**TAKAPUNA GRAMMAR SCHOOL**
**MATHEMATICS DEPARTMENT**

# SCHOLARSHIP CALCULUS

# WORD QUESTIONS

**AIM TO COMPLETE BY START OF TERM 3**

*Remember, the Scholarship paper will have at least one formal "word problem" part-question.*
*Write up to half a page on each of the following including diagrams or examples where appropriate.*
*This should be formally set out.*

Exercise 1 : Calculus

1 $y = r(x)$ is a relation in x. Describe the limitations on y if $y = r(x)$ is to be differentiable in the interval $[a, b]$. *(this only needs 3 brief statements)*

2 When $f(x) = \frac{ax+b}{cx+d}$ is differentiated, the differential takes the form : $f'(x) = \frac{k}{(cx+d)^2}$

Explain how the graph of $y = f'(x)$ illustrates the graph of $y = f(x)$.

3 Explain why it is unreliable to decide on the nature of a turning point by evaluating the y-values at the TPs.

4 For a smooth function f(x) to have a maximum at x = q, is it sufficient that $f'(q) = 0$?

5 The graph of a cubic function has a tangent at point P (x, y) on the curve. If the equations of the cubic and the tangent are solved simultaneously, describe the form of the resulting function in x.

6 Explain, with examples, how an integral might be zero but the area is not zero.

Exercise 2 : Co-ordinate Geometry

1 Make some general comments about the shape of a curve with 2 vertical asymptotes (at x=3 and at x=6). Discuss aspects such as the number and nature of any turning points, asymptotes, concavity.

2 For any numbers m and c, the set of points (x, y) which satisfy $y = mx + c$ lie on a straight line.

3 A relation is given parametrically by : $x = a + r\cos\theta$ and $y = b + r\sin\theta$ where a and b are constant. Either r or θ is also a constant and the other is a variable. Discuss how the resulting graph will change depending on which of r or θ varies.

Exercise 4 : Algebra

1 Explain in detail why the sum and product of two conjugate complex numbers are real.

2 Explain why the solutions of a quadratic equation with real coefficients must be conjugates.

3 Let $y = at^3 + bt^2 + ct + d = 0$, where a, b, c, d are all real constants. Explain carefully why this cubic equation must have at least one real root and at most three real roots. *(Schol Algebra 1988)*

4 $y = f(x)$ is a quadratic function. What restrictions are there on this function if it is positive definite (i.e. always lies above the x-axis)?

5 Show that the distance $d$ between two points $Z_1$ and $Z_2$ which represent $z_1 = x_1 + y_1i$ and $z_2 = x_2 + y_2i$ respectively in the complex plane, is given by $d = |z_1 - z_2|$.

6 Prove the proposition, known as the Triangle Inequality, that if $z_1$ and $z_2$ are two complex numbers then $|z_1 + z_2| \le |z_1| + |z_2|$.

**SOLUTIONS**

Exercise 1 : Differentiation

1 This question is pure book theory from notes. The definition of differentiability requires :

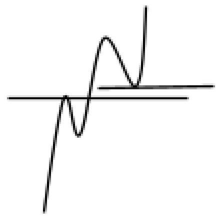
- y = r(x) must be a function not merely a relation.
- For all points x=k in the interval [a, b], both limit as $x \to k$ of the function and the value r(k) must exist and be equal (i.e. continuity).
- There must be a unique tangent at all points in the interval

2 (i) State that the graph is a rectangular hyperbola.
(ii) Checking the differential, k = ad – bc.
Consider how the graph changes depending on k.

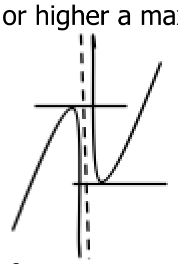
- For k > 0 $f'(x) > 0$ $\forall x$ so gradient is always positive / as x approaches $^{-d}/_{c}$ graph gets steeper / as $x \to \infty$ gradient→0
- For k < 0 $f'(x) < 0$ $\forall x$ so gradient is always negative / as x approaches $^{-d}/_{c}$ graph gets steeper / as $x \to \infty$ gradient→0
- Note asymptote at $x = {}^{-d}/_{c}$

3 Consider various cases :

- If the function is a polynomial, continuous in [a, b], maxima and minima follow alternately BUT if the power is 5th or higher a maximum can be below a minimum.

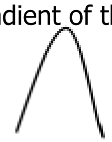
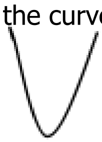


- ///ly for a complex trig wave.

