Analysis
Lesson 26
Natural Log and L'hopital's Rule

This lesson has two parts which can be done on different days.

For students who do not have their calculus textbook, you may wish to use the [free calculus textbook] that MIT has posted online. It is better to usethe calculus textbook you learned from because it will use the notation you are familiar with but if you are needing a reference the MIT one can be used.

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Part I: Natural Log,
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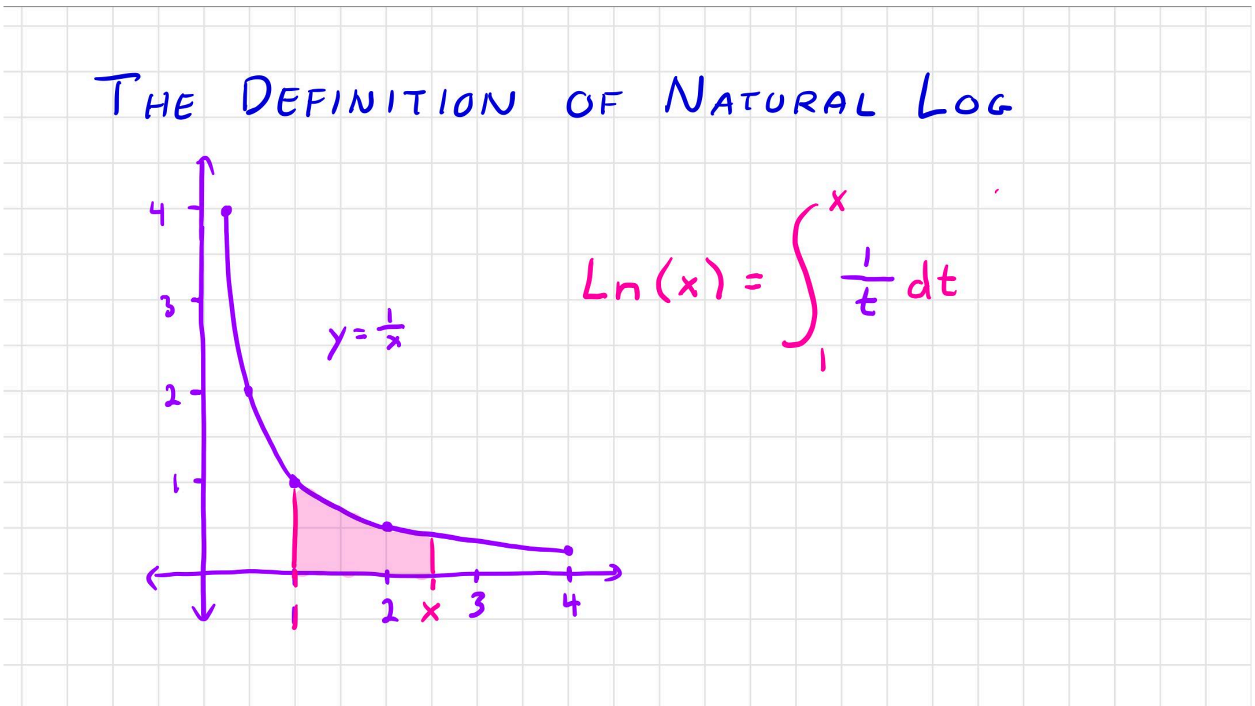
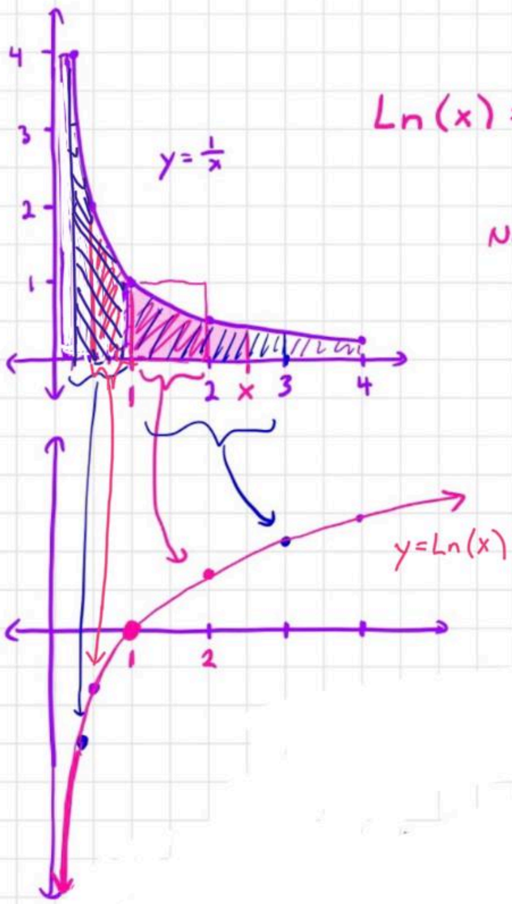
Watch [Playlist Ln-1to9].





