following displacement, 10 mm vertically down and 20 mm horizontally to the left. E = 200 GPa and I = 50 x 10^{-6} m^4.

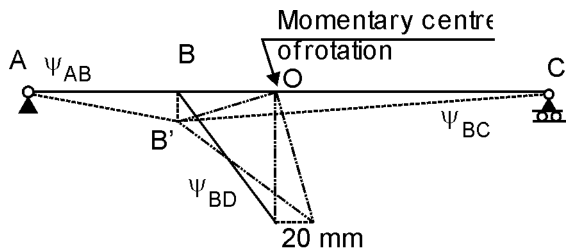


If one determines the number of independent sway mechanisms we see that there is one. The unknowns are thus θ_B and ⊖.

The sway angles will consist of a known angle as a result of the displacement of D and the unknown, ⊖.

Determine the known angles for the 10 mm and 20 mm displacement individually and add them together. In order to do this the unknown sway must be prevented.





For the 10 mm displacement, B drops vertically by 10 mm. The sway angles are thus equal to BB'/Length of the member:

For the 20 mm displacement, B may only move vertically so that both ends of member BD move and in this way we will find a momentary centre of rotation.

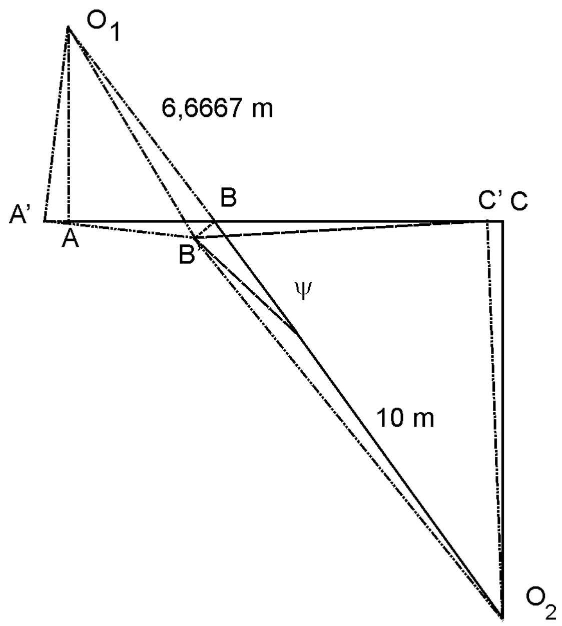
Therefore, $\ominus_{BD} = 5{,}0 \times 10^{-3}$

$BB' = \ominus_{OB} \times 3\ m = 0{,}015\ m$

The total sway as a result of the displacements is the sum of the individual sway angles. Therefore:

$\ominus_{BD} = 5{,}0 \times 10^{-3}$

To determine the relative sway angles:



Set $\ominus_{BD} = \ominus$, then $BB' = 5\ \ominus$

Equations required to solve the unknowns:

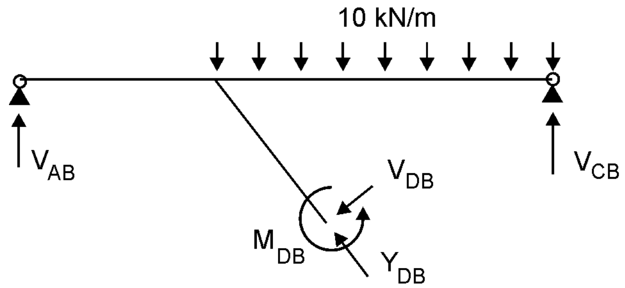
$M_{BA} = 15\ 000\ \theta_B + 11\ 250\ \ominus + 93{,}75$

$M_{BA} = 6\ 666{,}667\ \theta_B - 2\ 222{,}22\ \ominus + 82{,}7313$

$M_{BD} = 8\ 000\ \theta_B - 12\ 000\ \ominus - 60{,}00$

(1)

For the second equation one must look at all the external forces that are applied to the structure.



As it is very difficult to determine Y_{DB} take moments about a point where the moment of $Y_{DB} = 0$, i.e., O_2.

Take moments about B of the member AB:

To determine the force V_{DC}, take moments about B of the member BD:

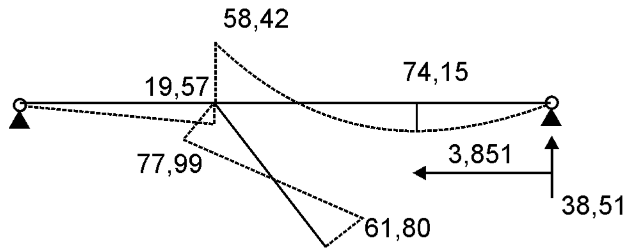
$M_{DB} = 4\ 000\ \theta_B - 12\ 000\ \ominus - 60{,}00$

$20\ 750\ \theta_B + 96\ 562{,}5\ \ominus + 199{,}6875 = 0$ (2)

Solve the two simultaneous equations:

θ_B = - 0,0040464
$\ominus$ = - 0,00119844

M_{BA} = + 19,572 kN.m
M_{BC} = + 58,418 kN.m
M_{BD} = - 77,990 kN.m
M_{DB} = - 61,804 kN.m


58,42
19,57
74,15
77,99
3,851
61,80
38,51