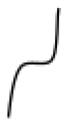
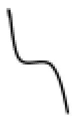


- For a non-rectangular hyperbola, often $y_{max} < y_{min}$ (a rectangular hyperbola has no turning points)

4 For a smooth function f(x) to have a maximum at x = q, is it sufficient that $f'(q) = 0$? No (obviously!) All that $f'(q) = 0$ indicates is that the gradient of the curve is momentarily zero i.e. the curve is horizontal.
There are **5** possible cases :

Two turning point cases : maximum  minimum 

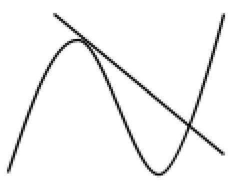
| | | |
|---|---|---|
| Must also have | gradient moving +→0→ - | gradient moving - →0→ + |
| Also <u>might</u> have | $f''(x) < 0$ | $f''(x) > 0$ |

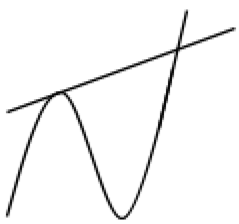
Two point of inflection cases :  or 

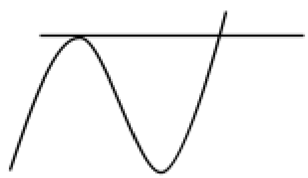
| | | |
|---|---|---|
| <u>Must</u> also have | gradient moving +→0→ + | gradient moving - →0→ - |
| <u>Must</u> also have | $f''(x) = 0$ | $f''(x) = 0$ |

Fifth case is where f(x) = k a horizontal line.

5 When a tangent is drawn to a cubic, it will cut the curve again :







Let the cubic be $y = ax^3 + bx^2 + cx + d$ and the tangent be $y = mx + c$.
Let $(p, ap^3 + bp^2 + cp + p)$ be the point on the cubic where the tangent touches the curve.
Equating the two produces a cubic equation in x : $ax^3 + bx^2 + (c - m)x + (d - c) = 0$
But this represents solutions for p which occur twice at the point where the tangent touches the curve (i.e. a perfect square) and a single value where the tangent meets the curve again.

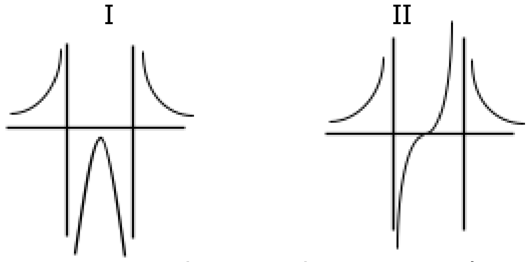
$\Rightarrow$ the cubic equation takes the form : $(x - p)^2(x - q) = 0$

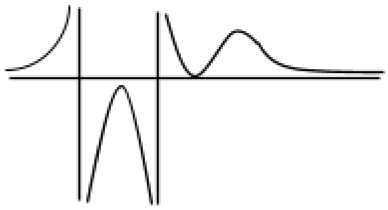
6 The definite integral $\int_a^b f(x)dx$ for a curve y = f(x) lying on the same side of OX between x = a and x = b is defined as the area enclosed by the curve, the x axis and the ordinates x = a and x = b. If this area is above the x axis the definite integral is positive; if below it is negative. Also, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any c such that a < c < b. Hence, if a curve y = f(x) crosses the axis between x = a and x = b at a point (c, 0) then the definite integral has a negative and a positive component. When areas are found, the absolute values of the definite integrals are taken as an area cannot be negative in "real" terms. So if one part had been negative it is added rather than subtracted so the result is different. ⇒ if the two components of area above and below the x axis are equal, there will be an area but the definite integral will be zero.

Exercise 2 : Co-ordinate Geometry

1 Two vertical asymptotes produces two basic types of function graph :

I II



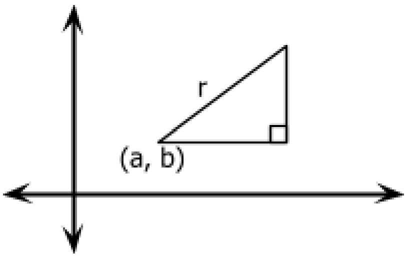


Type I has a turning point between the two vertical asymptotes. It may have the same concavity outside these asymptotes and the opposite between; it may or may not cross the horizontal line of tendency – if it does so concavity will alter :

Type II has no turning points between the vertical asymptotes but, as for type I, it may cross the horizontal line of tendency.

2 For any numbers m and c, the set of points (x, y) which satisfy y = mx + c lie on a straight line. Define the y-intercept as (0,c). Suppose the gradient of this line is = m ⇒ $\frac{y-c}{x-0} = m$ ⇒ rearranging y = mx + c

3




I Consider x = a + r cosθ and y = b + r sinθ and assume that a, b, and r are constant. The locus of the point represented by (x, y) will be a circle centre (a, b) and radius r

II Assume that a, b, and θ are constant then the gradient of the resulting locus is constant so the locus is a line such that (a, b) lies on it and the gradient is tanθ.