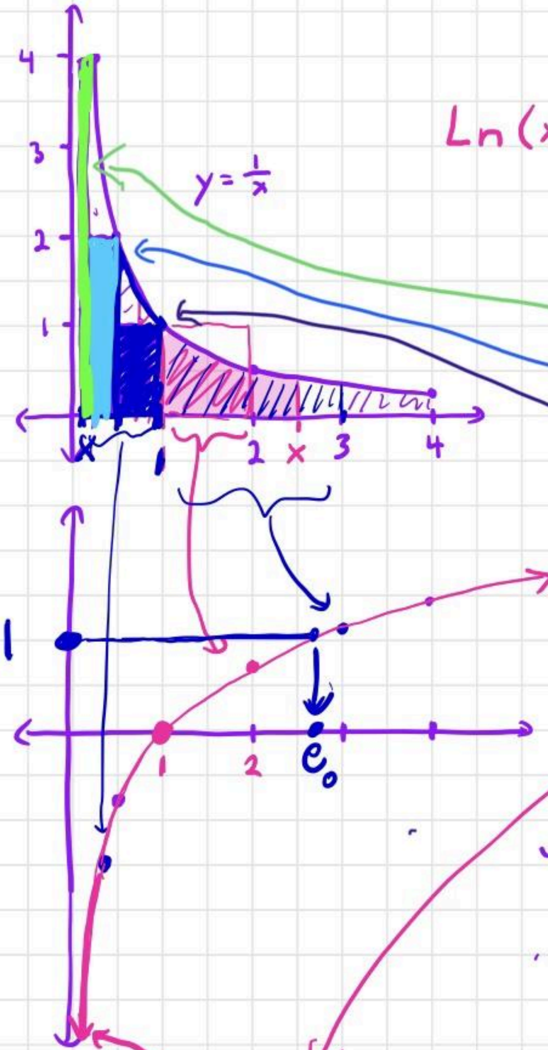
$$Ln(x) = \int_1^x \frac{1}{t}\,dt$$

Note that:

- $Ln(1) = \int_1^1 \frac{1}{t}dt = 0$ for $x = 1$
- $Ln(x) > 0$ for $x > 1$
- $Ln(x) < 0$ for $x < 1$

$$Ln(x) = \int_1^x \frac{1}{t}\,dt$$

has an asymptote as $x \to 0$



Prove this as classwork

x near 0, $x < 1$

$$Ln(x) = \int_1^x \frac{1}{t}\,dt$$

$$-Ln(x) = \int_x^1 \frac{1}{t}\,dt$$

$$= \begin{pmatrix}\text{area under } y = \frac{1}{t} \\ \text{from } t = x \text{ to } t = 1\end{pmatrix}$$

$$> \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(4)$$

if $x < \frac{1}{2}$ for $x < \frac{1}{4}$ for $x < \frac{1}{8}$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} +$ more as x gets smaller

$$\lim_{x \to 0} Ln(x) = -\infty$$

$-Ln(x) > R$

$\forall R > 0\ \exists \delta_R > 0$ s.t. $x \in (0, \delta_R) \Rightarrow Ln(x) < -R.$

$$-Ln\left(\left(\frac{1}{2}\right)^N\right) > \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(4) + \dots + \left(\frac{1}{2}\right)^N (2^{N-1})$$

$$= \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{N \text{ times}} = \frac{N}{2}$$

Scratch

Proof: (1) Given $R>0$ $\exists \delta_R = \boxed{} > 0$

(2) Whenever $x \in (0, \delta_R)$ we have

$$\underline{}$$

solve up

(final) $\ln(x) < -R$

Proof: ① Given $R > 0$ $\exists \delta_R = \boxed{\left(\frac{1}{2}\right)^{2R+1}} > 0$

② Whenever $x \in (0, \delta_R)$ we have

③ for $N > 2R$, $x < \left(\frac{1}{2}\right)^N$

④ $\int_x^1 \frac{1}{t}\,dt > \frac{N}{2} > R$ for $x < \left(\frac{1}{2}\right)^N$

⑤ $\int_x^1 \frac{1}{t}\,dt > R$

⑥ $-\ln(x) > R$

(final) $\ln(x) < -R$

fill in justifications

Classwork

Key Idea

Area under $y = \frac{1}{t}$

from 0 to 1

is infinite

So $\lim\limits_{x \to 0} -\ln(x) = \infty$

So $\lim\limits_{x \to 0} \ln(x) = -\infty$

Classwork: complete the above proof in two column form. Also carefully plot y=1/t and draw the rectangles beneath the curve.

