ON LINE COURSE WARE

STREAM: CSE

YEAR: SECOND YEAR

SEMESTER: IV

SUBJECT NAME: NUMERICAL METHODS AND STATISTICS

SUBJECT CODE: M(CSE) 401

TOTAL NO. OF LECTURES: 33

CONTACT HOURS: 33 HOURS.

CREDIT: 3

LESSION PLAN

MODULE NO	LECTURE NO	TOPIC	APPLICATION	REFERENCE BOOK
Module Number I: Numerical Methods I	**Lecture 1**	Approximation in Numerical Computation	• Certain real number is retained up to certain number of digits. • The rejection of unwanted digits is done in rational manner in order to minimize associated error.	1. Balaguruswamy : Numerical methods, Scitech, TMHE. 2. Baburam: Numerical methods, Pearson Education. 3. Dutta N.: Computer Programming & Numerical Analysis, Universities Press. 4. Soumen Guha & Raesh Srivastav: Numerical Methods, Oxford University Press. 5. Srimanta Pal, Numerical Methods, Oxford University Press. 6. Shastri: Numerical Analysis, PHI. 7. S. Ali Mollah, Numerical Analysis, New Central Book Agency. 8. Mathews: Numerical Methods for
	Lecture 2	Fixed and floating-point arithmetic	• Algorithm for representation of real presented in computer for numerical computation.	
	Lecture 3	Interpolation: Introduction to Interpolation	• Function not given in analytic form but the table of values is available at some fixed values. To evaluate the values at other points.	
	Lecture 4	Newton's Forward Interpolation formula	• A formula for evaluating variable at beginning of table.	
	Lecture 5	Newton's Backward Interpolation formula	• A formula for evaluating variable at end of table.	
	Lecture 6	Lagrange's Interpolation formula	• Interpolation formula for unequal intervals for any part of the table.	

	Lecture 7	Newton's Divided Difference Interpolation formula	• The interpolating formula is used mainly for unequally spaced arguments and can be done easily after introducing new arguments.	Mathematics, Science & Engineering PHI 9. Rao G. S.: Numerical Analysis, New Age International. 10. Rao, G. S.: Programmed Statistics (Q & A), New Age International. 11. Pradeep Niyogi: Numerical Analysis & Algorithm, TMH. 12. Dutta N.: Computer Oriented Numerical Mathematics, VIKAS. 13. Arumugam: Numerical Methods, Scitech Publication. 14. Rao: Probability and Statistics Engineers, Scitech Publication. 15. Wayse: Numerical Methods in Computer Application, EPH.
Module Number II:	**Lecture 8**		• Technique for determining the	1. Balaguruswamy : Numerical

Numerical Methods II		Numerical Integration. Trapezoidal Rule	approximate value of integrant $\int_a^b f(x).dx$. • First order simple	methods, Scitech, TMHE. 2. Baburam: Numerical methods, Pearson Education. 3. Dutta N.: Computer Programming & Numerical Analysis, Universities Press. 4. Soumen Guha & Raesh Srivastav: Numerical Methods, Oxford University Press. 5. Srimanta Pal, Numerical Methods, Oxford University Press. 6. Shastri: Numerical Analysis, PHI. 7. S. Ali Mollah, Numerical Analysis, New Central Book Agency. 8. Mathews: Numerical Methods for Mathematics, Science & Engineering PHI 9. Rao G. S.: Numerical Analysis, New Age International. 10. Rao, G. S.: Programmed Statistics (Q & A), New Age International.
	Lecture 9	Simpson's 1/3 Rule for Integration	• Parabolic integration formula respectively.	
	Lecture 10	Example of Trapezoidal Rule and Simpson's 1/3 Rule	• More Exercise of the integration rule.	
	Lecture 11	Numerical solution of system of linear equations Gauss Elimination method	• A system of equation to be solved by numerical technique. • Very much useful in various field of engineering. • Numerical system with one of the variable is eliminated stage wise.	
	Lecture 12	L-U Factorization method with example	• A logic based method with separating square matrix into an upper and lower triangular matrix.	
	Lecture 13	Gauss Seidel Method	• Improvement of Jacobi's method and diagonal dominant.	

				11. Pradeep Niyogi: Numerical Analysis & Algorithm, TMH. 12. Dutta N.: Computer Oriented Numerical Mathematics, VIKAS. 13. Arumugam: Numerical Methods, Scitech Publication. 14. Rao: Probability and Statistics Engineers, Scitech Publication. 15. Wayse: Numerical Methods in Computer Application, EPH.
	Lecture 14	Solution of algebraic and transcendental equations Bisection method	• An algebraic or polynomial Function of a variable (upto degree n) is very important in its application. • Simplest example solution depending of two points (Bisection). • The successive correction of initial approximation (Newton-Raphson).	
	Lecture 15	Regula-Falsi Method		
	Lecture 16	Newton Raphson method		

Module Number II: **Numerical Methods II**	**Lecture 17**	Numerical solution of Ordinary Differential Equation: Euler's method	• The solution of ODE formulated from Taylor's series expansion. • The solution of ODE formulated from dividing interval (a, b) into several sub-intervals (Euler's & modified Euler's method). • The solution of ODE formulated from dividing interval (a, b) avoiding higher derivatives computation & thus simple (Runge Kutta method). • Solution of Partial differential equation is application of various fields of science, engineering & is done with grid points. • A modern technique in evaluating solution of PDE .	
	Lecture 18	Modified Euler's method		
	Lecture 19	Example of above method		
	Lecture 20	Predictor Corrector method		
	Lecture 21	Runge Kutta method		
	Lecture 22	Introduction to Numerical solution of Partial Differential equation		
	Lecture 23	Numerical solution of Partial Differential equation **(Taylor series expansions in more than one dimension)**		
Module Number III: STATISTICS	**Lecture 24**	Mean, Variance and Standard Deviation	Measure of Tendency and deviation	1. Probability and Statisics for Engineers, Rao, Scitech Publication 2. N. G. Das: Statistical Methods, TMH
	Lecture 25	Median and Mode		
	Lecture 26	Preliminaries of Probability	Development of different types probabilistic models using computer simulations	
	Lecture 27	Conditional Probability and Bayes Theorem		
	Lecture 28	Problems on Bayes Theorem		

	Lecture 29	Binomial and Poisson Distribution		
	Lecture 30	Normal Distribution		
	Lecture 31	Probability Mass Function and Distribution		
	Lecture 32	Expectation		
	Lecture 33	Correlation Coefficient and Regression Lines	Data Analysis(Statistical Investigation) and Curve Fitting	

Module I: Numerical Methods I

Total Number of Lectures: 16L

LECTURE 1.

Approximation in Numerical Computation:

1. Introduction

The ultimate aim of the field of numerical analysis is to provide convenient methods for obtaining useful solutions to mathematical problems and for extracting useful information from available solutions which are not expressed in tractable forms. Such problems may each be formulated, for example, in terms of an algebraic or transcendental equation, an ordinary or partial differential equation, or an integral equation, or in terms of a set of such equations.

Significant Figures/Digits:

The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are all significant figures. But 0 is not always a significant figure. It is significant except when it is used to fix decimal point or to fill the place of unknown or discarded digits. As for example in the numbers 2407, 3500, 21780, 5.2307, zero's are significant. But in the numbers 0027, 0.00057, 0.12, 0.7800 zero's are not significant figures.

Errors:

In numerical calculation, sometimes methods are approximated and data are also inaccurate and in most cases result in numerical calculation will have some errors. The errors are basically two types of error is known as truncating error and other is computational error.

Truncation Error:

The truncation error or inherent error is that error which is inherent in numerical method itself. Inherent error or Truncation error is also observed when initial data are not exact. In this case also the error cannot be removed if the problem is solved correctly.

Computational Error:

The computational error arise during arithmetic calculation mainly due to the approximation of numbers. In arithmetic calculation, most numbers are used by retaining finite number of digits. Again in the case of subtraction of nearly two equal numbers, multiplication or division of very small numbers, sometimes loss of significant digits are observed. Accordingly computational error are again classified into two types –one is rounding off error and other is significant error.

Rounding off Error:

These types of error occur in arithmetic calculation due to rounding off numbers. When the given data are not exact number, the data are used in calculation by taking their approximation, some errors occur and they propagate in arithmetic calculation = these errors are known as rounding off error.

The general rule for rounding-off a number to n-significant figures are:

Discard all digit to the right of the nth place, if the discarded number is less than half a unit in the (n+1)th place, leave the nth digit is unchanged; if the discarded number is greater than half a unit in the (n+1)th place, add 1(one) to the nth digit. If the discarded digit is exactly half a unit in the (n+1)th place, leave the nth digit unaltered if it is an even number, but increase it by 1 (one) if it is an odd number.

Example: Correct up to six significant figures :

(i) 24.564932 becomes 24.5649 (ii) 24.564962 becomes 24.5650
(iii) 24.564652 becomes 24.5646 (iv) 24.564752 becomes 24.5648
(v) 24.564650 becomes 24.5646 (vi) 24.564750 becomes 24.5648
(vii) 24.564653 becomes 24.5647

Significant Error:

Significant error arrises due to loss of significant digits during subtraction of nearly two equal numbers, multiplication or division of small number by a second smaller number. As for example in the subtraction of two numbers 0.17381527 and 0.17381526123, if we take the values up to five decimal places, then serious error may be observed when the result is zero. Hence significant error arrises due to the loss of significant digits and not from the data.

Approximate Error:

Approximate error is defined as the difference between true value and approximate value. Let x_T and x_A denotes the true value and approximate/ experimental value of any quantity or number then approximate error is denoted by E_A and is given by

$$E_A = x_T - x_A$$

Absolute error:

The absolute value of the approximate error is known as absolute error. Hence absolute error is given by $|E_A| = |x_T - x_A|$

Relative error:

The ratio of absolute error and true value of a quantity is known as relative error. If E_R is denoted by relative error of a quantity or a number, then relative error is given by

$$E_R = \frac{|E_A|}{x_T}$$

Percentage error:

Percentage error is the error which is 100 times the relative error. If E_P is the percentage error then

$$E_P = \frac{|E_A|}{x_T} \times 100$$

Theorem:

If a number is correct upto n significant figures and the first significant digit is k, then the relative error E_R is less than $\frac{1}{k.10^{n-1}}$.

LECTURE 2.

Fixed and floating-point arithmetic:

Any rounded number can be written in the standard form:

$$0.b_{-1}b_{-2}.....b_{-m}\times 10^{n}$$

Where n is an integer, positive, negative or zero and m is a positive integer and $b_{-1}\neq 0.$ Then the rounded number is said to have m significant digits or figures. The above form of a rounded number is called the floating point representation of the given number.

Propagation of error:

We are aware that our given initial data are affected with round-off error which lies within the limit $\pm\frac{1}{2}$ in the last significant figure. Let us see how these initial round-off errors affect the successive differences in a difference table. If y_i denotes the exact value of $f(x_i)$, then its entered value is $y_i-\varepsilon_i$, where ε_i is the corresponding round-off error and the difference table proceeds as shown in the table below:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	
x_0	$y_0-\varepsilon_0$				
		$\Delta y_0-\varepsilon_1+\varepsilon_0$			
x_1	$y_1-\varepsilon_1$		$\Delta^2 y_0-\varepsilon_2+2\varepsilon_1-\varepsilon_0$		
		$\Delta y_1-\varepsilon_2+\varepsilon_1$		$\Delta^3 y_0-\varepsilon_3+3\varepsilon_2-3\varepsilon_1+\varepsilon_0$	
x_2	$y_2-\varepsilon_2$		$\Delta^2 y_1-\varepsilon_3+2\varepsilon_2-\varepsilon_1$		.....
		$\Delta y_2-\varepsilon_3+\varepsilon_2$		$\Delta^3 y_1-\varepsilon_4+3\varepsilon_3-3\varepsilon_2+\varepsilon_1$	
x_3	$y_3-\varepsilon_3$		$\Delta^2 y_2-\varepsilon_4+2\varepsilon_3-\varepsilon_2$		.....
		$\Delta y_3-\varepsilon_4+\varepsilon_3$		$\Delta^3 y_2-\varepsilon_5+3\varepsilon_4-3\varepsilon_3+\varepsilon_2$	
x_4	$y_4-\varepsilon_4$		$\Delta^2 y_3-\varepsilon_5+2\varepsilon_4-\varepsilon_3$		
		$\Delta y_4-\varepsilon_5+\varepsilon_4$			
x_5	$y_5-\varepsilon_5$				

Example1. Write down the approximate representation of 2/3 correct to four significant figures and hence find
(a)Absolute error (b)Relative error (c)Relative percentage error.

Solution: The approximate representation of 2/3 correct up to four significant figures is 0.6667.

Therefore $X_T = 2/3$ and $X_A = 0.6667$
Absolute error $(E_A) = |X_T - X_A| = |2/3 - 0.6667| = 0.000033$
Relative error $(E_R) = E_A/X_T = 0.000033/(2/3) = 0.0000495 \approx 0.00005$
Relative percentage error $(E_P) = E_R \times 100 = 0.005\%$

LECTURE 3:

Interpolation:

Introduction:
Consider a single valued continuous function y = f (x) defined over [a,b] where f (x) is known explicitly. It is easy to find the values of 'y' for a given set of values of 'x' in [a,b]. i.e., it is possible to get information of all the points (x, y) where a <x < b. But the converse is not so easy. That is, using only the points (x_0 , y_0), (x_1 , y_1) ,…,(x, y) where $a < x_i < b$, i = 0,1,2,..., n , it is not so easy to find the relation between x and y in the form y = f (x) explicitly. That is one of the problems we face in numerical differentiation or integration. Now we have first to find a simpler function, say g(x) , such that f (x) and g(x) agree at the given set of points and accept the value of g(x) as the required value of f (x) at some point x in between a and b. Such a process is called interpolation.

In linear interpolation, we are given with two pivotal values $f_0 = f(x_0)$ and $f_1 = f(x_1)$,
and we approximate the curve of f by a chord (straight line) P1 passing through the points (x_0,f_0) and (x_1, f_1). Hence the approximate value of f at the intermediate point $x = x_0 + sh$ is given by the linear interpolation formula

$f(x) \approx P1(x) = f_0 + s(f_1 - f_0) = f_0 + s\Delta f_0$
Where $s = \frac{x-x_0}{h}$ and $0 < s < 1$.

Forward Differences:
Let us consider a function y=f(x) defined for (n+1) distinct arguments of x say $x_0, x_1 = x_0+h$, $x_2 = x_0+2h$,……, $x_n = x_0+nh$ in [a,b] and the corresponding functional values are $y_0 = f(x_0)$, $y_1 = f(x_1) = f(x_0+h)$, $y_2 = f(x_2) = f(x_0+2h)$,……., $y_n = f(x_n) = f(x_0+nh)$. The values of x i.e $x_0, x_1 = x_0+h$, $x_2 = x_0+2h$,……, $x_n = x_0+nh$ are called the arguments or nodes and the values of the function corresponding to the function are called entries.

The forward difference of y=f(x) is denoted by

$\Delta f(x) = f(x+h)-f(x)$, i.e
$\Delta f(x_0) = f(x_0+h)-f(x_0) = y_1-y_0=\Delta y_0$
$\Delta f(x_0+h) = f(x_0+2h)-f(x_0+h) = y_2-y_1=\Delta y_1$
$\Delta f(x_0+2h) = f(x_0+3h)-f(x_0+2h) = y_3-y_2=\Delta y_2$ and so on
……

The back ward difference of y=f(x) is given by
$\nabla f(x) = f(x + h) - f(x)$ i.e

$\nabla f(x_0 + h) = f(x_0 + h) - f(x_0) = y_1 - y_{0=} \nabla y_1$

$\nabla f(x_0 + 2h) = f(x_0 + 2h) - f(x_0 + h) = y_2 - y_1 = \nabla y_2$

$\nabla f(x_0 + 3h) = f(x_0 + 3h) - f(x_0 + 2h) = y_3 - y_2 = \nabla y_3$ and so on

……

Example: Evaluate f (15), given that f(10) = 46, f(20) = 66.

Here $x_0 = 10$, $x_1 = 20$, $h = x_1 - x_0 = 20 - 10 = 10$,
$f_0 = f(x_0) = 46$ and $f_1 = f(x_1) = 66$.
Now to calculate f(15), take x = 15, so that where $u = \frac{x - x_0}{h}$
and hence
f (15) ≈ P1(15) = $f_0 + u(f_1 - f_0) = 46 + 0.5 (66 - 46) = 56$

LECTURE 4:

Newton's Forward Difference Interpolation Formula:

Using Newton's forward difference interpolation formula we find the n degree polynomial Pn which approximates the function f(x) in such a way that Pn and f agrees at (n+1) equally spaced arguments, so that $P_n(x_0) = f_0, P_n(x_1) = f_1 \ldots\ldots P_n(x_n) = f_n,$
where $f_1 = f(x_1), \ldots\ldots, f_n = f(x_n)$ are the values of f in the table.

Newton's forward difference interpolation formula is

$$f(x) \approx P_n(x) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \ldots\ldots + \frac{u(u-1)\ldots\ldots(u-n+1)}{n!}\Delta^n f_0$$

$$where\ x = x_0 + uh, u = \frac{x - x_0}{h} \quad , 0 \leq u \leq n.$$

Example Using Newton's forward difference interpolation formula and the following table evaluate f(15) .

