Activity Guide - Multiplication + Modulo

Summary
In this activity you’ll multiply numbers as input into the modulo operation and explore some interesting properties that relate to cryptography.

Goal: Understand how multiplication + modulo can be used to make computationally-hard-to-crack encryption.

Tools:
- Calculator: you probably want a calculator handy for multiplying big numbers
- The “Mod Clock” widget in code studio (pictured at right)

Assumption: You have been introduced to the modulo operation and the “clock” analogy for it.

Step 1: Experiment with the Mod Clock

Goal: familiarize yourself with properties of the Modulo operation

Get your feet wet - play
- Try inputting different values into the mod clock for both the “number” and the “clock size”.
- Try big numbers and small numbers for both

Questions:
1. Using a clock size of 50, write a list of 5 numbers that produce a result of 0.

2. With clock size of 50 how many total numbers are there that produce a result of 0? (If the list is short, write it out. If the list is long, describe a pattern of what the numbers are).

3. Using a clock size of 13, can you find a number to input that produces a result of 13? (If so, what is it? If not, why not?)

4. Using a clock size of 13, find the answers to the following:

<table>
<thead>
<tr>
<th>1 MOD 13</th>
<th>1,000 MOD 13</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 MOD 13</td>
<td>10,000 MOD 13</td>
</tr>
<tr>
<td>100 MOD 13</td>
<td>100,000 MOD 13</td>
</tr>
</tbody>
</table>

Are these results surprising or interesting? Why or why not?
Step 2: Toward encryption - Use multiplication to produce inputs

Experiment - Small changes to inputs, big changes to outputs.
Using a clock size of 37, let’s multiply two numbers (we’ll call them A and B) to use as input, then make small changes to each while holding the other constant. We’ll always use the formula \( A \times B \mod M \). We’ll start with \( A=20 \) and \( B=50 \) and \( M=37 \). So here is the first result...

\[
20 \times 50 \mod 37 = 1
\]

Now find in the rest of these values making small adjustments to A and B individually. Use a calculator, if necessary, to compute \( A \times B \). Use the Mod Clock to compute the modulus of the result.

<table>
<thead>
<tr>
<th>Increment A</th>
<th>Increment B</th>
</tr>
</thead>
<tbody>
<tr>
<td>[21 \times 50 \mod 37]</td>
<td>[20 \times 51 \mod 37]</td>
</tr>
<tr>
<td>[22 \times 50 \mod 37]</td>
<td>[20 \times 52 \mod 37]</td>
</tr>
<tr>
<td>[23 \times 50 \mod 37]</td>
<td>[20 \times 53 \mod 37]</td>
</tr>
</tbody>
</table>

Result: What do you notice about these results? Is there a pattern? Could you predict the result of \( 25 \times 50 \mod 37 \)?

Experiment 2 - Guessing inputs is hard?: Using a clock size of 101, we’ll give you the value of A and even hold the result of the modulo operation constant. Your task: find a value for B (the blank) that makes the math work out.

1) \[2 \times \_\_\_ \mod 101 = 1\]  
2) \[3 \times \_\_\_ \mod 101 = 1\]  
3) \[4 \times \_\_\_ \mod 101 = 1\]

Takeaways:
Solving modulus equations like \[2 \times \_\_\_ \mod 101 = 1\] is “hard” because you can’t solve it like a typical equation. There are no easy patterns or shortcuts like other equations you might see in a math class. As you learned in step 1 (hopefully) there is an infinite list of single values for which \( \_\_\_ \mod 101 = 1 \). (The list is 1, 102, 203, 304, 405...etc). With multiplication, to solve \[2 \times \_\_\_ \mod 101 = 1\] you end up randomly guessing to find some number to multiply by 2 that gives you a result in that list.

Things get especially “hard” when you use a prime number as the clock size. Thanks to some special properties of prime numbers with a prime clock size there’s only one solution to each modulus equation. You are guaranteed that there is the number less than the clock size itself, but there are still 100 different values you have to try. With a brute force search you could go through them all in a couple minutes. But what if the clock size were a 50-digit prime number?

Encryption!
Whenever you have a problem for which the only way to solve it is by random guessing or brute force search over a large range of values, you have a candidate for a encryption. Next you’ll get to try it!