Exercise 4 : Algebra

1 By definition if the first number is $z = x + yi$ then $\bar{z} = x - yi$. In this definition, x and y are real numbers.

Sum = $z + \bar{z} = x + yi + x - yi$ = 2x which is real because x is real.

Product = $z\,\bar{z} = (x + yi)(x - yi) = x^2 + y^2$ which is real because x and y are real.

2 Let the quadratic equation be $ax^2 + bx + c = 0$ with a, b and c all real.

I The quadratic formula gives $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ⇒ $x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$

⇒ the solutions take the form of x = A + B and x = A – B.

II If $\sqrt{b^2 - 4ac}$ is irrational, then (A + B) and (A – B) are conjugate surds.

III If $\sqrt{b^2 - 4ac}$ is imaginary, then (A + B) and (A – B) are conjugate complex numbers.

3 Consider $y = at^3 + bt^2 + ct + d = t^3\left(a + \frac{b}{t} + \frac{c}{t^2} + \frac{d}{t^3}\right)$

$\Rightarrow$ the sign of y = the sign of a as $t \to \infty$ and the sign of y = $(-)$ the sign of a as $t \to -\infty$

$\Rightarrow$ y changes sign in its range and there exists a value of t such that y = 0.

Suppose the value of t such that y = 0 is T $\Rightarrow$ (t - T) is a factor of y and $y = (t - T)(at^2 + b_1t + c_1)$ for some $b_1$ and $c_1$ constants $\Rightarrow$ The quadratic $at^2 + b_1t + c_1$ has 0, 1 or 2 real roots depending on whether the discriminant $b_1^2 - 4ac_1 < 0, = 0$ or $> 0$ respectively. Thus the cubic y has 1, 2 or 3 real roots.

4 Let $y = ax^2 + bx + c$ $\Rightarrow$ I the graph will be a parabola if $a \neq 0$.

II this parabola will have an orientation with a minimum if $a > 0$.

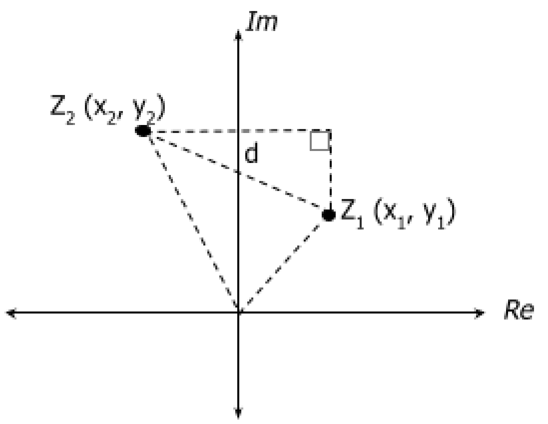
III to be positive definite, the graph must always lie above the axis

$\Rightarrow$ $ax^2 + bx + c = 0$ has no real solutions

5 By definition $|z_1|$ is the distance from $Z_1$ to the origin

i.e. $\sqrt{x_1^2 + y_1^2}$ and $|z_2|$ is the distance from $Z_1$ to the origin

i.e. $\sqrt{x_2^2 + y_2^2}$.




From the diagram : $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

But by definition : $|z_1 - z_2|^2 = |(x_1 - x_2) + (y_1 - y_2)i|$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

Hence $d = |z_1 - z_2|$

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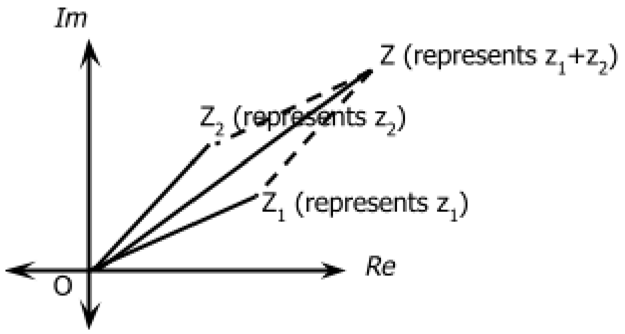


On the complex plane, when you draw an Argand Diagram, the point Z (a, b) represents the complex number z = a + bi.

By completing the parallelogram, $OZ \leq OZ_1 + Z_1Z$ (*equal if* $z_1 = z_2$)

But $Z_1Z = OZ_2$ (properties of the parallelogram)

$\Rightarrow$ $OZ \leq OZ_1 + OZ_2$

By definition, if z = a + bi then $|z| = \sqrt{a^2 + b^2}$ and since Z(a,b) then $|z|$ represents length OZ.

Also OZ = $|z_1 + z_2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$ $\Rightarrow$ $|z_1 + z_2| \leq |z_1| + |z_2|$.