X	f(x)	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$	$\Delta^4 f_0$
10	46				
		20			
20	66		-5		
		15		2	
30	81		-3		-3
		12		-1	
40	93		-4		
		8			
50	101				

Here x = 15, x_0 = 10,
x_1 = 20, h = x_1 - x_0 = 20 - 10 = 10,
u = (x - x_0)/h = (15–10)/10 = 0.5,
f_0 =-46, Δf_0 = 20, $\Delta^2 f_0$= -5, $\Delta^3 f_0$= 2, $\Delta^4 f_0$= -3.
Substituting these values in the Newton's forward difference interpolation formula for n = 4, we obtain $f(x) \approx P_n(x) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \ldots\ldots\ldots + \frac{u(u-1)\ldots\ldots(u-n+1)}{n!}\Delta^n f_0$
so that
f(15)= 56.8672, correct to 4 decimal places.

LECTURE 5:

Newton's Backward Difference Interpolation Formula:

Newton's backward difference interpolation formula is

$$f(x) \approx P_n(x) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \ldots\ldots\ldots + \frac{u(u+1)\ldots\ldots(u+n-1)}{n!}\nabla^n f_n$$

where $x = x_n + uh$, $u = \frac{x-x_n}{h}$ and $-n \le u \le 0$

.

Problem: For the following table of values, estimate f(7.5), using Newton's backward difference Interpolation formula.

x	F	∇f_n	$\nabla^2 f_n$	$\nabla^3 f_n$	$\nabla^4 f_n$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	343		42		
		169			
8	512				

Since the fourth and higher order derivatives are zero, so

$f(x) \approx P_n(x) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 f_n$

where $x = x_n + uh$, $u = \frac{x - x_n}{h}$ and $-n \le u \le 0$

$f(7.5) = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2!}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}6$

$= 421.875$

LECTURE 6:

Lagrange's Interpolation Formula:

If the function f(x) is known for (n+1) distinct values of x say $x_0, x_1, \ldots, x_n$ which are not equispaced then we used Lagrange's interpolation formula.

The Lagrange's interpolation formula is given by

$$f(x) \approx L(x) = w(x) \sum_{r=0}^{n} \frac{f(x_r)}{(x - x_r)w'(x_r)}$$

$$w(x) = (x - x_0)(x - x_1)(x - x_2)\ldots\ldots\ldots\ldots(x - x_n)$$

$$w'(x_r) = (x_r - x_0)(x_r - x_1)\ldots\ldots(x_r - x_{r-1})(x_r - x_{r+1})\ldots\ldots(x_r - x_n)$$

Example: Certain corresponding values of x and $\log_{10}(x)$ are
(300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871). Find $\log_{10}(301)$.

$\log_{10}(301)$

$$= \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)}(2.4771) + \frac{(1)(-4)(-6)}{(4)(-1)(-3)}(2.4829) + \frac{(1)(-3)(-6)}{(5)(1)(-2)}(2.4843) + \frac{(1)(-3)(-4)}{(7)(3)(2)}(2.4871)$$

$= 1.2739 + 4.9658 - 4.4717 + 0.7106$

$= 2.4786.$

LECTURE 7:

Newton's Divided Difference Interpolation Formula:

If $x_0, x_1, \ldots, x_n$ are arbitrarily spaced (i.e. if the difference between x_0 and x_1, x_1 and x_2 etc. may not be equal), then the polynomial of degree n through $(x_0, f_0), (x_1, f_1), \ldots\ldots (x_n, f_n)$ where $f_j = f(x_j)$ is given by the Newton's divided difference interpolation formula (also known as Newton's general interpolation formula) given by

$f(x) \approx f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ldots\ldots\ldots\ldots + \ldots\ldots$

$+ (x - x_0)\ldots(x - x_{n-1}) f[x_0, \ldots, x_n]$, with the remainder term after (n +1) terms is given by

$(x - x_0)\ldots(x - x_n) f[x_0, \ldots, x_n]$, where $f[x0, x1], f[x0, x1, x2], \ldots\ldots$ are the divided differences given by

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f\left[x_0, x_1, x_2\right] = \frac{f\left[x_1, x_2\right] - f[x_0, x_1]}{x_2 - x_0}$$

$$f\left[x_0, \ldots\ldots, x_k\right] = \frac{f\left[x_1, \ldots\ldots\ldots, x_k\right] - f[x_0, \ldots\ldots\ldots, x_{k-1}]}{x_k - x_0}$$

Therefore $f\left[x_0, \ldots\ldots, x_n\right] = \frac{f\left[x_1, \ldots\ldots\ldots, x_n\right] - f[x_0, \ldots\ldots\ldots, x_{n-1}]}{x_n - x_0}$

Example: Using the following table find f (x) as a polynomial in x

x:	-1	0	3	6	7
f(x):	3	-6	39	822	1611

The divided difference table is

				X	F(x)	1st	2nd	3rd	4th
				-1	3				
			1			-9			
		4		0	-6		6		
	7		3			15		5	
8		6		3	39		41		1
	7		3			261		13	
		4		6	822		132		
			1			789			
				7	1611				

Newton's Divided Difference Interpolation Formula is

$f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3)$
$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0, x_1, x_2, x_3, x_4)$

Therefore
$f(x) = 3+(x+1)(-9)+x(x+1)(6)+x(x+1)(x-3)(5)+x(x+1)(x-3)(x-6)$
$= x4 - 3x3 + 5x2 - 6$

Numerical Integration: (3L)

LECTURE 8:

Numerical Integration or Quadrature.

The problem of Numerical Integration is to evaluate definite integrals of the form

$$I = \int_a^b f(x)dx \text{ for any finite interval [a,b].}$$

The problem arises when the integration cannot be evaluated either when f(x) is not integrable or when f(x) is only known at finite number of points. Then we apply numerical integration method, which is quite stable. There is adequate number of formulae or rules. There are two types of formulae for numerical integration or quadrature, Newton-Cotes quadrature formula formula based on equally spaced and the Gaussian quadrature formula based on evaluation of least number of function values.

A general quadrature formula

Let us consider the definite integral $I = \int_a^b f(x)dx$

and divide the interval [a,b] of integration into n equal sub-intervals such that $a=x_0<x_1<x_2<\ldots..<x_n=b$ and xi = x0 +ih (i=0,1,2, …. n); h is the space length.

Also let the function f(x) be known at the nodes x_i, i.e. the values $y_i = f(x_i)$, I = 0,1,2,……..n are given.

Now putting $s = \frac{x - x_0}{h} \; i.e. x = x_0 + sh$, we get

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_0+sh} f(x)dx = h\int_0^n f(x_0 + sh)ds = h\int_0^n E^n f(x_0)ds \qquad [\because E^n f(x) = f(x+nh)]$$

$$= h\int_0^n (I+\Delta)^s f(y_0)ds \qquad [\because E = I + \Delta]$$

$$= h\int_0^n \left[1 + s\Delta + \frac{s(s-1)\Delta^2}{2!} + \frac{s(s-1)(s-2)\Delta^3}{3!} + \ldots\ldots\right] y_0 ds$$

Therefore we get,

$$I = \int_{x_0}^{x_n} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{2n^3 - 3n^2}{12}\Delta^2 y_0 + \frac{n^4 - 4n^3 + 4n^2}{24}\Delta^3 y_0 + \ldots\ldots\right] \quad \text{----- ----- ----- (1)}$$

This formula is known as general integration formula when the interval of integration is divided into n equal sub-intervals.

Now we can find different integration formulae from (1) as a particular cases by putting n=1,2,3….

Trapezoidal Rule:

By putting n=1 in (1) , we get

$$I_T = \int_{x_0}^{x_1} f(x)dx \approx h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{1}{2}h\left[y_0 + y_1\right] \quad \text{---- ---- ---- ---- ---- (2)}$$

This is called two point formula of Trapezoidal rule for numerical integration.

Composite Trapezoidal Rule:

$$I_T^C = \int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1}) + y_n\right]$$ ------- ------- ----- ------- **(3)**

Which is known as composite Trapezoidal Rule.

Trapezoidal Rule we can apply for any (even or odd) number of equal sub-intervals.

Error in Trapezoidal Rule :

Error in Trapezoidal Rule for 2 point formula is

$$E_T \approx -\frac{1}{12}h^3 f''(x_0)$$ ----- ---- ---- ---- (4)

Error in Trapezoidal Rule for composite formula is

$$E_T^C \approx -\frac{1}{12}h^3 n f''(\xi)\ ,\ x_0 < \xi < x_n$$ ---- ----- ---- ---- ----- (5)

In the Trapezoidal Rule, error involves with second order derivatives of the function f(x) and it gives the exact value of the integral if f(x) is only constant or a polynomial of degree one.
Hence the degree of precision of Trapezoidal Rule is one.

LECTURE 9:

Simpson's $\frac{1}{3}$ Rule:

By putting n=2 in (1) , we get

$$I_S = \int_{x_0}^{x_2} f(x)dx \approx 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$, neglecting the third and higher order differences.

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right] = \frac{1}{3}h\left[y_0 + 4y_1 + y_2\right]$$ ---- ---- ----- ----- (6)

This is called three points formula of Simpson's one-third rule for numerical integration.

Composite Simpson's $\frac{1}{3}$ Rule:

$$I_S^C = \int_{x_0}^{x_n} f(x)dx \approx \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2}) + y_n\right]$$ ------- ------- (7)

Which is known as composite Simpson's one-third rule.

Simpson's one-third Rule we can apply **only** for even number of equal sub-intervals.

Error in Simpson's $\frac{1}{3}$ Rule:

Error in Simpson's one-third Rule for 3 points formula is

$E_S \approx -\frac{1}{90}h^5 f^{iv}(x_0)$, in the interval $[x_0,x_2]$ ----- ---- ---- ---- (8)

Error in Simpson's one-third Rule for composite formula is

$E_S^C \approx -\frac{1}{90}h^5 \frac{n}{2} f^{iv}(\xi)\ ,\ x_0 < \xi < x_n$ ---- ----- ---- ---- ----- (9)

$\approx -\frac{(b-a)}{180}h^4 f^{iv}(\xi)\ ,\ x_0 < \xi < x_n$ ---- ---- ---- ---- ------ (10)

In the Simpson's one-third Rule, error involves with fourth order derivatives of the function f(x) and it gives the exact value of the integral if f(x) is only a polynomial of degree <= three.
Hence the degree of precision of Simpson's one-third Rule is three.

Example 1. Find the value of $\int_0^1 \frac{1}{1+x^2}dx$ by using (i) Trapezoidal Rule (ii) Simpson's $\frac{1}{3}$ Rule , taking number of equal sub-intervals(n) = 6 and hence find the approximate value of π in each case correct to four decimal places.

Solution: Here $f(x) = \frac{1}{1+x^2}$, again since number of equal sub-intervals(n) = 6 , then h = $\frac{1-0}{6} = \frac{1}{6}$
Now the tabulated values of f(x) corresponding different values of x are given below:

x:	0	1/6	1/3	1/2	2/3	5/6	1
Y=f(x):	1.00000	0.97297	0.90000	0.80000	0.69231	0.59016	0.50000

Here $\int_0^1 \frac{1}{1+x^2}dx = [\tan^{-1}x]_0^1 = \tan^{-1}(1) = \frac{\pi}{4}$

(i) Now applying composite formula of Trapezoidal Rule, we get

$$\int_0^1 \frac{1}{1+x^2}dx \approx \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3+y_4+y_5) + y_6\right]$$

$$= \frac{1}{2x6}\,[1 + 2(0.97297 + 0.90000 + 0.80000 + 0.69231 + 0.59016) + 0.50000]$$
$$= (1 + 2x3.95544 + 0.50000)/12 = 0.78424$$

$\therefore \frac{\pi}{4} =$ 0.78424 , Hence π = 3.136960 = 3.1370 correct to four decimal places.

(ii) Now applying composite formula of Simpson's one-third Rule, we get

$$\int_0^1 \frac{1}{1+x^2}dx \approx \frac{h}{3}\left[y_0 + 4(y_1+y_3+y_5) + 2(y_2+y_4) + y_6\right]$$

$= \frac{1}{3x6}$ [1 + 4(0.97297 + 0.80000 + 0.59016) + 2(0.90000 + 0.69231) +0.50000]

= (1 + 12.637139+ 0.50000)/18 = 0.78539 = 0.7854

$\therefore \frac{\pi}{4} =$ 0.7854 , Hence π = 3.1416 correct to four decimal places.

LECTURE 10:

Example 2. Find the value of $\int_0^{\frac{\pi}{2}} \sqrt{1-0.162\sin^2\theta}\, d\theta$ by using (i) Trapezoidal Rule (ii) Simpson's $\frac{1}{3}$ Rule , taking number of equal sub-intervals(n) = 6 in each case correct to five decimal places.

Solution: Here $f(\theta) = \sqrt{1-0.162\sin^2\theta}$, again since number of equal sub-intervals(n) = 6 ,

$\theta_0 = 0 \; and \; \theta_n = \frac{\pi}{2}$, then $h = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12} = 0.261799$, in radian. Now the tabulated values of $f(\theta)$ corresponding different values of θ are given below:

x:	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
Y=f(x):	1.000000	0.994559	0.979541	0.958645	0.937283	0.921332	0.915423

(i) Now applying composite formula of Trapezoidal Rule, we get

$$\int_0^{\frac{\pi}{2}} \sqrt{1-0.162\sin^2\theta}\, d\theta \approx \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3+y_4+y_5) + y_6\right]$$

$$= \frac{\pi}{2x12}\left[y_0 + 2(y_1+y_2+y_3+y_4+y_5) + y_6\right]$$

$$= \frac{\pi}{24}\,[1 + 2(0.994559 + 0.979541 + 0.958645 + 0.937283 + 0.921332) + 0.915423]$$

$$= \frac{3.141593}{24}\,(11.498143) = 1.50510356 = 1.50510$$ (correct to five decimal places)

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{1-0.162\sin^2\theta}\, d\theta \approx 1.50510$$

(ii) Now applying composite formula of Simpson's one-third Rule, we get

$$\int_0^{\frac{\pi}{2}} \sqrt{1-0.162\sin^2\theta}\, d\theta \approx \frac{h}{3}\left[y_0 + 4(y_1+y_3+y_5) + 2(y_2+y_4) + y_6\right]$$

$$= \frac{\pi}{3x12}\left[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6\right]$$

$$= \frac{3.141593}{36}$$ [1 + 4(0.994559 +0.958645+0.921332) +2(0.979541+0.937283)+ 0.915423]

$$= \frac{3.141593}{36}$$ (17.247215) = 1.505104 = 1.50510 (correct to five decimal places)

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{1 - 0.162\sin^2\theta}\, d\theta \approx$$ 1.50510

NUMERICAL SOLUTION OF A SYSTEM OF LINEAR EQUATIONS (3L)

LECTURE 11:

Introduction

A system of linear algebraic equations in n unknowns x_i(i=1,2,……n) can be expressed as follows

$a_{11}x_1+a_{12}x_2+\ldots\ldots + a_{1n}x_n = b_1$
$a_{21}x_1+a_{22}x_2+\ldots\ldots + a_{2n}x_n = b_2$ ……(1)
……
$a_{n1}x_1+a_{n2}x_2+\ldots\ldots+a_{nn}x_n = b_n$

Here, coefficients a_{ij} and the constants b_i are given. A solution of (1) is a set of values x_i which satisfies (1). (1) is said to be homogeneous if all b_i are equal or zero, otherwise the system is called non-homogeneous.

The system can be written as AX=B where A is the (nxn) coefficient matrix, where we assume that A is non-singular. The system of equations can be solved in two ways 1) direct method and 2) iterative method. Through finite steps of elementary arithmetic operations we obtain the solution in direct method where as in iterative method we assume an initial approximation and improve the solution by iteration. Here we will discuss the different methods. In direct method the following will be described

1) **Gauss Elimination**
2) **LU factorization method**
and
3) Gauss-Seidel method which is a iterative method will also be described.

<u>Gauss-Elimination Method</u>

In this method, the system is converted to an equivalent upper triangular system by a systematic elimination procedure.

To discuss about the method let us consider the following system

$a_{11}x_1+a_{12}x_2+\ldots\ldots +a_{1n}x_n=b_1$
$a_{21}x_1+a_{22}x_2+\ldots\ldots .+a_{2n}x_n=b_2$
……
$a_{n1}x_1+a_{n2}x_2+\ldots\ldots+a_{nn}x_n=b_n$ …….(2)

Let us take $a_{11}\neq 0$.Now multiplying the 1st equation of (2) by a_{i1}/a_{11} and subtract from i th equations we obtain

$$a_{11}x_1+a_{12}x_2 +\ldots\ldots\ldots\ldots+a_{1n}x_n=b_1$$
$$a_{22}^{(1)}x_2+\ldots\ldots\ldots+a_{2n}^{(1)}x_n=b_2^{(1)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n2}^{(1)}x_{2+\ldots\ldots} + a_{nn}^{(1)}x_n=b_n^{(1)} \quad \ldots\ldots\ldots..(3)$$

where $a_{ij}^{(1)}=a_{ij}-a_{1j}.a_{i1}/a_{11}$

$b_i^{(1)} = b_i-b_1\, a_{i1}/a_{11}$ where(i,j=2,…n)

the numbers a_{i1}/a_{11} are called row multipliers, the first equation of the system (3) contain x_1 while the other (n-1) equations are independent of x_1.