THM : $\ln(ab) = \ln(a) + \ln(b)$

PROOF : (Classwork)

① $\ln(ab) = \int_{t=1}^{ab} \frac{1}{t}\,dt$ ① BY DEFN OF LN.

② $= \int_{s=1/a}^{ab/a} \frac{1}{as}\,a\,ds$ ② CHANGE OF VARIABLES $s = t/a$ so $a\,ds = dt$

③ $= \int_{1/a}^{b} \frac{1}{s}\,ds$ ③ cancelling $\frac{ay}{a} = y$

④ $= \int_{s=1/a}^{1} \frac{1}{s}\,ds + \int_{1}^{b} \frac{1}{s}\,ds$ ④ ADDITION OF INTEGRALS

⑤ $= \int_{t=1}^{a} \frac{1}{(t/\cancel{a})}\frac{dt}{\cancel{a}} + \int_{1}^{b} \frac{1}{s}\,ds$ ⑤ CHANGE OF VARIABLES $s = t/a$ so $a\,ds = dt$, $t = s \cdot a$

⑥ $= \ln(a) + \ln(b)$ ⑥ DEFINITION OF LN.

QED

HW : $\ln(a/b) = \ln(a) - \ln(b)$

HW : $\ln(a^c) = c\ln(a)$

HW1: Prove the first HW above imitating the proof of the theorem we just proved.

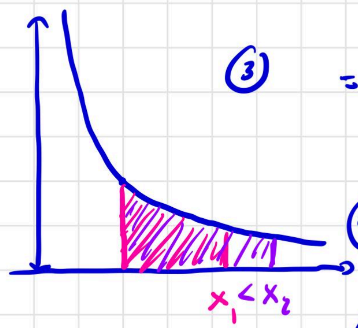
HW2: Prove the second HW above for natural numbers c by induction using the theorem we just proved above.

THM: Ln is a strictly increasing function.

If $x_1 < x_2$ THEN $Ln(x_1) < Ln(x_2)$.

Proof: ① Given any $x_1 < x_2$ ① Given.

② $Ln(x_2) - Ln(x_1) = \int_1^{x_2} \frac{1}{t}\,dt - \int_1^{x_1} \frac{1}{t}\,dt$ ② by defn of Ln

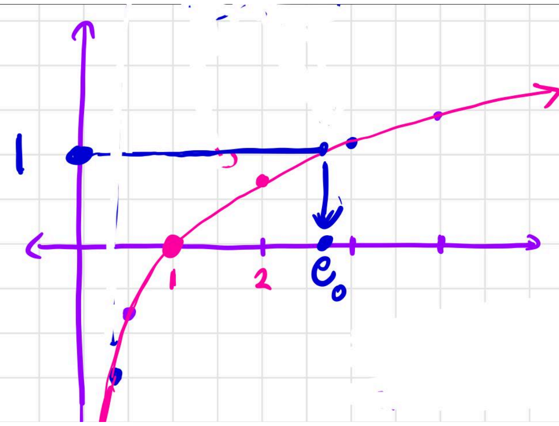


③ $= \int_{x_1}^{x_2} \frac{1}{t}\,dt$ ③ by subtraction of integrals

④ = Area under $y = \frac{1}{x} > 0$ between $x = x_1$ and $x = x_2$ ④ $x_1 < x_2$ and integration is area

⑤ > 0 ⑤ Area is positive

QED



because Ln is strictly increasing

$\exists!$ a unique real number where $Ln(x) = 1$

Call it e_0 $Ln(e_0) = 1$

Later prove $e_0 = e$

DEFN: $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ THM: This limit exists.

classwork

$\left(1+\frac{1}{1}\right)^1 = (1+1)^1 = 2$

$\left(1+\frac{1}{2}\right)^2 = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}\right) = 1+2\cdot1\cdot\frac{1}{2}+\frac{1}{4} = 1+1+\frac{1}{4} = 2.25$

$\left(1+\frac{1}{3}\right)^3 = \left(1+\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3}\right) = 1^3+3\cdot1^2\frac{1}{3}+3\cdot1\frac{1}{3}^2+\left(\frac{1}{3}\right)^3$