In the next step we take $a_{22}^{(1)}\neq 0$.Multiplying the 2nd equation of (3) by $a_{i2}^{(1)}/a^{(1)}_{22}$(where i=3,4,….n) we obtain

$$a_{11}x_1+a_{12}x_2+\ldots\ldots +a_{1n}x_n=b_1$$
$$a_{22}^{(1)}x_2+\ldots\ldots+a_{n2}^{(1)}x_n=b_2^{(1)}$$
$$a_{33}^{(2)}x_3+\ldots..+a_{3n}^{(2)}=b_3^{(2)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n3}^{(2)}x_{3+\ldots\ldots} + a_{nn}^{(2)}x_n=b_n^{(2)} \quad \ldots\ldots\ldots..(4)$$

where $a_{ij}^{(2)}=a_{ij}^{(1)}-a_{2j}^{(1)}.a_{i2}^{(1)}/a_{i2}^{(1)}$

$b_i^{(2)} = b_i^{(1)}-b_2^{(1)}\, a_{i2}^{(1)}/a_{i2}^{(1)}$ where(i,j=3,…n)

Here $a_{i2}^{(1)}/a_{i2}^{(1)}$ is also row multiplier. Proceeding as above we can reduce the given system to an upper triangular system as follows

$$a_{11}x_1+a_{12}x_2+\ldots\ldots+a_{1n}x_n=b_1$$
$$a_{22}^{(1)}x_2+\ldots\ldots+a_{n2}^{(1)}x_n=b_2^{(1)}$$
$$a_{33}^{(2)}x_3+\ldots..+a_{3n}^{(2)}=b_3^{(2)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{nn}^{(n-1)}x_n=b_n^{(n-1)} \quad \ldots\ldots\ldots\ldots(5)$$

The co-efficient of leading terms in (5) i.e. $a_{11,}a_{22}^{(1)},\ldots$is called pivotal elements and corresponding equations are pivotal equations. At the last stage by back substitution the solution occurs.

Example 1:

Solve the following system of equations by Gauss Elimination method

$$x+2y+z=0$$
$$2x+2y+3z=3$$
$$-x-3y=2$$

Solution: To solve the given system of linear equations by using Gauss Elimination method, let us consider augmented matrix to get upper triangular matrix by row operation

$$\begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 2 & 2 & 3 & | & 3 \\ -1 & -3 & 0 & | & 2 \end{bmatrix} \begin{matrix} R_2' \leftarrow R_2 - 2R_1 \\ R_3' \leftarrow R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & -2 & 1 & | & 3 \\ 0 & -1 & 1 & | & 2 \end{bmatrix} R_3' \leftarrow R_3 - \frac{1}{2}R_2 \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & -2 & 1 & | & 3 \\ 0 & 0 & \frac{1}{2} & | & \frac{1}{2} \end{bmatrix}$$

Thus the given system of equations reduces to the following upper triangular system as

$$x+2y+z=0$$
$$2y+z=3$$
$$\frac{z}{2}=\frac{1}{2}$$

By back substitution we obtain the required solution as z=1, y=-1, x=1.

Hence the required solution is x=1, y=-1, z=1.

LECTURE 12:

LU-FACTORIZATION METHOD:

This method is also said as triangular decomposition method. The method based on the fact that every square matrix can be expressed as the product of a lower and upper triangular matrix provided all the principal minors of the given square matrix A=(a_{ij}) are nonsingular i.e $a_{11}\neq 0$, $|a11\ a12\ a21\ a22\ |\neq 0$,……detA≠0.

Moreover, if the matrix can be factorized, then it is unique. Assume that it is possible to decompose the coefficient matrix A of the given system of equation (2) and is expressible as the product of a lower triangular matrix L and an upper triangular matrix U so that

A=LU …….(6)

Where

$$U=\begin{bmatrix} u_{11} & u_{12} & u_{13} & . & . & . & u_{1n} \\ 0 & u_{22} & u_{23} & & & & . \\ 0 & 0 & u_{33} & & & & . \\ . & & & . & & & . \\ . & & & & . & & . \\ . & & & & & . & . \\ 0 & . & . & . & . & 0 & u_{nn} \end{bmatrix} \qquad L=\begin{bmatrix} l_{11} & 0 & 0 & . & . & . & 0 \\ l_{21} & l_{22} & 0 & & & & . \\ l_{31} & l_{32} & l_{33} & & & & . \\ . & & & . & & & . \\ . & & & & . & & . \\ . & & & & & . & . \\ l_{n1} & . & . & . & . & . & l_{nn} \end{bmatrix}$$

…….(7)

Hence the system of equations AX=b become LUX=b ……..(8)

Putting UX=Y in (8) we get

LY=b …….(9)

Where $Y=(y_1,y_2,\ldots\ldots..,y_n)^T$

Thus using forward substitution ,the unknowns $y_1,y_2,\ldots\ldots..,y_n$ are determined from (9) and then the unknowns $x_1,x_2,\ldots\ldots,x_n$ are obtained from (8) by backward substitution.

Now we consider a system of three equations with three unknowns

$a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2$

$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3$ ……(10)

Here the coefficient matrix is A=[$a11\ a12\ a13\ a21\ a22\ a23\ a31\ a32\ a33$] can be written as A=LU

Where L=[$l11\ 0\ 0\ l21\ l22\ 0\ l31\ l32\ l33$] and U=[$1\ u12\ u13\ 0\ 1\ u23\ 0\ 0\ 1$]

Thus we have [$l11\ 0\ 0\ l21\ l22\ 0\ l31\ l32\ l33$][$1\ u12\ u13\ 0\ 1\ u23\ 0\ 0\ 1$]=

[$a11\ a12\ a13\ a21\ a22\ a23\ a31\ a32\ a33$]

Leading to

$l_{11}=a_{11},l_{21}=a_{21},l_{31}=a_{31}$

$l_{11}u_{12}=a_{12},l_{11}u_{13}=a_{13}$ which implies $u_{12}=a_{12}/l_{11},u_{13}=a_{13}/l_{11}$,

$l_{21}u_{12}+u_{22}=a_{22}$ which implies $l_{22}=a_{22}-l_{21}u_{12}=a_{22}-a_{21}.a_{12}/a_{11}$

$l_{31}u_{12}+l_{32}=a_{32}$ which implies $l_{32}=a_{32}-a_{31}.a_{12}/a_{11}$

$l_{21}u_{13}+l_{22}u_{23}=a_{23}$ which implies $u_{23}=a_{23}-l_{21}u_{13}/l_{22}$

and $l_{31}u_{13}+l_{32}u_{23}+l_{33}=a_{33}$ which implies $l_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}$.

Thus we obtained values of $l_{11},l_{21},\ldots\ldots$and $u_{12},u_{13},\ldots..$gives the matrices L and U.

Example 2:

Solve the following system

$3x_1+2x_2-4x_3=12$

$-x_1+5x_2+2x_3=1$

$2x_1-3x_2+4x_3=-3$

By LU-Factorization method.

Solution: Let, A=LU , therefore LU=A , then we can write as

$$[l11\ 0\ 0\ l21\ l22\ 0\ l31\ l32\ l33\ ][1\ u12\ u13\ 0\ 1\ u23\ 0\ 0\ 1\ ]=[3\ 2\ -4\ -1\ 5\ 2\ 2\ -3\ 4\ ]$$

Or,

$$[l11\ l11.u12\ l11.u13\ l21\ l21.u12+l22\ l21.u13+l22.u23\ l31\ l31.u12+l32\ l31.u13+l32.u23+l3$$

$$=[3\ 2\ -4\ -1\ 5\ 2\ 2\ -3\ 4\ ]$$

Therefore $l_{11}=3$

$l_{11}u_{12}=2 \rightarrow u_{12}=2/3$

$l_{11}.u_{12}=-4 \rightarrow u_{13}=-4/3$

$l_{21}=-1$

$l_{21}u_{21}+l_{22}=5 \rightarrow l_{22}=17/3$

$l_{21}u_{13}+l_{22}u_{23}=2 \rightarrow u_{23}=2/17$

$l_{31}=2$

$l_{31}u_{12}+l_{32}=-3 \rightarrow l_{32}=-13/3$

$l_{31}u_{13}+l_{32}u_{23}+l_{33}=4 \rightarrow l_{33}=366/51$

Thus $[3\ 0\ 0\ -1\ 17/3\ 0\ 2\ -13/3\ 366/51\ ][1\ 2/3\ -4/3\ 0\ 1\ 2/17\ 0\ 0\ 1\ ]=$

$[3\ 2\ -4\ -1\ 5\ 2\ 2\ -3\ 4\ ]$

From (9) we get

$$[3\ 0\ 0\ -1\ 17/3\ 0\ 2\ -13/3\ 366/51\ ][y1\ y2\ y3\ ]=12\ 1\ -3$$

Solving we obtain

$y_1=4$

$y_2=15/17$

$y_3=-1$

Again from (6) we get

$$[1\ 2/3\ -4/3\ 0\ 1\ 2/17\ 0\ 0\ 1\ ][x1\ x2\ x3\ ]=4\ 15/17\ -1$$

Solving we get $x_1=2$, $x_2=1$ and $x_3=-1$.

LECTURE 13:

Gauss-Seidel Method:

This method is an improvised form of Gauss Jacobi method, since the improved values of x_i are used here in each iteration rather than the values of the previous iteration and thus this method is also called Method of Successive Displacements.

To discuss the method, we rewrite the system of equations (1) as

$X_1 = (b_1-a_{12}x_2-a_{13}x_3-\ldots\ldots\ldots-a_{1n}x_n)/a_{11}$

$x_2 = (b_2-a_{21}x_1-a_{23}x_3-\ldots\ldots\ldots\ldots\ldots- a_{2n}x_n)/a_{22}$

…….

$X_n = (b_n-a_{n1}x_1-a_{n2}x_2\ldots\ldots\ldots\ldots\ldots- a_{nn-1}x_{n-1})/a_{nn}$ …….(11)

Provided $a_{11}\neq 0 (i=1,2,\ldots..n)$

To solve the equation (11) suppose $x_1^{(0)},\ldots\ldots x_n^{(0)}$ be the initial approximation of the equation (1).Now substituting those initial values in the RHS of the first equation of (11) and get the first approximation of x_1 as

$X_1^{(1)} =(b_1-a_{12}x_2^{(0)}-a_{13}x_3^{(0)}-\ldots\ldots\ldots-a_{1n}x_n^{(0)})/a_{11}$

In the second equation of (11) we substitute the improved value $x_1^{(1)}$ and initial values $x_3^{(0)}$, $x_4^{(0)},\ldots.$, $x_n^{(0)}$ and obtain the first approximation of x_2 as

$x_2^{(1)}=(b_2-a_{21}x_1^{(1)}-a_{23}x_3^{(1)}-\ldots\ldots\ldots\ldots\ldots- a_{2n}x_n^{(1)})/a_{22}$

Next we substitute in the third equation of (11) the improved values of $x_1^{(1)},x_2^{(1)}$ and the initial values of $x_4^{(0)},x_5^{(0)},\ldots..,x_n^{(0)}$ to obtain the first approximation of x_3 as

$X_3^{(1)}=(b_3-a_{31}x_1^{(1)}-a_{32}x_2^{(1)}-a_{34}x_4^{(0)}-\ldots..-a_{3n}x_n^{(0)})/a_{33}$

Proceeding in such manner we finally obtain

$X_n^{(1)}=(b_n-a_{n1}x_1^{(1)}-a_{n2}x_2^{(1)}\ldots\ldots\ldots\ldots\ldots-a_{nn-1}x_{n-1}^{(1)})/a_{nn}$

Thus the end of the first stage of iteration, we get first approximation $x_1^{(0)},x_2^{(0)}\ldots\ldots,x_n^{(1)}$ to the solution $x_1,x_2,\ldots,x_n$.

Now if $x_i^{(k)}(k=0,1,2,\ldots n)$ be the k th approximation to the solutions of $x_i(i=1,2,\ldots n)$ then the (k=1+ th approximation x_i^{k+1} of x_i are given by

$X_1^{(k+1)}=(b_1-a_{12}x_2^{(k)}-a_{13}x_3^{(k)}-\ldots\ldots\ldots-a_{1n}^{(k)}x_n^{k})/a_{11}$

$x_2^{(k+1)}=(b_2-a_{21}x_1^{(k+1)}-a_{23}x_3^{(k}-\ldots\ldots\ldots\ldots-a_{2n}x_n^{(k))})/a_{22}$

$\ldots\ldots$

$X_n^{(k+1)}=(b_n-a_{n1}x_1^{(k+1)}-a_{n2}x_2^{(k+1)}\ldots\ldots\ldots\ldots\ldots-a_{nn-1}x_{n-1}^{(k+1)})/a_{nn}$ .....(11)

The process is continued until we obtain the solution $x_1,x_{2,\ldots,}x_n$ with sufficient degree of accuracy.

Th sequence $\{x_i^{(n)}\}$ generated from(11) can be shown to be convergent to the solution $\{x_i^*\}$ if

$|a_{ii}|>\sum_{j=1\,j\neq i}^{n}|aij|,(i=1,2,\ldots\ldots,n)$ .....(12)

Hence the Gauss-Seidel iteration is convergent if the system of equations (1) is strictly diagonally dominant.

Example 3:

Solve the following system of equation by Gauss-Seidel method

$20x_1+5x_2-2x_3=14$

$3x_1+10x_2+x_3=17$

$x_1-4x_2+10x_3=23$

Correct upto 4 decimal places.

Solution:

The system is obviously diagonally dominant.

We write the corresponding iteration formulae as:

$X_1^{(k+1)}=\frac{1}{20}[14-5x_2^{(k)}+2x_3^{(k)}]$

$X_2^{(k+1)}=\frac{1}{10}[17-3x_1^{(k+1)}-x_3^{(k)}]$

$X_3^{(k+1)}=\frac{1}{10}[23-x_1^{(k+1)}+4x_2^{(k+1)}]$

Computation table is given below

k	$X_1^{(k)}$	$X_2^{(k)}$	$X_3^{(k)}$
0	0	0	0
1	0.7	1.5	3.5
2	0.68	1.15	2.69
3	0.682	1.226	2.722
4	0.6667	1.2278	2.7244
5	0.6665	1.2279	2.7246
6	0.6655	1.2279	2.7246

Thus $x_1=0.6655$, $x_2=1.2279$, $x_3=2.7246$ correct up to 4 decimal places.

Numerical Solutions Of Algebraic And Transcendental Equations (3L)

LECTURE 14:

BISECTION METHOD:

Working principle: It is an direct method and is based on a well known theorem which states that if f(x) be a continuous function in a closed interval [a,b] and f(a) and f(b) are of opposite signs [i.e. $f(a)f(b) < 0$], then there exists at least one real root of the equation f(x) = 0, between a and b. If further f'(x) exists and f'(x) maintains same sign in [a,b], i.e., f(x) strictly monotonic, then there is only one root of f(x) = 0, in (a,b).

The computational scheme of the method is find an interval $[a_0,b_0]$ where $f(a_0)f(b_0) < 0$ and f'(x) maintains same sign. To generate the sequence $\{x_n\}$, let us put $x_0=a_0$ or b_0 and $x_1 = (a_0+b_0)/2$ and find $f(x_1)$. If $f(a_0)$ and $f(x_1)$ are of opposite in signs, then we can set $a_1=a_0$ and $b_1=x_1$so that $[a_1,b_1] = [a_0,x_1]$. On the other hand, if $f(x_1)$ and $f(b_0)$ are of opposite in signs the we can set $a_1=x_1$ and $b_1=b_0$, i.e. $[a_1,b_1] = [x_1,b_0]$. Thus we see that there is a real root α in $[a_1,b_1]$ in either case. Similarly next we can set $x_2=(a_1+b_1)/2$,
Repeating the above process we get $x_{n+1} = (a_n+ b_n)/2$.

Advantage of bisection method:
a) The iteration using bisection method always produces a root, since the method brackets the root between two values.
b) As iterations are conducted, the length of the interval gets halved. Thus this method surely convergent and converge to a real root (i.e. the solution) of the equation.
c) The Bisection Method is easily programmable in any programming language.

Disadvantage of bisection method:
a) The convergence of the bisection method is slow as it is simply based on halving the interval.
b) Bisection method cannot be applied over an interval where there is a discontinuity.
c) Bisection method cannot be applied over an interval where the function takes always values of the same sign.
d) The method fails to determine complex roots.
e) If one of the initial guesses $[a_0,b_0]$ is closer to the exact solution, it will take larger number of iterations to reach the root.

Example.1. Find a real root of the equation $x^3-3x + 1.06 = 0$ using bisection method correct to three decimal places.

Solution: The given equation is $x^3-3x + 1.06 = 0$, Let $f(x) = x^3-3x + 1.06$.
Now f(0) = 1.06 (+) ive, f(1) = -0.9400 , thus a (+) real root lies in the interval (0,1).
Hence $x_1 = (0+1)/2 = 0.5$ and f(0.5) = -0.3150, the root lies in the interval (0,0.5). Bisection method tabulated as:

n	a_n(+)	b_n(-)	$x_n =(a_n+b_n)/2$	$f(x_n)$
0	0.0000	1.0000	0.5000	-0.3150
1	0.0000	0.5000	0.2500	+0.3256
2	0.2500	0.5000	0.3750	-0.0123
3	0.2500	0.3750	0.3125	+0.1530
4	0.3125	0.3750	0.3438	+0.0694
5	0.3438	0.3750	0.3594	+0.0283
6	0.3594	0.3750	0.3672	+0.0079
7	0.3672	0.3750	0.3711	-0.0022

8	0.3672	0.3711	0.3691	+0.0029
9	0.3691	0.3711	0.3701	+0.0003
10	0.3701	0.3711	0.3706	-0.0009
11	0.3701	0.3706	0.3704	-0.0003
12	0.3701	0.3704	0.3702	+0.0000
13	0.3702	0.3704	0.3703	-0.0001

Hence a positive real root of the given equation is 0.370 in (0,1) correct to 3 decimal places.

Example.2. Find a smallest real root of the equation $e^x = 4\sin(x)$ using bisection method correct to three decimal places.

Solution: The given equation is $e^x = 4\sin(x)$ i.e. $4\sin(x) - e^x = 0$, Let $f(x) = 4\sin(x) - e^x$
Now f(0) = -1.0 (-)ive , f(1) = 0.6476 (+)ve , thus a (+) real root lies in the interval (0,1).
Hence x1 = (0+1)/2 = 0.5 and f(0.5) = 0.26898, the root lies in the interval (0,0.5). Bisection method tabulated as:

n	a_n	b_n	$x_n=(a_n+b_n)/2$	$f(x_n)$
0	0.0000	1.0000	0.5000	+0.2690
1	0.0000	0.5000	0.2500	-0.2944
2	0.2500	0.5000	0.3750	+0.0101
3	0.2500	0.3750	0.3125	-0.1371
4	0.3125	0.3750	0.3438	-0.0621
5	0.3438	0.3750	0.3594	-0.0257
6	0.3594	0.3750	0.3672	-0.0077
7	0.3672	0.3750	0.3711	+0.0012
8	0.3672	0.3711	0.3691	-0.0032
9	0.3691	0.3711	0.3701	-0.0010
10	0.3701	0.3711	0.3706	+0.0001
11	0.3701	0.3706	0.3704	-0.0004
12	0.3704	0.3706	0.3705	-0.0002
13	0.3705	0.3706	0.3705	-0.0000

Hence a smallest positive real root of the given equation is 0.371 in (0,1) correct to 3 decimal places.

LECTURE 15:

REGULA-FALSI METHOD (Method of False Position / Method of Chords):

Regula-Falsi Method (Method of False Position) for finding a real root of an equation f(x)=0 is an oldest method. In this method we need approximate location of the root in the given interval (a,b). This method is also based on the intermediate value theorem as like bisection method. Like bisection method, we choose two points a_n and b_n such that $f(a_n)$ and $f(b_n)$ are of opposite signs (i.e., $f(a_n)f(b_n) < 0$) .

Then, intermediate value theorem suggests that a zero of $f(x)$ lies in between a_n and b_n, if f is a continuous function.

Given a function $f(x)$ continuous on an interval $[a_0, b_0]$ and satisfying $f(a_0)f(b_0) < 0$.

For n = 0, 1, 2, … until termination do:

Compute $x_1 = a_0 - \dfrac{f(a_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$. Then compute $f(x_1)$.

If $f(a_0)f(x_1) < 0$, then we set $a_1 = a_0$ & $b_1 = x_1$ and compute x_2 and hence $f(x_2)$.

If $f(a_0)f(x_1) > 0$, then we set $a_1 = x_1$ & $b_1 = b_0$ and compute x_2 and hence $f(x_2)$.

Maintaining the same procedure

$x_{n+1} = a_n - \dfrac{f(a_n)(b_n - a_n)}{f(b_n) - f(a_n)}$. If $f(x_{n+1}) = 0$ then x_{n+1} is the root of the given equation.

Advantage and disadvantage of Regula Falsi Method:

1. The formula is very simple for computation'
2. The method is evidently convergent on the other hand the disadvantage of the method is very slow.
3. The interval in which the root lies must be very small.

Example.1. Find a real root of the equation $x^2 - \sin(x) = 0$ using Regula-Falsi method correct to four(4) decimal places.

Solution: The given equation is $x^2 - \sin(x) = 0$. Let $f(x) = x^2 - \sin(x)$.
Now f(0) = 0 , f(1) = 0.1585 and f(0.5) = -0.2294. Thus a (+) real root lies in the interval (0.5,1).
Let us consider x0= 0.5 and x1=1.0. Then by Regula-Falsi iterative formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$$

, n = 1,2,3….. This is we can write in the form as

$$x_{n+1} = x_n - h_n, \quad where\ h_n = \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$$

, n = 1,2,3, …. ….. ….

Therefore $x_2 = x_1 - \dfrac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1) = x_1 - \dfrac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1) = 1 - \dfrac{(1 - 0.5)}{f(1) - f(0.5)} f(1)$

$$x_2 = 1 - \frac{(1 - 0.5)}{0.1585 - (-0.2294)} (0.1585) = 1 - \frac{0.5}{0.3880} (0.1585) = 0.7957$$

Since f(0.7957) = -0.0812,

Regula-Falsi method tabulated as:

NI	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	h_n	x_{n+1}	$f(x_{n+1})$

0	0.50000	1.00000	-0.22943	0.15853	0.20431	0.79569	-0.08123
1	0.79569	1.00000	-0.08123	0.15853	0.13509	0.86491	-0.01297
2	0.86491	1.00000	-0.01297	0.15853	0.12488	0.87512	-0.00178
3	0.87512	1.00000	-0.00178	0.15853	0.12349	0.87651	-0.00024
4	0.87651	1.00000	-0.00024	0.15853	0.12330	0.87670	-0.00003
5	0.87670	1.00000	-0.00003	0.15853	0.12328	0.87672	-0.00000

Hence a real root of the given equation is 0.8767 correct to four(4) decimal places.

Example.2. Find a real root of the equation $2x - \log_{10}x - 7 = 0$ using Regula-Falsi method correct to three decimal places in the interval (3,4).

Solution: The given equation is $2x - \log_{10}x - 7 = 0$. Let $f(x) = 2x - \log_{10}x - 7$.
Now f(0) = 0 , f(1) = 0.1585 and f(0.5) = -0.2294. Thus a (+) real root lies in the interval (0.5,1).
Let us consider x0= 0.5 and x1=1.0. Then by Regula-Falsi iterative formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$$, n = 1,2,3….. This is we can write in the form as

$$x_{n+1} = x_n - h_n, \quad where\ h_n = \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$$, n = 1,2,3, …. ….. ….

Therefore, $$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1) = 4 - \frac{(4-3)}{f(4) - f(3)} f(4) = 4 - \frac{1}{f(4) - f(3)} f(4)$$

$$x_2 = 4 - \frac{1}{(0.39794) - (-1.47712)}(0.39794) = 4 - 0.21223 = 3.78777$$

Since f(3.78777) = -0.00284,

Regula-Falsi method tabulated as:

NI	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	h_n	x_{n+1}	$f(x_{n+1})$
0	3.00000	4.00000	-1.47712	0.39794	0.21223	3.78777	-0.00284
1	3.78777	4.00000	-0.00284	0.39794	0.21072	3.78928	-0.00000
2	3.78928	4.00000	-0.00000	0.39794	0.21072	3.78928	-0.00000

Hence a real root of the given equation is 3.789 correct to three(3) decimal places.

LECTURE 16:

Newton-Raphson Method (Method of Tangents):

Working principle: The Newton-Raphson method is an iterative method and is used to find isolated roots of an equation f(x)=0. The object of this method is to correct the approximate root x_0 (say) successively to its exact value α. Initially a crude approximation small interval (a_0,b_0) is found out in which only one root α (say) of f(x) = 0 lies.

If f'(x) is a simple expression, the real root of f(x) =0 can be computed very easily by this method, provided $f'(x) \neq 0$.

Firstly find a crude approximation $[a_0, b_0]$ where $f(a_0).f(b_0) < 0$ and f'(x) maintains same sign. Take x_0 $(a_0 \leq x_0 \leq b_0)$ as an initial approximation of the root α of the equation f(x) =0. The working formula of Newton-Raphson method is:

$x_{n+1} = x_n + h_n$ where $h_n = - f(x_n)/f'(x_n)$ (n= 0,1,2,3…..) and $f'(x_n) \neq 0$ (n= 0,1,2,3…..).

The method is also known as tangent method.

Advantages:
Since the rate of Convergence of the method is quadratic the method converges more rapidly.
If the function is nearly vertical when it crosses the x-axis, the correct value of the root can be found with great rapidity and very little execution.

Disadvantages:

It is evident from the formula that larger derivative $f'(x)$ the smaller is the correction which must be applied to get the correct value of the root.

If $f'(x)$ is very small in the neighborhood of the root the values of $h = -\frac{f(x_k)}{f'(x_k)}$ would be large and the computation of the root by this method would be a slow process or might even fail together. The sequence converges provided that the initial approximation is chosen sufficiently close to the root otherwise the sequence may diverge.

Example.1. Find a real root of the equation $e^x = 2x + 1$ correct to four(4) decimal places using Newton-Raphson method near x=1.

Solution: The given equation is $e^x = 2x + 1$ i.e. $e^x - 2x - 1 = 0$.
Let $f(x) = e^x - 2x - 1$, $\therefore$ $f'(x) = e^x - 2$

Then by Newton-Raphson iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0]$$, n = 0,1,2,3, …. ….. …. This is we can write in the form as

$$x_{n+1} = x_n - h_n, \ where\ h_n = \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0]$$, n = 0,1,2,3, …. ….. …

Let us consider $x_0 = 1$
Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-0.2817}{0.7183} = 1.3922$$

Newton-Raphson iterative method tabulated as:

NI	x_n	$f(x_n)$	$f'(x_n)$	$h=f(x_n)/f'(x_n)$	x_{n+1}
0	1.00000	-0.28172	0.71828	-0.39221	1.39221
1	1.39221	0.23932	2.02374	0.11825	1.27396
2	1.27396	0.02706	1.57497	0.01718	1.25678
3	1.25678	0.00052	1.51408	0.00035	1.25643
4	1.25643	0.00000	1.51286	0.00000	1.25643

A real root of the given equation is 1.2564 correct to four decimal places.

Example.2. Find a real root of the equation xsin(x) + cos(x) = 0 correct to five(5) decimal places using Newton-Raphson.

Solution: The given equation is xsin(x) + cos(x) = 0
Let f(x) = xsin(x) + cos(x), $\therefore$ f'(x) = xcos(x) + sin(x) – sin(x) = xcos(x)

Then by Newton-Raphson iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0]$$

, n = 0,1,2,3, …. ….. …. This is we can write in the form as

$$x_{n+1} = x_n - h_n, \ \ where\ h_n = \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0]$$

, n = 0,1,2,3, …. ….. …

Now f(0) = 1 , f(1) = 1.3818 , f(2) = 1.4024 and f(3) = -0.5666
Therefore a real root of the given equation lies in the interval (2,3).
Let us choose $x_0 = 2.5$

Therefore $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{0.6950}{-2.0029} = 2.8470$$

Newton-Raphson iterative method tabulated as:

NI	x_n	$f(x_n)$	$f'(x_n)$	$h=f(x_n)/f'(x_n)$	x_{n+1}
0	2.500000	0.695037	-2.002859	-0.347022	2.847022
1	2.847022	-0.130355	-2.724392	0.047847	2.799175
2	2.799175	-0.002080	-2.636671	0.000789	2.798386
3	2.798386	-0.000001	-2.635186	0.000000	2.798386

A real root of the given equation is 2.79839 correct to five decimal places.

Newton-Raphson Method for finding an assigned root of a positive real number.

Find the N^{th}. Root of a real number R, i.e. find the value of $\sqrt[N]{R}$.

Let x = $\sqrt[N]{R}$, $\therefore$ $x^N = R$ or $x^N - R = 0$.

Let $f(x) = x^N - R$. $\therefore$ $f'(x) = Nx^{N-1}$

Now by Newton-Raphson iterative formula we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0]$$, n = 0,1,2,3,……

$$= x_n - \frac{x_n^N - R}{Nx_n^{N-1}} = \frac{Nx_n^N - x_n^N + R}{Nx_n^{N-1}} = \frac{(N-1)x_n^N + R}{Nx_n^{N-1}}$$, n = 0,1,2,3,……

Example.3. Find the cube root of 35 correct up to four decimal places by using Newton-Raphson Method

Solution : Let x = $\sqrt[3]{35}$, $\therefore$ $x^3 = 35$ or $x3 - 35 = 0$, Here N = 3 and R = 35

Let initial approximation $x_0 = 3.5$

Therefore,

$$x_1 = \frac{(N-1)x_0^N + R}{Nx_0^{N-1}} = \frac{(3-1)x_0^3 + 35}{3x_0^{3-1}} = \frac{2(3.5)^3 + 35}{3(3.5)^2} = 3.2857143$$

$$x_2 = \frac{2(3.2857143)^3 + 35}{3(3.2857143)^2} = 3.2711315$$

$$x_3 = \frac{2(3.2711315)^3 + 35}{3(3.2711315)^2} = 3.271066$$

$$x_4 = \frac{2(3.271066)^3 + 35}{3(3.271066)^2} = 3.271066$$

Using Newton-Raphson Method, obtain iteration formula for the reciprocal of a number N and hence find the value of $\frac{1}{27}$, correct to four decimal places.

Solution: Let x = $x = \frac{1}{N}$

Then $\frac{1}{x}=N$, Therefore $\frac{1}{x}-N=0$

Let $f(x)=\frac{1}{x}-N, \quad \therefore f'(x)=-\frac{1}{x^2}$

Now by Newton-Raphson iterative formula we get,

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \quad [f'(x_n)\neq 0]$, n = 0,1,2,3,……

Then,

$$x_{n+1}=x_n-\frac{\frac{1}{x_n}-N}{-\frac{1}{x_n^2}}=x_n+\frac{(1-Nx_n)}{x_n}x_n^2=x_n+(1-Nx_n)x_n$$

$$=(2-Nx_n)x_n \quad \text{, n = 0,1,2,3,……}$$

This is the required iterative formula.

Since the value of $\frac{1}{27}$ lies between $\frac{1}{30}$ *and* $\frac{1}{25}$ i.e. between 0.033 and 0.040.

For this problem let us take N = 27 and $x_0 = 0.03$

Therefore,

$x_1=(2-Nx_0)x_0=(2-27\text{x}0.03)\text{x}0.03=0.0357$

$x_2=(2-Nx_1)x_1=(2-27\text{x}0.0357)\text{x}0.0357=0.0369888$

$x_3=(2-Nx_2)x_2=(2-27\text{x}0.0369888)\text{x}0.0369888=0.0370370$

$x_4=(2-Nx_3)x_3=(2-27\text{x}0.0370370)\text{x}0.0370370=0.0370370$

Hence the value of $\frac{1}{27}$ is 0.0370 correct to four decimal places.

MODULE II

Module I: Numerical Methods II

Total Number of Lectures: (7L + 10L)= 17L

Numerical Solution of Ordinary Differential Equation (ODE) (7L)

LECTURE 17:

Euler's Method:

Working formula for the ODE $\frac{dy}{dx} = f(x,y)\ with\ \ y(x_0) = y_0$, where x_0 is the initial value of x and y_0 is the initial value of y.

$x_1=x_0+h$, where h is the step length , now approximate value of y at $x=x_1 (=x_0+h)$ written as

$$y(x_1) = y_1 = y_0 + hf(x_0, y_0)$$

Again the approximate value of y at $x=x_2 (=x_1+h)$ written as

$$y(x_2) = y_2 = y_1 + hf(x_1, y_1)$$

Then the general formula we written as

$$y(x_{n+1}) = y_{n+1} = y_n + hf(x_n, y_n) \quad (n = 0,1,2,3,.....) \qquad \ldots\ \ldots\ \ldots. \quad (.1)$$

Where $x_n = x_0 + nh$

Example.1. Find y(0.2) for the ordinary differential equation (ODE) $\frac{dy}{dx} = x + y$,with the initial condition y=1 when x=0, correct up to 4 decimal places, taking step length h=0.05 by Euler's Method.

Solution : Here $x_0 = 0$, $y_0 = 1$, h=0.05 and $\frac{dy}{dx} = f(x,y) = x + y$

Using the Euler's successive approximations formula, we get,

$x_1 = x_0 + h = 0 + 0.05 = 0.05$
$y_1 = y(x_1) = y(0.05) = y(0) + 0.05[0 + 1] = 1 + 0.05 = 1.05$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.10$
$y_2 = y(x_2) = y(0.1) = y(0.05) + 0.05[x_1 + y_1] = 1.05 + 0.05[0.05 + 1.05]$
$= 1.05 + 0.05(1.1) = 1.05 + 0.0550 = 1.1050$

$x_3 = x_2 + h = 0.10 + 0.05 = 0.15$
$y_3 = y(x_3) = y(0.15) = y(0.10) + 0.05[x_2 + y_2] = 1.1050 + 0.05[0.10 + 1.1050]$
$= 1.1050 + 0.05(1.2050) = 1.1050 + 0.0603 = 1.1653$

$x_4 = x_3 + h = 0.15 + 0.05 = 0.20$
$y_4 = y(x_4) = y(0.20) = y(0.15) + 0.05[x_3 + y_3] = 1.1653 + 0.05[0.15 + 1.1653]$
$= 1.1653 + 0.05(1.3153) = 1.1653 + 0.0658 = 1.2311$

Hence y(0.2) = 1.2311 correct to four(4) decimal places.

Table of Example 1:

NI	x_0	y_0	$f(x_0,y_0)$	y1	$x_1 = x_0+h$
0	0.0000	1.000000	+1.000000	1.050000	0.0500
1	0.0500	1.050000	+1.100000	1.105000	0.1000
2	0.1000	1.105000	+1.205000	1.165250	0.1500

3	0.1500	1.165250	+1.315250	1.231013	0.2000

Hence the value of y(0.20) = 1.2310

Example 2. . Find y(4.4) for the ordinary differential equation (ODE) $\frac{dy}{dx} = \frac{2-y^2}{5x}$, with the initial condition y=1 when x=4, correct up to 4 decimal places, taking step length h=0.2, by Euler's Method.

Solution : Here $x_0 = 4$, $y_0 = 1$, h=0.2 and $\frac{dy}{dx} = f(x,y) = \frac{2-y^2}{5x}$

Using the Euler's successive approximations formula, we get, $x_1 = x_0 + h = 4 + 0.2 = 4.2$

$y_1 = y(x_1) = y(4.2) = y(4) + 0.2[(2.0 – 1x1)/(5x4)] = 1 + 0.2(0.05000)= 1.0100$

$x_2 = x_1 + h = 4.2 + 0.2 = 4.4$

$y_2 = y(x_2) = y(4.4) = y(4.2) + 0.2[(2.0 - 1.01x1.01)/(5x4.2)] = 1.01 +0.2(0.0467)= 1.0193$

Hence y(4.4) = 1.0193 correct to four(4) decimal places.