$= 1+1+\frac{1}{3}+\left(\frac{1}{3}\right)^3 = 2.\ldots$

```
    1
   1 1
  1 2 1
 1 3 3 1
1 4 6 4 1
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Pascal's Triangle

$\left(1+\frac{1}{4}\right)^4 = 1^4+4\cdot1^3\left(\frac{1}{4}\right)^1+6\cdot1^2\left(\frac{1}{4}\right)^2+4\cdot1\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4$

$= 1+1+\frac{3}{8}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^4$

Increasing Sequence

Prove $\left(1+\frac{1}{n}\right)^n$ is increasing and bounded above

Thus $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists.

by Monotone Conv Thm.

HW3 Write the show for proving this is an increasing sequence and make this the final line. Extra Credit for proving this. Do not use Ln in the proof but you may use Log base 10. Also Extra Credit for proving the sequence is bounded above (very difficult)

Today we will show

$$\ln(e) = 1$$

$$\int_1^e \frac{1}{t}\,dt = 1$$

THM $\displaystyle\int_1^e \frac{1}{t}\,dt = 1$ DEFN: $\displaystyle e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

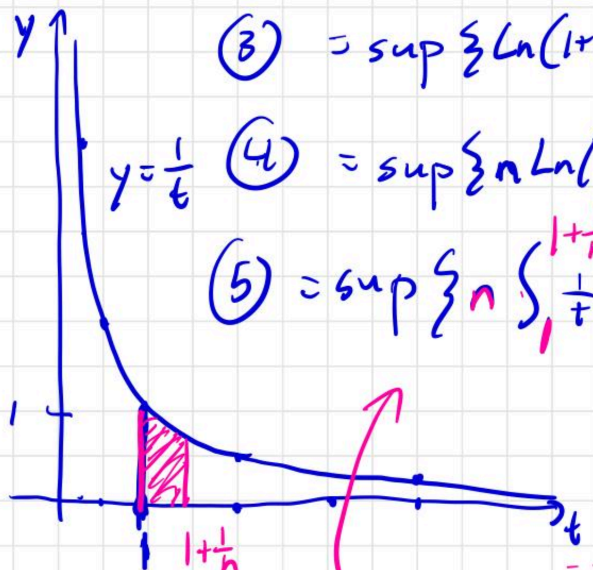
PROOF: ① $\displaystyle\int_1^e \frac{1}{t}\,dt = \lim_{n\to\infty}\int_1^{(1+\frac{1}{n})^n} \frac{1}{t}\,dt$

② $\displaystyle = \sup\left\{\int_1^{(1+\frac{1}{n})^n} \frac{1}{t}\,dt \;\middle|\; n\in\mathbb{N}\right\}$

③ $= \sup\left\{\operatorname{Ln}\left(1+\frac{1}{n}\right)^n \;\middle|\; n\in\mathbb{N}\right\}$

④ $= \sup\left\{n\operatorname{Ln}\left(1+\frac{1}{n}\right) \;\middle|\; n\in\mathbb{N}\right\}$

⑤ $\displaystyle = \sup\left\{n\int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \;\middle|\; n\in\mathbb{N}\right\}$



① $\lim_{n\to\infty} b_n = b$ then $\displaystyle\lim_{n\to\infty}\int_a^{b_n} f(x)\,dx = \int_a^b f(x)\,dx$

when f is continuous
Prove this for Extra Credit (very challenging)

② Monotone Inc Sequence because the upper bound is strictly increasing
the integrand $\frac{1}{t}>0$

③ by defn of Ln

④ by $\operatorname{Ln}(a^c) = c\operatorname{Ln}(a)$ that was proven for HW
could prove by induction for $c\in\mathbb{N}$

⑤ by defn of Ln

Show this sup = 1 by showing

- 1 is an upper bound
- 1 is the smallest upper bound.

THM $\int_1^e \frac{1}{t}\,dt = 1$ DEFN: $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

PROOF: ① $\int_1^e \frac{1}{t}\,dt = \lim_{n\to\infty} \int_1^{(1+\frac{1}{n})^n} \frac{1}{t}\,dt$

② $= \sup\left\{ \int_1^{(1+\frac{1}{n})^n} \frac{1}{t}\,dt \;\middle|\; n\in\mathbb{N} \right\}$

③ $= \sup\left\{ Ln\left(1+\frac{1}{n}\right)^n \;\middle|\; n\in\mathbb{N} \right\}$

④ $= \sup\left\{ n\,Ln\left(1+\frac{1}{n}\right) \;\middle|\; n\in\mathbb{N} \right\}$

⑤ $= \sup\left\{ n \int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \;\middle|\; n\in\mathbb{N} \right\}$

① $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} \int_a^{b_n} f(x)\,dx = \int_a^b f(x)\,dx$ when f is continuous

Prove this for Extra Credit (very challenging)

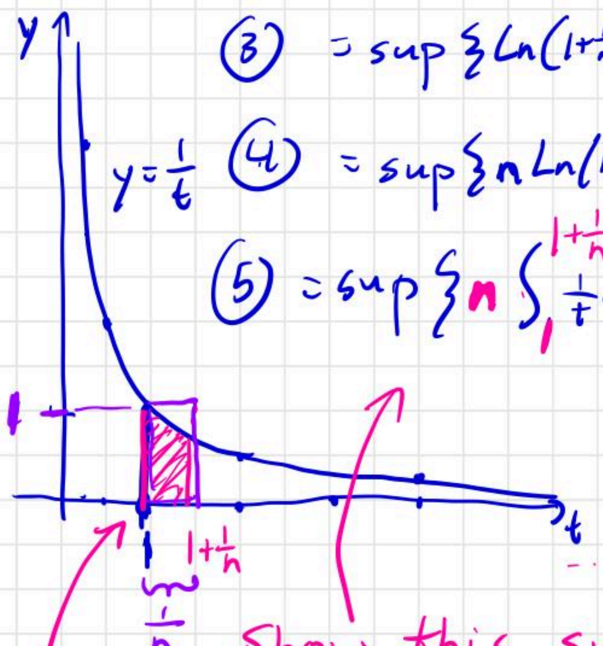
② Monotone Inc Sequence because the upper bound is strictly increasing