Table of Example 2:

NI	x_0	y_0	$f(x_0,y_0)$	y1	$x_1 = x_0+h$
0	4.0000	1.000000	+0.050000	1.010000	4.2000
1	4.2000	1.010000	+0.046662	1.019332	4.4000

Hence the value of y(4.40) = 1.0193

LECTURE 18:

Euler's Modified Method:

Euler's Modified Method:

Working formula for the ODE $\frac{dy}{dx} = f(x,y) \ with \ \ y(x_0) = y_0$, where x_0 is the initial value of x and y_0 is the initial value of y. To minimize the error we use Euler's Modified Method instead of Euler's Method. To get the value of y at $x_1 = x^0 + h$, where h is the step length. Here we are using two formulas:

1) One is Euler's Method (as we learn earlier is known as Predictor formula), approximate value of y at $x = x_1 (= x_0 + h)$ written as

$$y(x_1) = y_1^{(0)} = y_0 + hf(x_0, y_0) \quad \ldots\ldots\ldots\ldots (.2)$$

And to get the second approximation for y_1 we replace $f(x_0, y_0)$ by average of $f(x_0, y_0)$ and $f(x_1, y_1^{(0)})$

2) Another one is second approximation for y_1 (is known as Corrector formula)at $x=x_1$ $(=x_0+h)$ written as

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})] \quad \ldots\ \ldots\ \ldots\ \ldots (\ .3)$$

Similarly, we get the third approximation for y1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] \quad \ldots\ \ldots\ \ldots\ \ldots (\ .4)$$

In general we can write $2^{nd}.$, $3^{rd}.$, $4^{th}.$ …… as

$$y_n^k = y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(k-1)})] \quad [(n = 1,2,3,.....); \textit{for particular } n\,(k = 1,2,3....)]$$

is used to approximate y_n.

Note : Euler's Modified Method is also known as Predictor-Corrector Method.

LECTURE 19:

Example 3: Evaluate y(1.2) for the ODE $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ with y(1) = 1 by Euler's Modified Method correct up to 4 decimal places.

Solution: Here $x_0 = 1$, $y_0 = 1$, and $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, therefore $\frac{dy}{dx} = f(x,y) = \frac{1}{x^2} - \frac{y}{x} = \frac{1-xy}{x^2}$

Let h=0.1 , so that $x_1 = x_0 + h = 1+0.1 = 1.1$

$y_1^{(0)} = y_0 + hf(x_0, y_0)$ = 1 + 0.1[(1 – 1)/(1x1)] = 1 + 0 = 1

Now from (.3) we get

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$ = 1 + 0.1[(1-1)/(1x1) + (1-1.1x1)/(1.1x1.1)]/2

= 1 + 0.1[0 - 0.082645]/2 = 1 - 0.0041322 = 0.99587

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$ = 1 + 0.1[0 + (1-1.1x0.99587)/(1.1x1.1)]/2

= 1 + 0.05(-0.07889) = 0.99606

$y_1^{(3)}$ = 1 + 0.1[0 + (1-1.1x0.99606)/(1.1x1.1)]/2 = 1 + 0.05(-0.079063) = 0.99605

$y_1^{(4)}$ = 1 + 0.1[0 + (1-1.1x0.99605)/(1.1x1.1)]/2 = 1 + 0.05(-0.079054) = 0.99605

Hence $y_1 = y(1.1) = 0.99605 = 0.9961$ (Approximately correct to 4 decimal places)

Now $x_2 = x_1 + h = 1.1 + 0.1 = 1.2$ and $f(x_1,y_1) = (\ 1 - 1.1 \text{ x } 0.9961)/(1.1 \text{ x } 1.1) = -0.07910$

$y_2^{(0)}$ = $y_1 + hf(x_1,y_1) = 0.9961 + 0.1(-0.07910) = 0.98819$

$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$ = 0.9961 + 0.05[-0.07910 + (1 – 1.2 x 0.98819)/(1.2 x 1.2)]

= 0.9961 + 0.05(- 0.07910 - 0.12905) = 0.98569

$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$ = 0.9961 + 0.05[-0.07910 + (1 – 1.2 x 0.98569)/(1.2 x 1.2)]
= 0.9961 + 0.05(-0.07910 -0.12696) = 0.9858

$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$ = 0.9961 + 0.05[-0.07910 + (1 – 1.2 x 0.9858)/(1.2 x 1.2)]
= 0.9961 + 0.05(-0.07910 -0.12706) = 0.98579

Hence $y_2 \approx$ 0.9858 (correct to 4 decimal places)

$\therefore y(1.2) \approx$ 0.9858

NI	x_0	y_0	y_1	y_2	$x_1=x_0+h$
0	1.0000	1.00000	1.00000		1.1000
			1.00000	0.99587	1.1000
			0.99587	0.99606	1.1000
			0.99606	0.99605	1.1000
			0.99605	0.99605	1.1000
1	1.1000	0.99605	0.98814		1.2000
			0.98814	0.98564	1.2000
			0.98564	0.98575	1.2000
			0.98575	0.98574	1.2000
			0.98574	0.98574	1.2000

Hence y(1.20)=0.98574

Example 4: Evaluate y(0.02) for the ODE $\frac{dy}{dx} = x^2 + y$ with y(1) = 0 by Euler's Modified Method correct up to 4 decimal places, by taking step length h=0.01.

Solution: Here $x_0 = 0$, $y_0 = 1$, and $\frac{dy}{dx} = x^2 + y$, therefore $\frac{dy}{dx} = f(x, y) = x^2 + y$
h=0.01 , so that $x_1 = x_0 + h = 0+0.01 = 0.01$

$y_1^{(0)} = y_0 + hf(x_0, y_0)$ = 1 + 0.01(0x0 + 1) = 1 + 0.01 = 1.01

Now from (.3) we get

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$ = 1 + 0.01[(0x0 + 1) + (0.01x0.01 + 1.01)]/2
= 1 + 0.005[1 + 0.001 + 1.01] = 1 + 0.010055 = 1.010055

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$ = 1 + 0.005[(0x0 + 1) + (0.01x0.01 + 1.010055)]

$= 1 + 0.005(1 + 1.010155) = 1.010051$

$y_1^{(3)}$ = 1 + 0.005[(0x0 + 1) + (0.01x0.01 + 1.010051)] = 1 + 0.005(1+1.010151) = 1.010051

Hence $y_1 = y(0.01) = 1.010051$

Now $x_2 = x_1 + h = 0.01 + 0.1 = 0.02$ and $f(x_1,y_1) = (0.01x0.01+ 1.010051) = 1.010151$

$y_2^{(0)} = y_1 + hf(x_1,y_1) = 1.010051 + 0.01x1.010151 = 1.0201525 \approx 1.02015$

$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$

$= 1.010051 + 0.005[1.010151 + (0.02x0.02 + 1.020150)]$

$= 1.010051 + 0.010153505 \approx 1.020205$

$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 1.010051 + 0.005[1.010151 + (0.02x0.02 + 1.020205)]$

$= 1.010051 + 0.01015378 \approx 1.02025$

Hence $y_2 \approx 1.02025$ (correct to 5 decimal places)

$\therefore y(0.02) \approx 1.02025$

NI	x_0	y_0	y_1	y_2	$x_1=x_0+h$
0	0.0000	1.00000	1.01000		0.0100
			1.01000	1.01005	0.0100
			1.01005	1.01005	0.0100
1	0.0100	1.01005	1.02015		0.0200
			1.02015	1.02020	0.0200
			1.02020	1.02020	0.0200

Hence y(0.02)=1.02020

LECTURE 20:

Predictor-Corrector method:

To find more accurately numerical solution of an ordinary differential equation we use **Predictor-Corrector method.** Firstly we use Predictor formula to get approximate value of $y_{n+1} = y(x_{n+1})$, then to improve the approximate value, we use Corrector formula. Corrector formula is more accurate than the predictor formula. Predictor-Corrector method is a multi-step method.

Milne's method: This multi-step method obtained by Milne by integrating over more than one step.

$\frac{dy}{dx} = f(x,y) = f \quad and \quad f(x_n, y_n) = f_n$ with

$y(x_n) = y_n,\ y(x_{n-1}) = y_{n-1},\ y(x_{n-2}) = y_{n-2}, y(x_{n-3}) = y_{n-3}$

Milne's predictor formula of order 4 is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \quad \text{--- --- --- ---} \quad (1)$$

which is the 4-step explicit recursion formula of order four(4).

Milne's corrector formula of order 4 is

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}) \quad \text{--- --- --- ---} \quad (2)$$

which is 2-step implicit recursion formula of order four(4).

By using (1) and (2) we have to find out y_{n+1} where x_{n+1} is given.

Example 5: Find the value of y(0.4) for the Ordinary Differential Equation $\frac{dy}{dx} = xy + y^2$, with y(0) = 1, y(0.1)=1.1169, y(0.2)=1.2773, y(0.3)=1.5040 correct to three decimal places.

Solution: For the given problem f(x,y)= xy + y^2 and also given that x_0=0.0, x_1=0.1, x_2=0.2, x_3=0.3, h=0.1 and y_0=1.0, y_1=1.1169, y_2=1.2773, y_3=1.5040

Therefore f_0=f(x0,y0)=f(0.0,1)=1, f_1=f(x_1,y_1)=f(0.1,1.1169)=0.1x1.1169 +$(1.1169)^2$=1.359156

f_2=f(x_2,y_2)=f(0.2,1.2773)=0.2x1.2773+$(1.2773)^2$ = 1.886955

f_3=f(x_3,y_3)=f(0.3,1.5040)=0.3x1.5040+$(1.5040)^2$ = 2.713216

Now putting n=3 in equation (1) and (2) we get as

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \text{--- --- --- ---} \quad (3)$$

$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \quad \text{--- --- --- ---} \quad (4)$$

Now from (3) we get,

$y_4^p = 1 + \frac{4x0.1}{3}$ (2x1.359156 – 1.886955 + 2x2.713216) = 1.834372

Hence $f_4^p = f(x_4, y_4^p) =$ 0.4x1.834372 + $(1.834372)^2$ = 4.098669

$\therefore y_4^{c1} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p) =$ 1.2773 + $\frac{0.1}{3}$ (1.886955 + 4x2.713216 + 4.098669)= 1.838583

$\therefore$ Again $f_4^{c1} = f(x_4, y_4^{c1}) =$ 0.4x1.838583 + $(1.838583)^2$ = 4.115821

Hence $y_4^{c2} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{c1}) =$ 1.2773 + $\frac{0.1}{3}$ (1.886955 + 4x2.713216 + 4.115821)= 1.839155

$\therefore$ Again $f_4^{c2} = f(x_4, y_4^{c2}) =$ 0.4x1.839155 + $(1.839155)^2$ = 4.118153

Hence $y_4^{c3} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{c2}) =$ 1.2773 + $\frac{0.1}{3}$ (1.886955 + 4x2.713216 + 4.118153)= 1.839232

$\therefore$ Again $f_4^{c3} = f(x_4, y_4^{c3}) =$ 0.4x1.839232 + $(1.839232)^2$ = 4.118467

Hence $y_4^{c4} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{c3}) =$ 1.2773 + $\frac{0.1}{3}$ (1.886955 + 4x2.713216 + 4.118467)= 1.839243

$\therefore$ $y_4^{c3} = y_4^{c4}$ = 1.8392 , correct up to four(4) decimal places.

Hence y(0.4) = 1.8392 correct up to four(4) decimal places.

LECTURE 21:

Runge-Kutta Methods:

Rumge-Kutta Methods are used to finding numerical solution of an ordinary differential equation. Rumge-Kutta Methods gives us more accurate result than Euler's and Euler's Modified Method. Here we discussed two methods as:

a) Second Order Runge-Kutta Method:

b) Fourth Order Runge-Kutta Method:

Fourth Order Runge-Kutta Method:

Computational formula for an ordinary differential equation

$$y' = \frac{dy}{dx} = f(x, y) \ with \ \ y(x_0) = y_0$$ is

$y1 = y0 + k$ where $k = (k1 + k2)/2$ and $k1 = h.f(x0,y0)$; $k2 = h.f(x0+h,y0+k1)$. The error in this formula is O(h3).

Example 1: Compute y(0.2) for the ordinary differential equation $\frac{dy}{dx} = x - y \ \ with \ y(0) = 1,$ taking h = 0.1, by Runge-Kutta Method of order 4, correct to six decimal places.

Solution: For the given problem Here $x_0 = 0$, $y_0 = 1$, and $\frac{dy}{dx} = x - y$, therefore $f(x, y) = x - y$ and h=0.1. now for y(0.1)

$k_1 = hf(x_0,y_0) = 0.1(0-1) = -0.1$

$k_2 = hf(x_0+h/2, y_0+k_1/2) = 0.1[(0+0.05) –(1-0.05)] = 0.1(-0.9) = -0.09$

$k_3 = hf(x_0+h/2, y_0+k_2/2) = 0.1[(0+0.05) –(1-0.045)] = 0.1(1-0.095) = 0.1(-0.9050) = -0.0905$

$k_4 = hf(x_0+h, y_0+k_3) = 0.1[(0+0.1) –(1-0.0905)] = 0.1(1-0.1905) = 0.1(-0.8095) = -0.08095$

$k = (k_1 + 2k_2 + 2k_3 + k_4)/6 = [-0.1 +2(-0.09) +2(-0.0905) -0.08095] = (-0.54195)/ 6$

$= -0.090325$

$y_1 = y_0 + k = 1 - 0.090325 = 0.909675$

$\therefore y(0.1) = y_1 = 0.909675$

Now for y(0.2) : $x_1 = 0.1$ and $y_1 = 0.909675$

$k_1 = hf(x_1,y_1) = 0.1f(0.1,0.909675) = 0.1(0.1-0.909675) = -0.0809675$

$k_2 = hf(x_1+h/2, y_1+k_1/2) = 0.1f[(0.1+0.05),(0.909675-0.04048375)] = 0.1f(0.15, 0.86919125)$
$= 0.1(0.15 - 0.86919125) = -0.0719191$
$k_3 = hf(x_1+h/2, y_1+k_2/2) = 0.1f[(0.1+0.05),(0.909675-0.03595955)] = 0.1f(0.15, 0.87371545)$
$= 0.1(0.15 - 0.87371545) = -0.0723715$
$k_4 = hf(x_1+h, y_1+k_3) = 0.1f(0.1+0.1, 0.909675-0.0723715) = 0.1f(0.2, 0.8373035)$
$= 0.1(0.2 - 0.8373035) = -0.0637303$
$k = (k_1 + 2k_2 + 2k_3 + k_4)/6 = [-0.0809675+2(-0.0719191) +2(-0.0723715) -0.0637303]$
$= (-0.433279)/ 6 = -0.07221317$
$y_2 = y_1 + k = 0.909675 -0.07221317 = 0.837462$ (correct to 6 decimal places)
$\therefore\ y(0.2) = y_2 = 0.837462$

Numerical Solution of Partial Differential Equations (2L)

LECTURE 22:

The ***general form*** of a linear PDE in 2 independent variables is

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + Fu = G$$

The coefficients A, B, C, *etc*. may be constants or variables.

The ***order*** of a pde is the order of the highest partial derivative present.

For example, $u_t = u_{xx}$ is second order.

In the general form above, u(x,y) is the ***dependent variable*** and x and y are the ***independent variables***.

Pde's are ***linear*** if no products of the dependent variable or its derivatives, or functions such as sine involving the dependent variable, are present. Otherwise the pde is ***nonlinear***. Generally, linear pde's are much easier to solve, except for particular nonlinear pdes where analytic solutions are known.

Pde's may be ***classified*** according to whether they are:

Hyperbolic	$B^2 > 4AC$	Vibrating systems, waves, *etc*.
Parabolic	$B^2 = 4AC$	Heat flow, diffusion, *etc*.
Elliptic	$B^2 < 4AC$	Equilibrium problems, electrostatics, *etc*.

Finite Difference Approximations(FDA)

Taylor Series Expansion Approximations to first and second derivatives of f(x)

The Taylor series expansion for a function of one variable about the point x is

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + O(h^3)$$

The notation $O(h^3)$ indicates that the series, when truncated at the quadratic term in h, contains errors that scale as h^3 and higher powers of h.

We can immediately obtain an approximation to the derivative of f(x) from the first two terms of the expansion.

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h^1)$$

Note that, even though we neglect terms of $O(h^2)$ in the expansion, since we divide through by h to obtain the derivative expression the approximation is correct to $O(h^1)$ only. Note that there is an ***asymmetry*** in this approximation to the derivative at x, since the function at x and x+h occur, but not the function at x-h. This is therefore referred to as a ***forward difference approximation***. It is possible to expand f(x) in the negative direction in the Taylor expansion and hence to obtain a ***backward difference approximation***

$$f(x-h) = f(x) - h\,f'(x) + \frac{h^2}{2!} f''(x) + O(h^3)$$

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h^1)$$

By combining forward and backward difference approximations it is possible to obtain a ***central difference approximation*** to the derivative of f(x) at x that contains errors of order $O(h^2)$. Note the difference in scaling of errors for the central difference approximation when compared with the forward and backward difference approximations.

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + O(h^3)$$

$$f(x-h) = f(x) - h\,f'(x) + \frac{h^2}{2!} f''(x) + O(h^3)$$

$$f(x+h) - f(x-h) = 2\,h\,f'(x) + 2\frac{h^3}{3!} f''(x) + O(h^4)$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

By retaining terms in the Taylor series to order h^3 we can obtain an approximation for the second derivative which contains errors of order h^2

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + O(h^4)$$

$$f(x-h) = f(x) - h\,f'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + O(h^4)$$

add these expansions to obtain

$$h^2 f''(x) = f(x+h) - 2\,f(x) + f(x-h) + O(h^4)$$

$$f''(x) = \frac{f(x+h) - 2\,f(x) + f(x-h)}{h^2} + O(h^2)$$

LECTURE 23:

Taylor series expansions in more than one dimension

A PDE contains at least two independent variables and so we need to approximate differential operators in at least two dimensions. This is done using Taylor series expansions in more than one dimension. Suppose

u = u(x,y)

The Taylor series expansion of u about the point (x,y) is

$$u(x+h, y+k) = u(x,y) + h\,u_x(x,y) + k\,u_y(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + \frac{2hk}{2!}u_{xy}(x,y) + \frac{k^2}{2!}u_{yy}(x,y) + O(|\mathbf{h}|^3)$$

$$\mathbf{h} = \begin{pmatrix} h \\ k \end{pmatrix}$$

The vector notation for this expansion is

$$u(\mathbf{x}+\mathbf{h}) = u(\mathbf{x}) + \mathbf{h}^{\mathbf{T}}.\nabla u(\mathbf{x}) + \frac{1}{2!}\mathbf{h}^{\mathbf{T}}.\nabla\nabla u(\mathbf{x}).\mathbf{h} + O(|\mathbf{h}|^3)$$

$$\mathbf{h} = \begin{pmatrix} h \\ k \end{pmatrix} \quad \mathbf{h}^{\mathbf{T}} = \begin{pmatrix} h & k \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \nabla\nabla = \frac{\partial^2}{\partial x_i \partial x_j}$$

Returning to the long-hand notation, the expansion of u(x,y) in the x direction is

$$u(x+h,y) = u(x,y) + h\,u_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + O(h^3)$$

$$u(x-h,y) = u(x,y) - h\,u_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + O(h^3)$$

If we subtract these two equations and rearrange to make u_x the subject of the equation we find that the ***central difference approximation*** to u_x is

$$u_x(x,y) = \frac{u(x+h,y) - u(x-h,y)}{2h} + O(h^2)$$

We can also obtain ***forward and backward difference approximations*** which contain errors of order $O(h^1)$ from the Taylor series expansion in either direction.

Computational Molecules

It is convenient to write the function u at points on a grid for numerical solution with subscripted indices rather than arguments. Thus we make the equivalence

$$u(x,y+k) = u_{i+1,j}$$

k is the stepsize or distance between gridpoints in the y direction in the numerical solution. Perversely, Farlow reverses the order of the arguments/indices in going over to the gridpoint index. We will do the same to maintain consistency with Farlow. The indicial representation of the first and second order partial derivatives is given below and illustrated by corresponding '***computational molecules***'.

$$u_x(x,y) = \frac{1}{2h}\left(u_{i,j+1} - u_{i,j-1}\right) + O(h^2)$$

$$u_y(x,y) = \frac{1}{2k}\left(u_{i+1,j} - u_{i-1,j}\right) + O(k^2)$$

$$u_{xx}(x,y) = \frac{1}{h^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) + O(h^2)$$

$$u_{yy}(x,y) = \frac{1}{k^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + O(k^2)$$

Provided that the stepsizes in the two directions (h and k) are equal, then we obtain the following approximation for the ***Laplacian*** operator in 2 dimensions

$$\nabla^2 u(x,y) = \frac{1}{h^2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right) + O(h^2)$$

.

MCQ NUMERICAL METHODS:

If y=f(x) be defined at (n+1) distinct interpolating points, then the degree of the Lagrangin function is

a) at least n b) at most n c) n+1 d) n

The order of convergence of the Newton-Raphson's method is

a) 0 b) 1 c) 2 d) 3 .

.

When Gauss-Jordan method is used to solve the system of equation which can be written in the form AX=b, A is transform to a

a) Lower triangular matrix b) Upper triangular matrix
c) Diagonal identity matrix d) Symmetric matrix

If dy/dx = xy and y(0) = 2, then k_2 according to Runge-Kutta method is [h = 0.2]

a) 0.4 b) 0.04 c) 0.004 d) 0.0004

The order of convergence of Newton Raphson method is

a) 3 b) 2 c) 1 d) 4

$\frac{1}{\sqrt{3}}$

The degree of precession of simpson's 1/3 rule is

a) 3 b) 2 c) 1 d) 4

The first order forward difference of constant function is

a) 0 b) 2 c) 1 d) none of these

Newton's forward interpolation formula is used for

a) equispaced nodes b) unequispaced nodes $\frac{n(n+1)}{2}$

c) both (a) and (b) $\frac{n(n-1)}{2}$ d) none of these

Newton Raphson method is also known as

a) straight line b) tangent $\frac{n(n+1)}{2}$ c) normal $\frac{n(n-1)}{2}$ d) cyclic

The relation between Δ and ∇ is

a) $\Delta\nabla = \Delta - \nabla$ b) $\Delta\nabla = I$ c) $\Delta = I / \nabla$ d) none of these

Correct to 5 decimal places, 1.004355 is rounded to

a) 1.00436 b) 1.00435 $\frac{n(n+1)}{2}$ c) 1.004355 $\frac{n(n-1)}{2}$ d) none of these

Which is the direct method?

a) Gauss elimination method b) Gauss Jacobi method

c) Gauss Seidel method d) none of these

Modified Euler method has a truncation error of order

a) 0 b) 2 c) 1 d) none of these

Which of the following is not a computational error?

a) truncation error b) round off error c) inherent error d) none of these

Runge Kutta 4th order method has a truncation error of order

a) 4 b) 5 c) 3 d) none of these

Module Number III: Statistics (10L)

LECTURE 24:

Mean, Variance and Standard Deviation

Frequency distribution table is $\begin{bmatrix} x: & x_1 & x_2 & x_3 & ..... & x_n \\ f: & f_1 & f_2 & f_3 & ..... & f_n \end{bmatrix}$

Mean $$\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + ... + f_nx_n}{f_1 + f_2 + f_3 + ... + f_n} = \frac{1}{\sum_{i=1}^{n} f_i} \sum_{i=1}^{n} f_i x_i = \frac{1}{N} \sum_{i=1}^{n} f_i x_i \ \ where\ N = \sum_{i=1}^{n} f_i$$

Variance $$\text{var}(x) = \frac{f_1x_1^{\,2} + f_2x_2^{\,2} + f_3x_3^{\,2} + ... + f_nx_n^{\,2}}{f_1 + f_2 + f_3 + ... + f_n} - (\bar{x})^2 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2$$

Standard Deviation $sd(x)\ or\ \sigma_x = +\sqrt{\text{var}(x)} = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$

If $f_1 = f_2 = ... = f_n = 1$ then

Mean= $\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{N}$ where N=number of data sets

$$\text{var}(x) = \frac{x_1^{\,2} + x_2^{\,2} + x_3^{\,2} + ... + x_n^{\,2}}{N} - (\bar{x})^2$$

Standard Deviation $sd(x)\ or\ \sigma_x = +\sqrt{\text{var}(x)}$

<u>Problem</u>

Find mean, variance and standard deviation of first n natural numbers.

Mean $\bar{x} = \frac{1 + 2 + 3 + ... + n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$

Variance
$$\begin{aligned}\text{var}(x) &= \frac{1^2 + 2^2 + 3^2 + ... + n^2}{n} - (\bar{x})^2 \\ &= \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2 \\ &= \frac{(n+1)(n-1)}{12} \\ &= \frac{n^2 - 1}{12}\end{aligned}$$

Standard Deviation $sd(x)\ or\ \sigma_x = +\sqrt{\text{var}(x)} = \sqrt{\frac{n^2 - 1}{12}}$

<u>Problem</u>: Find mean, variance and standard deviation of 1,3,4,5,6,6,10

Mean $\bar{x} = \frac{1 + 3 + 4 + 5 + 6 + 6 + 10}{7} = \frac{35}{7} = 5$

Variance

$$\text{var}(x) = \frac{1^2 + 3^2 + 4^2 + 5^2 + 6^2 + 6^2 + 10^2}{7} - (\bar{x})^2$$
$$= 6.86$$

Standard Deviation

$$sd(x) = +\sqrt{\text{var}(x)}$$
$$= +\sqrt{6.86}$$
$$= 2.6$$

Problem: Find mean, variance and standard deviation for the following frequency distribution table

X	5	15	25	35	45	55	65	75
F	3	7	9	23	15	8	6	4

Mean

$$\bar{x} = \frac{3X5 + 7X15 + 9X25 + 23X35 + 15X45 + 8X55 + 6X65 + 4X75}{3 + 7 + 9 + 23 + 15 + 8 + 6 + 4}$$
$$= \frac{2955}{75}$$
$$= 39.4$$

Variance

$$\text{var}(x) = \frac{3X5^2 + 7X15^2 + 9X25^2 + 23X35^2 + 15X45^2 + 8X55^2 + 6X65^2 + 4X75^2}{3 + 7 + 9 + 23 + 15 + 8 + 6 + 4} - (39.4)^2$$

$$= 285.61$$

Standard Deviation $sd = \sqrt{285.61} = 16.9$

<u>Problem:</u> Show that if $\bar{x}$ is mean of the values x_i with weight f_i ($i = 1, 2, 3, \ldots\ldots, n$), then $\sum_{i=1}^{n} f_i(x_i - \bar{x}) = 0$.

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots\ldots + f_nx_n}{f_1 + f_2 + f_3 + \ldots\ldots + f_n}$$

$$\Rightarrow (f_1 + f_2 + f_3 + \ldots\ldots + f)\bar{x} = f_1x_1 + f_2x_2 + f_3x_3 + \ldots\ldots + f_nx_n$$

$$\Rightarrow f_1\bar{x} + f_2\bar{x} + \ldots\ldots + f_n\bar{x} = f_1x_1 + f_2x_2 + f_3x_3 + \ldots\ldots + f_nx_n$$

$$\Rightarrow f_1(\bar{x} - x_1) + f_2(\bar{x} - x_2) + \ldots\ldots + f_n(\bar{x} - x_n) = 0$$

$$\Rightarrow \sum_{i=1}^{n} f_i(\bar{x} - x_i) = 0$$

Exercise Find mean , variance and standard deviation from the following table

X	1	2	3	4	5	…….	N
F	1	2	3	4	5	……..	N

LECTURE 25:

Calculation of median in a series of individual observations

For calculating median in a series of individual observations, the following steps are taken:

i) Arrange the data in ascending or descending order

ii) Median is the size of the (n+1)/2 th item

Example: Find out the median from the data recorded on the number of clusters per plant in a pulse crop

Number of clusters=10,18,17,19,10,15,11,17,12

Sl no.	Data in ascending order
1	10
2	10
3	11
4	12
5	15
6	17
7	17
8	18
9	19

Median=size of (9+1)/2 th item

=size of 5^{th} item =15

Calculation of median in a continuous series

While computing the value of the median in a continuous series, first determine the particular class in which the value of median lies. Use N/2 as the rank of the median. The following formula is used for determining the exact value of the median

Median= $l_1 + \frac{N/2 - F}{f_m} Xc$

Where l_1 is lower boundary of median class

N=sum of all frequencies

F=cumulative frequency of the class just preceding the median class

f_m is frequency of median class

C=width of the median class

Problem: Find median from the following data

Class	Frequency
10-25	7
25-40	20
40-55	40
55-70	26
70-85	5
85-100	2

Solution :

Class	Frequency	Cumulative Frequency (cuf)
10-25	7	7
25-40	20	27
40-55	40	67
55-70	26	93
70-85	5	98
85-100	2	100

Here sum of all frequency N=100

N/2=50

Therefore the median class is 40-55

Median= $l_1 + \frac{N/2 - F}{f_m} Xc$

Where l_1 is lower boundary of median class=40

F=cumulative frequency of the class just preceding the median class=27

f_m is frequency of median class=40

C=width of the median class=15

Median= $40 + \frac{100/2 - 27}{40} X15$ =48.63

Exercise: Calculate the median from the following data recorded on a variety of rice

Number of spikes per panicle	50-60	60-70	70-80	80-90	90-100	100-110	110-120	120-130	130-140
Number of branches	5	13	17	25	29	22	17	12	5

Mode

Problem: Calculate the mode of the following distribution

Number of grains	100-110	110-130	130-140	140-160	160-170	170-180
Number of plants	11	40	27	34	12	6

Solution

Number of grains	Number of Plants
100-110	11
110-120	20
120-130	20
130-140	27
140-150	17
150-160	17
160-170	12
170-180	6

Here the highest frequency is 27 in the class 130-140

Mode class=130-140

Mode= $l_1 + \frac{f_0 - f_{-1}}{2f_0 - f_1 - f_{-1}} Xc$

Where l_1 is lower boundary of modal class=130

f_0 is frequency of modal class=27

f_{-1} is frequency of the class just preceding the modal class=20

f_1 is frequency of the class just following the modal class=17

C=width of the modal class=10

Mode= $130 + \frac{27 - 20}{2X27 - 17 - 20} X10$

=134.12

Coefficient of variation $cv = \frac{s\tan dard\ deviation}{mean} X100$

Mean deviation from mean= $\frac{1}{\sum_{i=1}^{n} f_i} \sum_{i=1}^{n} f_i \mid x_i - \bar{x} \mid$

If $f_1 = f_2 = ... = f_n = 1$ then

Mean deviation from mean= $\frac{1}{n}\sum_{i=1}^{n}|x_i-\bar{x}|$

Example: Find mean deviation from the following data

30,40,70,20,60,50,10

Ans:

$$\bar{x}=\frac{30+40+70+20+60+50+10}{7}=40$$

$$\frac{1}{7}\sum_{i=1}^{7}|x_i-\bar{x}|=\frac{1}{7}(|30-40|+|40-40|+|70-40|+|20-40|+|60-40|+|50-40|+|10-40|)$$

=17.14

LECTURE 26:

INTRODUCTION TO PROBABILITY

Probability and chance both study the possibilities of different things occurring based on a few known factors. Almost every possible activity or result has a probability. For example, someone might wonder about the probability they will get a low grade on a test they have taken or if they will be accepted for a job they applied for. Some of these things can be modeled and measured numerically with probability. So **probability** is the numerical measure of the chance that an event will occur and hence **probability** is quantified as a number between 0 and 1 (where 0 indicates impossibilities and 1 indicates certainty). The probability of an event *A* is written as P(A). So $0 \leq P(A) \leq 1$.

- **BASIC TERMINOLOGY:**
  1. **Die**: it is a small cube with the 1,2,3,4,5 and 6 dots marked on its six faces. The outcome of throwing a die is the number that appears on its upper face.
  2. **Cards or playing cards:** A pack of cards or a deck of cards contains four different suits, which are Spades, Hearts , Diamond and Clubs. Each suit has 13 cards- nine cards with numbers 2,3,…., 10, then Jack, Queen, King and Ace. Spades and Clubs are black in colour and Hearts and Diamonds are red in colour. Ace, King, Queen and Jack are known as Face cards.
  3. **Random experiment:** An experiment is called random if the following conditions are satisfied:

i. All the possible outcomes are known in advance
ii. It is not possible to predict the outcome at any particular instant or experiment
iii. it can be infinitely repeated under identical conditions

Random experiment is denoted by E. Performing a random experiment is called trial.

Example: Tossing an unbiased coin is a simple example of a random experiment. The outcomes are HEAD and TAIL but it is not possible to predict which face of the coin will turn up at a particular toss, under identical conditions.

4. **Event Space**: The set of all the possible outcomes of a random experiment constitute its event space or sample space. It is denoted by S.

Example: if E is tossing an unbiased coin then S={H,T}, where H and T correspond to outcomes HEAD and TAIL respectively.

5. **Event**: An event is essentially a subset of the event space i.e. an event is a particular outcome of a random experiment.

Example: If unbiased die is thrown, then the event space is S={1,2,3,4,5,6}. A set A={1,3,5} is an event, which corresponds to “odd number appears on throwing a die”.

6. **Impossible Event**: If a certain event can never take place during trials of a random experiment under identical conditions,it is called impossible event.

Example: The event “number 7 appears on upper face” is an impossible event for the random experiment “throwing a die”.

7. **Certain Event**: If an event is bound to take place in every trial of a random experiment under identical conditions, it is called certain event.

Example: The event HEAD or TAIL is a certain event for the random experiment “tossing an unbiased coin”.

A special case of certain event is elementary event or simple event. If the event contains exactly one element, it is called elementary event.

Example: The event “number 2 appears on the upper face” is a simple event of the random experiment of throwing an unbiased die.

8. **Compound Event**: An event is called composite event or compound event if the event contains more than one element.

Example: The event “odd number appears on the upper face” is a compound event of the random experiment of throwing an unbiased die as the event may have die faces with numbers 1,3 or 5.

9. **Complementary Event**: An event which is, in effect, the opposite of a certain event of a random experiment is called the complementary event of that certain event.

Example: The event “odd number appears on upper face” is complementary of the event “even number appears on upper face”, for the random experiment of throwing an unbiased die.

10. **Mutually Exclusive Events**: Two events related to a random experiment are called mutually exclusive if these can not occur simultaneously.

Example: for the random experiment of throwing an unbiased die, the events “odd number appears on upper face” and “even number appears on upper face” are mutually exclusive.

11. **Exhaustive Event**: A collection of events is called exhaustive events if at least one event belonging to the collection is bound to occur in every trial.

Example: For the random experiment of throwing an unbiased die, the event “ a number appears on upper face” is exhaustive.

12. **Equally Likely Events**: If the events take place without any preference to any of them i.e. the events are equally probable, then the events are called equally likely.

CLASSICAL DEFINITION OF PROBABILITY

Let A be an event associated with a random experiment E. By the classical definition, the probability that A occurs, denoted by P(A), is defined by:

Probability of A= P(A)= $\frac{favourable\ number\ of\ elementary\ events}{Total\ number\ of\ elementary\ events}$

Example: for the random experiment of throwing an unbiased die, if the event A= multiple of number 3 appears on upper face i.e. A={3,6}

Then P(A)= $\frac{2}{6} = \frac{1}{3}$

- However, the classical definition of probability fails to provide result for the following cases:

i) If the event space is not finite
ii) If the events are not equally likely

In order to overcome this drawback, another definition of probability has been developed which is known as the frequency definition of probability. This definition is given below:

- **Frequency Definition of Probability:**

Let E be a random experiment which is repeated N number of times out of which the event A occurs N(A) times.