the integrand $\frac{1}{t} > 0$

③ by defn of Ln

④ by $Ln(a^c) = c\,Ln(a)$ that was proven for HW could prooue by induction for $c\in\mathbb{N}$

⑤ by defn of Ln



Show this sup = 1 by showing

* 1 is an upper bound
* 1 is the smallest upper bound.

⑥ $\int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \le$ Area of (Rectangle of height = 1 and width = $\frac{1}{n}$) $= 1\cdot\left(\frac{1}{n}\right) = \frac{1}{n}$

So $n\int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \le n\cdot\frac{1}{n} = 1$

⑥ Integrals are areas

(7) Thus $\sup\left\{ n\int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \;\middle|\; n\in\mathbb{N} \right\} \le 1$

(7) by defn of sup is the smallest upper bound.

(8) To prove the sup = 1 we can use a proof by contradiction:

(8.1) Assume on the contrary $\exists$ a upper bound $B < 1$

(8.1) indirect hyp

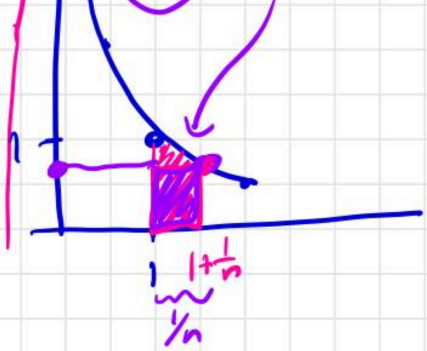
(8.2) so $n\int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \le B$

(8.2) by defn upper bound

(8.3) $\int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \le \frac{B}{n}$

(8.3) $a \le b$ and $c = \frac{1}{n} > 0 \Rightarrow ac \le bc$

(8.4) Area $\square \le \frac{B}{n}$



(8.4) Area $\square \le \int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt$

height is $\frac{1}{t}$ where $t = 1+\frac{1}{n}$

width is $\frac{1}{n}$

(8.5) $\frac{1}{n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)} \le \frac{B}{n}$

(8.5) area of rectangle is height × width

(8.6) $1 \le \frac{B}{\cancel{n}}\,\cancel{n}\left(1+\frac{1}{n}\right)$

(8.6) $a \le b$ and $c = n\left(1+\frac{1}{n}\right) > 0 \Rightarrow ac \le bc$

(8.7) $1 \le B + B\frac{1}{n}$

(8.7) distribution

Solve for n to get a $\otimes$

Fill in Justification

8.8 $1 - B \le B \cdot \frac{1}{n}$

Classwork

8.9 $(1-B)n \le B$

$1 > B$

$\Rightarrow 1 - B > 0$

$\Rightarrow \frac{1}{1-B} > 0$

8.10 $\forall n \in \mathbb{N}$ $n \le \frac{B}{(1-B)}$

⊗ Archimedes Principle

Thus $\sup\left\{ n \int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \mid n \in \mathbb{N} \right\} = 1$. Done with step 8.

9 $\int_1^e \frac{1}{t}\,dt = \sup\{\cdots\} = 1$ (9) by steps 1-8.

Thus $\boxed{\ln(e) = \int_1^e \frac{1}{t}\,dt = 1}$ QED

DEFN: $\mathrm{Ln}(x) = \int_1^x \frac{1}{t}\,dt$ DEFN: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

THM: $\mathrm{Ln}(x)$ is the inverse function for e^x

Thus: $\mathrm{Ln}(x) = \mathrm{Log}_e(x)$ where $\mathrm{Log}_b(a) = c \Leftrightarrow b^c = a$

BEFORE PROVING THIS THM RECALL:

Given $f: A \to B$ has an inverse if $\exists\, f^{-1}: B \to A$

such that $\begin{cases} \forall a \in A\ f^{-1}(f(a)) = a \\ \forall b \in B\ f(f^{-1}(b)) = b \end{cases}$

Thus to prove our thm that $\mathrm{Ln}: (0,\infty) \to (-\infty,\infty)$ is the inverse of $\exp: (-\infty,\infty) \to (0,\infty)$ where $\exp(x) = e^x$

We must show $\begin{cases} \forall a \in (-\infty,\infty)\ \mathrm{Ln}(\exp(a)) = a \\ \forall b \in (0,\infty)\ \exp(\mathrm{Ln}(b)) = b \end{cases}$