The probability of A= P(A)= $f(A) = \frac{N(A)}{N}$, where $f(A) = \frac{N(A)}{N}$ is called frequency ratio of the event A.

WORKED OUT PROBLEMS

1. What is the probability that a leap year, selected at random, will contain 53 Sunday ?

Solution. A leap year contain 366 days, i.e. , 52 full weeks (=364 days) and two days extra. These extra two days will be either i) (Sunday, Monday) or ii) (Monday , Tuesday) or iii) (Tuesday, Wednesday) or iv) (Wednesday, Thursday) or v) (Thursday, Friday) or vi) (Friday, Saturday) or vii) (Saturday, Sunday) .

Therefore a leap year contains 53 Sundays if one of the extra days is Sunday. Now one of the above seven cases two are favourable.

Then, required probability = $\frac{2}{7}$

2. Two cards are drawn in succession from a pack of 52 cards. Find the probability that the first is a king and the second is a queen, if the first card is i) replaced, ii) not replaced.

Solution. i) The first card can be drawn in $^{52}C_1 = 52$ ways

The second can be drawn in $^{52}C_1 = 52$ ways

Then, the total no. of outcome is 52x52 = 2704

Let A denote the event 'First is a king and the second is a queen'.

Now the pack contains 4 kings and 4 queens. So the first card king may come in

$^4C_1 =$ 4 ways and the second queen card may come in $^4c_1$=4 ways

Then, m (A)= total number of event points=4x4=16

Therefore P (A) =$\frac{16}{2704} = \frac{1}{169}$.

ii) If the first card is not replaced, then the first card in drawn in $^{52}C_1$= 52 ways, and the

second card in $^{51}C_1$= 51 ways.

Therefore, the total no. of outcome =- 52x51

Let B denote the event "First is a king and the second is a queen".

Now the pack contain 4 king cards and 4 queen cards. So the first king card is drawn in

$^4C_1$=4 ways.

Therefore, m(B)= total no. of event points= 4x4 = 16

So, P (B) = $\frac{16}{52x51} = \frac{4}{13x51} = \frac{4}{663}$.

3. Among the digits 1,2,3,4,5 at first one is chosen , and then a second selection is made among the remaining four digits . Assuming that all possible outcomes are equally likely , find the probability that an odd digit will be selected at i) the first time , ii) second times, iii) both times.

Solution. The total number of outcomes = ${}^5C_1 \times {}^4C_1 = 5 \times 4 = 20$

i) Let A denote the event "odd digits will be selected at the first time". So

A= { (1,2), (1,3), (1,4), (1,5), (3,1), (3,2), (3,4), (3,5), (5,1), (5,2), (5,3), (5,4) }

Therefore the no. of event points = 12

and P(A) = $\frac{12}{20} = \frac{3}{5}$, which is the required probability.

ii) Let B denote the event "odd digits will be selected at the second time. So

B= { (2,1), (2,3), (2,5), (4,5), (4,3), (4,1), (1,3), (1,5), (3,1), (3,5), (5,1), (5,3) }

Therefore the no. of event points = 12

and P(A) = $\frac{12}{20} = \frac{3}{5}$, which is the required probability

iii) Let C denote the event "both the chosen digits are odd". So

C= { (3,1), (1,3), (1,5), (5,1), (3,5), (5,3) }

Therefore the no. of event points = 6

and P(A) = $\frac{6}{20} = \frac{3}{10}$, which is the required probability

LECTURE 27:

AXIOMATIC DEFINITION OF PROBVABILITY

Let E be random experiment and S be the corresponding event space. Let A be an event associated with the random experiment E. Then the Probability of A, denoted by **P(A)**, is a real number which satisfies the following axioms:

(i) $P(A) \geq 0$ for every $A \in S$

(ii) $P(S) = 1$.

(iii) If $A_1, A_2, A_3, \ldots, A_n, \ldots$ be a finite or infinite number of pair wise mutually exclusive events,

i.e., $A_i A_j = \varphi$ for $i \neq j$ and $A_i, A_j \in S$, then

$$P(A_1 + A_2 + A_3 + \ldots + A_n + \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n) + \ldots$$

SOME RESULTS FROM AXIOMATIC DEFINITION OF PROBVABILITY

- Probability of an impossible event is 0, i. e., $P(\varphi)=0$.
- Probability of the complementary event $\bar{A}$ of A is $P(\bar{A})=1-P(A)$.
- For any two events A, B, $P(\bar{A}B) = P(B)-P(AB)$

  Addition Theorem:
  - For any two events A, B; $P(A+B) = P(A) + P(B) - P(AB)$
  - For any three events A, B, C;

    $P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$
- **BOOLE'S INEQUALITY:** For any events $A_1, A_2, A_3, \ldots, A_n$ connected to a random experiment E, $P(A_1+A_2+A_3+\ldots+A_n) \leq P(A_1)+P(A_2)+P(A_3)+\ldots+P(A_n)$.

CONDITIONAL PROBABILITY

Let A, B be two events connected to a given random experiment E. The ***conditional probability*** of the event A on supposition that B has already occurred, denoted by P(A/B), is defined by

$$P(A/B) = \frac{P(AB)}{P(B)}, \text{ provided } P(B) \neq 0.$$

Similarly the ***conditional probability*** of the event B on supposition that A has already occurred, denoted by P(A/B), is defined by

$$P(B/A) = \frac{P(AB)}{P(A)}, \text{ provided } P(A) \neq 0.$$

NOTE: (i) From definition of Conditional probability, we have $P(AB) =$ P(A/B) P(B) = P(B/A) P(A).

(ii) Two events A, B connected to a given random experiment E are called **Independent Events** if $P(AB) =$ P(A) P(B).

WORKED OUT PROBLEMS

1. Let A and B be two events such that P(A+B) = $\frac{7}{8}$. P(AB) = $\frac{1}{4}$ and P(□A)= $\frac{5}{8}$. Find the conditional

probability of A on the hypothesis that B does not occur.

Solution. We have P(A+B) = $\frac{7}{8}$. P(AB) = $\frac{1}{4}$ and P(□A)= $\frac{5}{8}$.

We have to find P(A/□B) = $\frac{P(AB)}{P(B)}$

= $\frac{P(A)-P(AB)}{1-P(B)}$ …. (1)

Now, P(A) = 1-P(□A) = 1- $\frac{5}{8}$ = $\frac{3}{8}$.

and P(A+B) =P(A) + P(B) –P(AB)

or, $\frac{7}{8}$ = $\frac{3}{8}$ + P(B) – $\frac{1}{4}$

So, P(B) = $\frac{3}{4}$.

From (1) we get P(A/□B) = $\frac{1}{2}$

2. For any two events A and B , then prove that, P(AB)<=P(A)<=P(A+B)<=P(A)+P(B)

Ans. We know that P(B/A)=$\frac{P(AB)}{P(A)}$

therefore, P(AB)=P(A) P(B/A)

Since, 0<=P(B/A)<=1

Therefore, P(AB)<=P(A)…..(1)

Again, P(A+B)= P(A)+P(B)-P(AB)……(2)

=P(A)+P(□AB)

Since, 0<=P(□AB)<=1, SO P(A+B)>=P(A)…AB)<=1, SO P(A+B)>=P(A)…(3)

Combining (1) and (2) we get,

P(AB)<=P(A)<=P(A+B)…..(4)

Also from (2) we get,

P(A)+P(B)>=P(A+B)….(5) as 0<=P(AB)<=1

Therefore, from (4) and (5) it follows that

P(AB)<=P(A)<=P(A+B)<=P(A)+P(B).

- **BONFERRONI'S INEQUALITY:** For any n events A_1, A_2,…..,A_n connected to a random experiment E,

i) $P(A_1A_2A_{3\ldots}A_n) \geq 1- \sum^n_{i=1} P(\overline{A}_i)$

ii) $P(A_1A_2A_{3\ldots}A_n) \geq \sum^n_{i=1} P(A_i)-(n-1)$

BAYE'S THEOREM:

If $A_1, A_2, A_3,\ldots,A_n$ be pair wise mutually exclusive events connected to a given random experiment one of which certainly occurs and A is any event connected to the same random experiment, then

$$P(A) = \sum^n_{i=1} P(A_i)P(A/A_i)$$

and $P(A_i/A) = \frac{P(Ai)P(A/Ai)}{P(A)}$, provided the conditional probabilities are defined.

Proof. Will be done in class.

WORKED OUT PROBLEMS

1. The probabilities of X, Y, Z becoming the Principal of a college are respectively 0.3, 0.5, 0.2. The probabilities that "Student-Aid Fund" will be introduced in the college if X, Y, Z become Principal, are 0.4, 0.6, 0.1 respectively. Given that "Student-Aid Fund" has been introduced. Find the probability that Y has been appointed as the Principal.

Solution. We have P(X) = 0.3, P(Y)=.5 and P(Z) = 0.2.

Let A be the event that "Student-Aid Fund is introduced". So by the given conditions we have,

P(A/X) = 0.4, P(A/Y) = 0.6 and P(A/Z) = 0.1. We have to find P(Y/A).

By Baye's theorem,

$P(A)= P(X)\,P(A/X) + P(Y)\,P(A/Y) + P(Z)\,P(A/Z) = 0.44$

Also by Baye's theorem,

$$P(Y/A) = \frac{P(Y)P(A/Y)}{P(A)} = \frac{0.5\times 0.6}{0.44} = \frac{15}{22}$$

LECTURE 28:

RELATED PROBLEMS

1. There were three candidates A, B, C for the position of a manager whose chances of getting the appointment are in the proportion 4 : 2 : 3 respectively. The probability that A, if selected, would launch a new product in the market is 0.3. The probabilities of B and C doing the same are 0.5 and 0.8 respectively. What is the probability that the new product was launched in the market by C?

Solution. Let,

The event A= Candidate A gets appointment
The event B= Candidate B gets appointment
The event C= Candidate C gets appointment
The event X= The new product is launched

We have, P(A) = $\frac{4}{9}$, P(B) = $\frac{2}{9}$, P(A) = $\frac{3}{9}$ = $\frac{1}{3}$, P(X/A) = 0.3, P(X/B) = 0.5 and P(X/C) = 0.8.

We have to find P(X).

By Baye's theorem,

P(X) = P(A) P(X/A) + P(B) P(X/B) + P(C) P(X/C) = $\frac{23}{45}$.

Ex 2. In how many of the permutations of 10 different things taken 4 at a time , will one particular thing (a) never occur ? (b) always occur ?

Solution: (a) We leave aside the particular thing and permute three from remaining 9 things which can be done in $^9P_4$ = 504 ways. (b) Keep the particular thing at 1st place . Then the remaining three places can be permuted from 9 things in $^9P_3$ ways . Similarly keeping the particular thing at 2nd place the three things can be permuted in $^9P_3$ ways also. In this way keeping the particular thing at 3rd place and 4th place the no. of permutation be $^9P_3$ each .

Hence the total no. of required permutation = $4\times{}^9P_3$ = 2016.

Ex 3. How many odd numbers greater than 600000 can be formed from the digits 5,6,7,8,9,0 if

(a) Repetitions are allowed

Solution: (a) The first place can be filled by 6,7,8,9 i.e. in 4 ways as the number is to be greater

than 600000. The last place can be filled in by 5,7,9 i.e. 3 ways s the number is to be odd and because of repetition. Hence the number of ways of filling the 1^{st} and last place is 4×3 = 12 ways

We have to fill in the remaining 4 places of the six digit number i.e. 2^{nd} , 3^{rd} , 4^{th}, and 5^{th} place . Since repetition is allowed each place can be filled in 6 ways. Hence the 4 places can be filled in 6×6×6×6 = 1296 ways. Hence by fundamental theorem the total numbers will be 1296 ×12= 15552.

(b) Repetitions are not allowed

Total no. of ways of filling of filling the first and the last place under given condition is

3+3+2+2 = 10 ways. Having filled in the first and the last places in 10 ways , we can fill in the remaining 4 places out of 4 digits (repetition not allowed) in 4! = 24 ways. Therefore total no. of ways is 10×24 = 240.

Ex 4. In how many ways can 10 books be arranged on a shelf such that a particular pair of books should always be together?

***Solution*:**We have a total of 10 books.

Given that a particular pair of books should always be together. Hence, just tie these two books together and consider as a single book.

Hence we can take total number of books as 9. These 9 books can be arranged in ${}^{9}P_{9}$ =9!ways.

We had tied two books together. These books can be arranged among themselves in ${}^{2}P_{2}$ =2! ways.

Hence, required number of ways
=9!×2!

Ex 5. In how many ways can 4 different balls be distributed among 5 different boxes when any box can have any number of balls?

Solution: Here both balls and boxes are different.

1^{st} ball can be placed into any of the 5 boxes.
2^{nd} ball can be placed into any of the 5 boxes.
3^{rd} ball can be placed into any of the 5 boxes.
4^{th} ball can be placed into any of the 5 boxes.

Hence, required number of ways
$= 5\times5\times5\times5=5^4$

Ex 6. A company has 10 software engineers and 6 civil engineers. In how many ways can a committee of 4 engineers be formed from them such that the committee must contain at least 1 civil engineer?

***Solution*:** The committee should have 4 engineers. But the committee must contain at least 1 civil engineer.

Initially we will find out the number of ways in which a committee of 4 engineers can be formed from 10 software engineers and 6 civil engineers.

Total engineers $=10+6=16=10+6=16$
Total engineers in the committee $=4=4$
Hence, number of ways in which the committee of 4 engineers can be formed
$= {}^{16}C_4$ ...(A)

Now we will find out the number of ways in which a committee of 4 engineers can be formed from 10 software engineers and 6 civil engineers such that the committee must not contain any civil engineer.

For this, select 4 software engineers from 10 software engineers.

Hence, number of ways in which the committee of 4 engineers can be formed such that the committee must not contain any civil engineer
$= {}^{10}C_4$ ...(B)

From (A) and (B),
Number of ways in which the committee of 4 engineers can be formed from 10 software engineers and 6 civil engineers such that the committee must contain at least 1 civil engineer
$= {}^{16}C_4 - {}^{10}C_4 = 1610.$

Ex 7. A box contains 4 different black balls, 3 different red balls and 5 different blue balls. In how many ways can the balls be selected if every selection must have at least 1 black ball and one red ball?

Solution: Number of ways in which one or more objects can be selected from n distinct objects*(i.e., we can select 1 or 2 or 3 or … or n objects at a time)*
$= {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$

It is explicitly given that all the 4 black balls are different, all the 3 red balls are different and all the 5 blue balls are different. Hence this is a case where all are distinct objects.

Initially let's find out the number of ways in which we can select the black balls. Note that at least 1 black ball must be included in each selection.

Hence, we can select 1 black ball from 4 black balls
or 2 black balls from 4 black balls.
or 3 black balls from 4 black balls.
or 4 black balls from 4 black balls.

Hence, number of ways in which we can select the black balls
$= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$
$=2^4-1$ ···(A)

Now let's find out the number of ways in which we can select the red balls. Note that at least 1 red ball must be included in each selection.

Hence, we can select 1 red ball from 3 red balls
or 2 red balls from 3 red balls
or 3 red balls from 3 red balls

Hence, number of ways in which we can select the red balls

$= {}^3C_1 + {}^3C_2 + {}^3C_3$
$=2^3-1 \cdots (B)$

Now let's find out the number of ways in which we can select the blue balls. There is no specific condition given here.

Hence, we can select 0 blue ball from 5 blue balls *(i.e, do not select any blue ball. In this case, only black and red balls will be there)*
or 1 blue ball from 5 blue balls
or 2 blue balls from 5 blue balls
or 3 blue balls from 5 blue balls
or 4 blue balls from 5 blue balls
or 5 blue balls from 5 blue balls.

Hence, number of ways in which we can select the blue balls
$= {}^5C_0 + {}^5C_1 + {}^5C_2 + \ldots + {}^5C_5$
$=2^5 \cdots (C)$

From (A), (B) and (C), required number of ways
$=(2^4-1)(2^3-1)2^5$.

LECTURE 29:

Binomial Distribution:

A discrete random variable X is said to be a binomial distribution with parameters $p(0 < p < 1)$ and n (a positive integer) if its distribution is given by

$$X : \quad 0 \quad 1 \quad 2 \quad 3 \quad \ldots \quad \ldots \quad n$$

$$f_i : \quad f_0 \quad f_1 \quad f_2 \quad f_3 \quad \ldots \quad \ldots \quad f_n$$

Where the $pmf f_i = P(X = i) = (n\,i)p^i(1-p)^{n-i}$, $i = 0, 1, 2, \cdots, n$

For instant, one probability mass

$f_1 = (n\ 1)p(1-p)^{n-1} = np(1-p)^{n-1}$ etc.

Illustration:

The efficiency of a fighter-plane is such that the probability of a bomb hitting a target is 2/5. The fighter is assigned to completely destroy a camp of enemy-side. The plane carries 6 bombs, i.e., 6 bombs can be aimed at the camp. Here throwing a bomb is experiment. It can be repeated 6 times; ' A bomb hits the camp' = Success and X= number of success in 6 trials. Then X has the Binomial distribution,

here $f_i = (6\ i)\left(\frac{2}{5}\right)^i\left(1-\frac{2}{5}\right)^{6-i} = (6\ i)\left(\frac{2}{5}\right)^i\left(\frac{3}{5}\right)^{6-i}$

If it is known that the last four direct hits are necessary to destroy the camp then the probability of complete destruction of the camp

$=P(X\geq 4) = P(X=4) + P(X=5) + P(X=6) = f_4 + f_5 + f_6$

$=(6\ 4)\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2 + (6\ 5)\left(\frac{2}{5}\right)^5(3\ 5) + (6\ 6)\left(\frac{2}{5}\right)^6 = \frac{112}{625}$

Theorem: If X Binomial Distribution with parameter *n* and *p* then (i) its mean is *np* (ii)its variance is *npq* where *q=1-p*.