Thus to prove our thm that $Ln: (0,\infty) \to (-\infty,\infty)$
is the inverse of $\exp: (-\infty,\infty) \to (0,\infty)$
where $\exp(x) = e^x$

We must show $\begin{cases} \forall a \in (-\infty,\infty) \ \ Ln(\exp(a)) = a & \leftarrow \text{Part I} \\ \forall b \in (0,\infty) \ \ \exp(Ln(b)) = b & \leftarrow \text{Part II} \end{cases}$

Part I Prove $\forall a \in (-\infty,\infty)$ $Ln(e^a) = a$

① $Ln(e^a) = \int_1^{e^a} \frac{1}{t}\,dt$	① by defn of Ln
② $= a\,Ln(e)$	② by property $Ln(a^c) = c\,Ln(a)$ proven for HW
③ $= a \cdot 1 = a$	③ we proved $Ln(e) = 1$

QED Part I

Part II Prove $\forall b \in (0,\infty)$ $e^{Ln(b)} = b$

Extra Credit complete this part
very tricky.

Hint for Part II extra credit:

First prove the derivative of exp(x)=exp(x) using the limit definition of derivative and the limit defn of e.

Next use the fundamental theorem of calculus to show
the derivative of Ln(x) is 1/x

Let F(x) =exp(Ln(x)) and work to show F(x)=x as follows

First note F(1)=exp(Ln(1))=exp(0)=1 by defn of exp and Ln

Use the chain rule to show F'(x)=exp(Ln(x))(1/x)=F(x)/x for all x>0

Thus F'(x)/F(x)=1/x for all x>0

Integrate both sides from 1 to b

On the left integral do u sub u=F(x)

integral from F(1) to F(b) of 1/u du = integral from 1 to b of 1/x dx

Since F(1)=0 we have Ln(F(b))=Ln(b)

Since Ln is strictly increasing it is one to one thus F(b)=b

Thus exp(Ln(b))=b.

For extra credit write this as a two column proof.

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Part 2: L'hopital's Rule

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Watch Playlist L'hop-1to6. If you are far behind schedule you may watch only the first three videos in this playlist.

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MAT320S21

Scratchwork MAT320S21 Axiomatic Geometry

L'hopital's Rule

"If "$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$" then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ "

Not written rigorously

Theorem: If ① f and g are continuously differentiable

and if ② $\lim_{x \to x_0} f(x) = 0$ and $\lim_{x \to x_0} g(x) = 0$

and if ③ $\lim_{x \to x_0} f'(x) = F$ and $\lim_{x \to x_0} g'(x) = G$

<u>then</u> $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{F}{G}$. ← is a rigorous statement

Proof: (classwork)

Given: ①, ② and ③ above

Show: $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{F}{G}$

Proof:

① $\lim_{x \to x_0} \frac{f(x)}{g(x)} =$

$=$

$=$

final $= \frac{F}{G}$

try

Proof:

(1) $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)-0}{g(x)-0}$$

(1) $a - 0 = a$

(2) $$= \lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)}$$

(2) $f(x_0) = \lim_{x \to x_0} f(x) = 0$

given (1) f and g are contin diffble   given (2)

$g(x_0) = \lim_{x \to x_0} g(x) = 0$

(3) $$= \lim_{x \to x_0} \frac{f'(c_{x,x_0})(x-x_0)}{g'(a_{x,x_0})(x-x_0)}$$

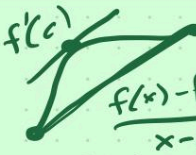
where $c_{x,x_0} \in (x, x_0)$ or (x_0, x)

$a_{x,x_0} \in (x, x_0)$ or (x_0, x)

(3) by mean valu theorem which holds for f and g by given (1) f, g are contin diffble

(4) $$= \lim_{x \to x_0} \frac{f'(c_{x,x_0})}{g'(a_{x,x_0})}$$

(4) by $\frac{ab}{cb} = \frac{a}{c}$



$$\frac{f(x)-f(x_0)}{x-x_0} = f'(c)$$

$$f(x)-f(x_0) = f'(c)(x-x_0)$$

(5) $$= \frac{\lim_{x \to x_0} f'(c_{x,x_0})}{\lim_{x \to x_0} g'(a_{x,x_0})}$$

(5) true as long as $g' \neq 0$ by div of limits

(6) $$= \frac{\lim_{c \to x_0} f'(c)}{\lim_{a \to x_0} g'(a)}$$

(6) $c_{x,x_0} \to x_0$

$a_{x,x_0} \to x_0$ by squeeze thm when $x \to x_0$

(7) $$= \frac{F}{G}$$

(7) by third given

QED

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SCREEN RECORDING now

Screen Recording video saved to Photos

Scratchwork | MAT320S21 | Axiomatic Geometry

More General L'hopital's Rule

$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$ then diff top + bottom

$= \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \frac{0}{0}$ repeat

$= \lim_{x \to x_0} \frac{f''(x)}{g''(x)} = \frac{0}{0}$ repeat

$\cdots = \lim_{x \to x_0} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \frac{F}{G}$

How to prove this rule works?

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MAT320S21

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Theorem: If $f, g \in C^{(n)}$ which means n continuous derivatives

and if $\lim_{x \to x_0} f^{(j)}(x) = 0$ for $j = 0, 1, 2 \ldots n-1$

and if $\lim_{x \to x_0} g^{(j)}(x) = 0$ for $j = 0, 1, 2 \ldots n-1$

and if $\lim_{x \to x_0} f^{(n)}(x) = F$ and $\lim_{x \to x_0} g^{(n)}(x) = G \neq 0$

then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{F}{G}$ True $\forall n \in \mathbb{N}$.

We will prove this using proof by induction (classwork)

Theorem: If $f, g \in C^{(n)}$ which means n continuous derivatives

and if $\lim_{x\to x_0} f^{(j)}(x) = 0$ for $j = 0, 1, 2 \ldots n-1$

and if $\lim_{x\to x_0} g^{(j)}(x) = 0$ for $j = 0, 1, 2 \ldots n-1$

and if $\lim_{x\to x_0} f^{(n)}(x) = F$ and $\lim_{x\to x_0} g^{(n)}(x) = G \neq 0$

then $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{F}{G}$ True $\forall n \in \mathbb{N}$.

We will prove this using proof by induction (classwork)

Proof by Induction

Base Case $\boxed{n=1}$ We proved this in Thm I

Given $f+g$ have $n=1$ contin derivative $\quad n-1=0$

$f^{(0)}(x) \to 0 \quad g^{(0)}(x) \to G$

$f'(x) \to F \quad g'(x) \to G$

Show $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{F}{G}$

Induction Step n^{th} step; Ind Hyp

f, g have n contin deriv

$\lim_{x\to x_0} f^{(j)}(x) = 0 \quad \lim_{x\to x_0} g^{(j)}(x) = 0$ for $j = 1 \ldots n-1$

$\lim_{x\to x_0} f^{n}(x) = F \quad \lim_{x\to x_0} g^{n}(x) = G \neq 0$

any f and g with these properties

then $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{F}{G}$

$n+1^{st}$ step

$\tilde{f}, \tilde{g}$ have $n+1$ contin deriv

$\lim_{x\to x_0} \tilde{f}^{(j)}(x) = 0 \quad \lim_{x\to x_0} \tilde{g}^{(j)}(x) = 0$ for $j = 1 \ldots (n+1-1)$

$\lim_{x\to x_0} \tilde{f}^{n+1}(x) = \tilde{F} \quad \lim_{x\to x_0} \tilde{g}^{n+1}(x) = \tilde{G}$

then $\lim_{x\to x_0} \dfrac{\tilde{f}(x)}{\tilde{g}(x)} = \dfrac{\tilde{F}}{\tilde{G}}$

$\tilde{f}, \tilde{g}$ here are new functions

Proof of the induction step:

(1) Given $\tilde{f}, \tilde{g}$ have n+1 contin deriv

$\lim_{x \to x_0} \tilde{f}^{(j)}(x) = 0 \quad \lim_{x \to x_0} \tilde{g}^{(j)}(x) = 0$ for $j = 1 \ldots (n+1-1)$

$\lim_{x \to x_0} \tilde{f}^{n+1}(x) = \tilde{F} \quad \lim_{x \to x_0} \tilde{g}^{n+1}(x) = \tilde{G}$

(2) $\lim_{x \to x_0} \frac{\tilde{f}(x)}{\tilde{g}(x)} =$

cannot apply the Induction Hyp to $\tilde{f}$ and $\tilde{g}$ because $\lim_{x \to x_0} \tilde{g}^{(n)}(x) = 0$ and $G \neq 0$

(final) $= \frac{\tilde{F}}{\tilde{G}}$

Instead Imitate the proof of Thm I.

Proof of the induction step:

(1) Given $\tilde{f}, \tilde{g}$ have $n+1$ contin deriv

$\lim_{x \to x_0} \tilde{f}^{(j)}(x) = 0 \quad \lim_{x \to x_0} \tilde{g}^{(j)}(x) = 0$ for $j = 1 \ldots (n+1-1)$

$\lim_{x \to x_0} \tilde{f}^{n+1}(x) = \tilde{F} \quad \lim_{x \to x_0} \tilde{g}^{n+1}(x) = \tilde{G}$

(1) Given

(2) $\lim_{x \to x_0} \dfrac{\tilde{f}(x)}{\tilde{g}(x)} = \lim_{x \to x_0} \dfrac{\tilde{f}(x) - 0}{\tilde{g}(x) - 0}$

(2) $a - 0 = a$

(3) $= \lim_{x \to x_0} \dfrac{\tilde{f}(x) - \tilde{f}(x_0)}{\tilde{g}(x) - \tilde{g}(x_0)}$

(3) $\tilde{f}(x_0) = \lim_{x \to x_0} \tilde{f}(x) = 0$

given (1) $\tilde{f}$ and $\tilde{g}$ are contin diffble

given (2)

$\tilde{g}(x_0) = \lim_{x \to x_0} \tilde{g}(x) = 0$

(4) $= \lim_{x \to x_0} \dfrac{\tilde{f}'(c_{x,x_0})(x - x_0)}{\tilde{g}'(a_{x,x_0})(x - x_0)}$

where $c_{x,x_0} \in (x, x_0)$ or (x_0, x)

$a_{x,x_0} \in (x, x_0)$ or (x_0, x)

(4) by mean valup theorem which holds for f and g by given (1) f, g are contin diffble

(5) $= \lim_{x \to x_0} \dfrac{\tilde{f}'(c_{x,x_0})}{\tilde{g}'(a_{x,x_0})}$

(5) by $\dfrac{b}{cb} = \dfrac{a}{c}$

(6) $= \dfrac{\lim_{x \to x_0} \tilde{f}'(c_{x,x_0})}{\lim_{x \to x_0} \tilde{g}'(a_{x,x_0})}$

(6) true as long as $g \neq 0$ by div of limits

Copied from proof of Thm I

(5) $= \lim_{x \to x_0} \dfrac{\tilde{f}'(c_{x,x_0})}{\tilde{g}'(a_{x,x_0})}$ — (5) by $\frac{b}{cb} = \frac{a}{c}$

Copied from proof of Thm I

(6) $= \dfrac{\lim_{x \to x_0} \tilde{f}'(c_{x,x_0})}{\lim_{x \to x_0} \tilde{g}'(a_{x,x_0})}$ — (6) true as long as $\tilde{g} \neq 0$ by div of limits

could be done better

(7) $= \dfrac{\lim_{c \to x_0} \tilde{f}'(c)}{\lim_{a \to x_0} \tilde{g}'(a)}$ — (7) $c_{x,x_0} \to x_0$, $a_{x,x_0} \to x_0$ by squeeze thm when $x \to x_0$

(8) $= \lim_{x \to x_0} \dfrac{\tilde{f}'(x)}{\tilde{g}'(x)}$ — (8) lim of fraction is fraction of limit.

(9) $= \lim_{x \to x_0} \dfrac{\tilde{f}'^{(n)}(x)}{\tilde{g}'^{(n)}(x)}$ — (9) by Ind Hyp taking $f(x) = \tilde{f}'(x)$, $g(x) = \tilde{g}'(x)$

Must check the rules* work (classwork)

(10) $= \lim_{x \to x_0} \dfrac{\tilde{f}^{(n+1)}(x)}{\tilde{g}^{(n+1)}(x)}$ — (10) by defn of n+1 st derivative

(11) $= \dfrac{\tilde{F}}{\tilde{G}}$

QED

$\boxed{HW}$ Go to your calculus textbook and do a few examples of applications of L'hopital's Rule