Proof. Here X : 0 1 2 … *n*

And its pmf is $f_i = P(X=i) = (n\ i)p^i(1-p)^{n-i}$

(i) Mean = $E(X) = \sum_{i=1}^{n} if_i = \sum_{i=1}^{n} i(n\ i)p^i(1-p)^{n-i}$

$=np\sum_{i=1}^{n}(n-1\ i-1)p^{i-1}(1-p)^{n-i}$

$=np\sum_{r=0}^{n-1}(n-1\ r)p^{i-1}(1-p)^{n-1-r}$, replacing $(i-1)$ by *r*.

$=np(p+1-p)^{n-1} = np$

(ii) Now

$$E\{X(X-1)\} = \sum_{i=0}^{n} i(i-1)(n\,i\,)p^{i}(1-p)^{n-i}$$

$$= \sum_{i=0}^{n} i(i-1)\frac{n(n-1)}{i(i-1)}(n-2\,i-2\,)p^{i}(1-p)^{n-i}$$

$$= n(n-1)p^{2}\sum_{i=2}^{n}(n-2\,i-2\,)p^{i-2}(1-p)^{n-i}$$

$= n(n-1)p^{2}\sum_{r=0}^{n-2}(n-2\,r\,)p^{r}(1-p)^{n-2-r}$, replacing $i-2$ by *r*

$= n(n-1)p^{2}(p+1-p)^{n-2} = n(n-1)\,p^{2}$

Therefore, $Var(X) = E\{X(X-1)\} - m(m-1)$

$= n(n-1)p^{2} - np(np-1) = np(1-p) = npq$

Therefore, standard deviation $\sigma = \sqrt{npq}$.

Illustration:

An unbiased die is tossed four times. Let 'multiple of three' be success; otherwise it is failure. Here *p*=probability of success of a single trial = 2/6=1/3. Let X = number of 'multiple of three' appeared among these four trial. Then, as we discussed before, X has Binomial distribution with parameter *n*=4 and *p*=1/3. The expected number of 'multiple of 3' = mean of X= $4\times\frac{1}{3} = \frac{4}{3}$. The standard deviation of

$X = \sqrt{4\times\frac{1}{3}\,(1-\frac{1}{3})}$ $=\sqrt{4\times\frac{1}{3}\times\frac{2}{3}} = \frac{2\sqrt{2}}{3}$

Example:

Comment on the statement "a binomial variate has mean 4 and s.d. 3".

Here , $np = 4$ and $\sqrt{np(1-p)}$= 3 i.e. $np(1-p) = 9$

Therefore , $4(1-p) = 9$

Therefore, $1 - p = \frac{9}{4}$

Therefore, $p = -\frac{5}{4}$ which is not possible since $0 < p < 1$. So the statement is false.

Example:

If the mean of a binomial distribution is 3 and the variance is 3/2, find the probability of obtaining at most 3 success.

Let X r.v. corresponding to the number of success. Then the *pmf* of X is

$f_i = P(X = i) = inCp^i(1 - p)^{n-i}$, $i = 0, 1, 2, \ldots, n$

Or $\frac{np(1-p)}{np} = \frac{3}{2} \times \frac{1}{3}$

So, p=1/2 and n=6

Now probability of at most 3 success = $P(X) \leq 3$

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

=

$$06C\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^6 + 16C\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 + 26C\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 + 36C\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{21}{32}$$

Poisson Distribution:

A discrete random variable X is said to have a Poisson distribution with parameter μ(>0) if its distribution is given by

X :	0	1	2	3	…	…
f_i :	f_0	f_1	f_2	f_3	…	…

Where the *pmf* , $f_i = P(X = i) = \frac{e^{-\mu}\mu^i}{i!}$

e.g one probability mass, $f_2 = \frac{e^{-\mu}\mu^2}{2!}$ Etc.

Theorem: If X has a random variate with parameter μ then

(i) Mean of X is μ
(ii) Variance of X is μ

Proof: The values assumed by X are 0, 1, 2, …, … with probability

$$P(X = i) = \frac{e^{-\mu}\mu^i}{i!}$$

(i) Mean $= E(X) = \sum_{i=0}^{\infty} if_i = \sum_{i=0}^{\infty} \frac{i\, e^{-\mu}\mu^i}{i!}$

$$= e^{-\mu} \sum_{i=0}^{\infty} \frac{\mu^i}{(i-1)!} = e^{-\mu}\left(\frac{\mu}{0!} + \frac{\mu^2}{1!} + \frac{\mu^3}{2!} + \ldots upto\ \infty\right)$$

$$= \mu e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \ldots upto\ \infty\right)$$

$$= \mu e^{-\mu} e^{\mu} = \mu$$

(ii) Now, $E\{X(X-1)\} = \sum_{i=0}^{\infty} i(i-1)e^{-\mu}\frac{\mu^i}{i!} = e^{-\mu}\mu^2 \sum_{i=2}^{\infty} \frac{\mu^{i-2}}{(i-2)!}$

$$= \mu^2 e^{-\mu}\left(1 + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \ldots upto\ \infty\right)$$

$$= e^{-\mu}\mu^2 e^{\mu} = \mu^2$$

So , $Var(X) = E\{X(X-1)\} - m(m-1)$, where m=mean

$$= \mu^2 - \mu(\mu - 1) = \mu$$

Ex. 1. For Poisson variate if *P(X=2)=P(X=1)*, find *P(X=1 or 0)*. Find also mean of X.

Let m be the parameter of Poisson variate.

So, $P(X = i) = f_i = \frac{e^{-m}m^i}{i!}$

Now $f_2 = f_1$, or, $\frac{e^{-m}m^2}{2!} = \frac{e^{-m}m^1}{1!}$ Therefore m=2

Therefore $P(X = 1 \text{ or } 0) = f_1 + f_0 = e^{-m}(1 + m) = e^{-2}(1 + 2) = 3e^{-2}$

Mean of *X=m=2.*

Ex. 2. A radio active source emits on the average 2.5 particles per second. Calculate the probability that 2 or more particles will be emitted in an interval of 4 seconds.

Here λ= number of changes (which is particles emitted) per unit time on an average=2.5.

Let X be the random variable denoting the number of particles emitted in the given interval. Then X is Poisson distributed with parameter μ= average number of particles in 4 seconds.

So the *pmf*, $f_i = P(X = i) = \frac{\mu^i e^{-\mu}}{i!} = \frac{10^i e^{-10}}{i!}$

So the required Prob. $= P(X \geq 2) = 1 - P(X < 2)$

$$= 1 - \{P(X = 0) + P(X = 1)\}$$

$$= 1 - \left\{e^{-10} + 10e^{-10}\right\} = 1 - 11e^{-10}$$

LECTURE 30:

Normal Distribution:

A continuous random variable X is said to have a normal distribution if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}} \quad ,-\infty < x < \infty$$

Where μ and $\sigma > 0$ are the two parameters of the distribution.

Ex. 1. If weekly wage of 10,000 workers in a factory follows normal distribution with a mean and *s.d*. Rs. 70 and Rs. 5 respectively, find the expected number of workers whose weekly wages are (i) between Rs. 66 and Rs. 72 (ii) less than Rs. 66 (iii) more than Rs. 72

[Given that $\frac{1}{\sqrt{2\pi}}\int_0^{\pi} e^{\frac{-t^2}{2}}\, dt$ =0.1554 and 0.2881 according as z=0.4 and z=0.8]

Let X= wage of worker. X has a normal distribution with μ=70, σ=5.

Therefore $Z = \frac{X-70}{5}$ has standard normal distribution.

(i) When X=66, Z= (66-70)/5 = -0.8 ; and when X=72 , Z= (72-70)/5 = 0.4

So, P (66 < X < 72) = P (-0.8 <Z < 0.4)

= area under the standard normal curve enclosed between the two ordinates Z=-0.8 and Z=0.4

$$= \int_{-0.8}^{0.4} \phi(t)\, dt = \int_{-0.8}^{0} \phi(t)dt + \int_{0}^{0.4} \phi(t)dt$$

$= \int_0^{0.8} \phi(t)dt + \int_0^{0.4} \phi(t)dt$, since ϕ curve is symmetric about Y axis.

= 0.2881 + 0.1554 = 0.4435

So the probability that the wage of the worker lies between Rs. 66 and Rs. 72 is 0.4435.

The number of worker whose wage lie between Rs. 66 and Rs. 72 = 0.4435 × 10,000 = 4435.

(ii) When X= 66 , Z = -0.8

So, P (X < 66) = P (Z < -0.8)

= area under the standard normal curve enclosed on the left side of the ordinate Z = -0.8

$$= \int_{-\infty}^{-0.8} \phi(t)\, dt = \int_{-0.8}^{0} \phi(t) dt \text{ (by symmetry)}$$

$$= 0.5 - \int_{0}^{0.8} \phi(t) dt = 0.5 - 0.2881 = 0.2119$$

The expected number of workers = 0.2119 × 10, 000 = 2119

(iii) When X = 72 , Z = 0.4

So, $P\,(X > 72) = P(Z > 0.4) = \int_{0.4}^{\infty} \phi(t) dt$

$$=0.5 - \int_{0}^{0.4} \phi(t) dt = 0.5 - 0.1554 = 0.3446$$

The required number = 0.3446 × 10,000 = 3446

Ex. 2. The mean of a normal distribution is 50 and 5% of values are greater than 60. Find the standard deviation of the distribution (Given that the area under normal curve between z=0 and z= 1.64 is 0.45)

Let X be the normal variate. Let its s.d. be σ.

Z= (X-50) / σ is the standard normal variate. By problem

P(X> 60) = 5/100 = 0.05.

When X= 60 , Z = (60-50) / σ = 10/σ

Therefore, from above P (Z > 10 / σ) =0.05 ---------- (1)

From the supplied data we have P (0 < Z < 1.64) = .45

Or, P (Z > 164) = 0.5 – 0.45 = 0.05 ---------- (2)

Comparing (1) and (2) we get (10 / σ) = 1.64

Or, σ = 10 / 1.64 = 6.097 = 6.1 and the s.d. is 6.1

Probability Mass Function and Probability density Functiom

Discrete Random Variable : A random variable X is said to be discrete if the spectrum of X is finite or countably infinite i.e. an infinite sequence of distinct values.

Continuous Random Variable : A random variable X is said to be continuous if it can assume every value in an interval.

Illustration. (i) Let us consider the random experiment of tossing two (unbiased) coins. Then the sample space S contains 4 sample points.

i.e. , S= { HH , HT, TH, TT}

Let the random variable X be such that X (an outcome) = "the number of heads". Then X is a function over S defined by

X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 0.

Thus the spectrum of X is { 0, 1, 2} which is a finite set. Hence X is a discrete random variable here. Here the event (X=1) = { TH, HT} = 'One Head' , the event (-1 < X ≤ 0) = {TT} = 'Two tails'.

(ii).Let the random variable X denote the weights (in kg) of a group of individuals. Then X can assume every value in an interval say (30, 100), supposing there is no individual having weight less than 30 and greater than 100. Hence X is a continuous random variable. Here event (42 < X ≤ 50) = the group of individual whose weight lie between 42 and 50, including 50; the event (X = 70) = The group of individuals whose weight is 70 kg.

LECTURE 31:

Probability Mass Function and Discrete Distribution :

Let X be a discrete random variable which assumes the values $x_0, x_1, x_2, \ldots, x_n, \ldots$ Let $P(X = x_i) = f(x_i) = f_i$. So the value of f_i depends on x_i i.e. i. Thus the function f_i is called *Probability Mass Function* **(p.m.f)** of the random variable X. a particular value of f_i is called probability mass.

The set of ordered pairs $(x_i,\ f_i)$ is called discrete probability distribution of the random variable X.

Discrete distribution is presented in the following way :

$$X \ : \quad 0 \quad 1 \quad 2 \quad \ldots \quad \ldots$$

$$f_i \ : \quad f_0 \quad f_1 \quad f_2 \quad \ldots \quad \ldots$$

Illustration. For the random experiment of tossing two coins given in illustration (i), we see X assumes the values 0, 1 and 2.

Moreover $(X = 0) = \frac{1}{4},\ P(X = 1) = \frac{1}{2},\ P(X = 2) = \frac{1}{4}$.

So, the distribution of the number of heads is given by

$$X \quad : 0 \quad 1 \quad 2$$

$$f_i \quad : \ \frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4}$$

Fundamental Properties of pmf:

$$\text{If} \quad X \ : \quad 0 \quad 1 \quad 2 \quad \ldots \quad \ldots$$

$$f_i \ : \quad f_0 \quad f_1 \quad f_2 \quad \ldots \quad \ldots$$

is a discrete distribution of X, then the pmf has the following two properties:

(i) $f_i \geq 0$

(ii) $\sum_i f_i = 1$

Distribution Function or Cumulative Distribution Function.

The distribution function (d.f) of a random variable X (discrete or continuous) is given by

$$F(x) = P(-\infty < X < x),\ -\infty < x < \infty$$

Thus, if $x_i \leq x \leq x_{i+1}$, then

$$F(x) = P\left(X = x_0\right) + P\left(X = x_i\right) + \ldots + P\left(X = x_i\right) = \sum_{\alpha=0}^{i} f_\alpha .$$

Illustration. In the discrete distribution

$$X \quad : \quad 0 \qquad 1 \qquad 2$$

$$f_i \quad : \frac{1}{4}\ \frac{1}{2}\ \frac{1}{4}$$

The d.f is F(x)=0 , x <0

= 1/4 , 0 ≤ x < 1

= (1/4) + (1/2) , 0 ≤ x < 2

= (1/4) + (1/2) + (1/4) , 2 ≤ x

Properties of Distribution Function:

(i) The distribution function F(x) is a monotonic non-decreasing function.

(ii) $F(-\infty) = 0$ and $F(\infty) = 1$ and $0 \leq F(x) \leq 1$.

(iii) $F(x)$ is a continuous on the right of all points and has a jump discontinuity on the left at x=a , the height of jump being equal to
$P(X = a) i.e. , F(a) = F(a)\ and\ F(a) = F(x) = P(x = a)$

(iv) Suppose *a* and *b* are any real numbers such that *a<b*
Then *P* (*a* < X ≤ b) = *F*(*b*) – *F* (*a*) ,
P (a< X< b)= F(b) – F(a)– P(X=b)
And *P(a ≤ X < b)= F(b) – F(a) – P(X=b) + P(X=a)*

Illustration (i) Let X be a random variable denoting the number of points appearing in a toss of a die. The distribution of X is

$$X \quad : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$f_i \quad : \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}$$

Now, if *x<1, F(x)= P(X≤ x) = 0*

If 1 ≤ x < 2 , F(x)= P(X ≤ x) = $f_1 = \frac{1}{6}$

If 2 ≤ x < 3 , F(x)= P(X ≤ x)= $f_1 + f_2 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

and so on.

Thus the distribution function *F(x)* is given by :

F(x)= 0 , $-\infty < x < 1$

= 1/6 , 1 ≤ x < 2

= 2/6 , 2 ≤ x <3

= 3/6 , 3 ≤ x < 4

= 4/6 , 4 ≤ x < 5

= 5/6 , 5 ≤ x < 6

= 1 , $6 \leq x < \infty$

Density Curve : The curve given by *y=f(x) (f(x) is pdf)* is the probability density curve which gives the graphical representation of the corresponding continuous distribution.

Illustration : consider a function *y=f(x) (f(x) is pdf)* which is defined as

$f(x) = \frac{2}{x^3}$, $1 \leq x < \infty$

= 0, elsewhere

As $f(x) > 0$ everywhere and $\int_{-\infty}^{\infty} f(x)dx$

$$= \int_1^{\infty} \frac{2}{x^3} dx = \int_1^{P} \frac{2}{x^3} dx = \left(1 - \frac{1}{P^2}\right) = 1 - 0 = 1$$

So this *f(x)* is a probability density function of some random variable.

Now, $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0 \cdot dx = 0 \; when \; -\infty < x < 1$

And $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{1}^{x} \frac{2}{x^3} dx = 0 \; when \; 1 < x < \infty$

$$= 1 - \frac{1}{x^2}$$

So the distribution function of *pdf* is

$$F(x) = 0, \quad -\infty < x < 1$$

$$= 1 - \frac{1}{x^2}, \quad 1 \leq x < \infty$$

Expectation of a Discrete Random Variable:

Let X be a discrete random variable whose distribution is

$$X : \quad 0 \quad 1 \quad 2 \quad 3 \quad \ldots \quad \ldots \quad n \ \ldots$$

$$f_i : \quad f_0 f_1 f_2 \quad f_3 \quad \ldots \quad \ldots \quad f_n \quad \ldots$$

Then the mean or expectation or expected value of X , denoted by *E(X) or m(X)* or simply *m* is defined as

$E(X) = x_0 f_0 + x_1 f_1 + x_2 f_2 + \ldots = \sum_i x_i f_i$, provided the series is absolutely convergent if the above sum is an infinite series.

Expectation of Continuous Random Variable :

For a continuous random variable X with probability density function *f(x)* , the mean or expectation of X is defined as

$E(X) = \int_{-\infty}^{\infty} xf(x)dx$,

Provided the infinite integral converges absolutely.

Similarly, the mean of a function $\Psi(X)$ on the random variable X denoted by $E\{\Psi(X)\}$ is defined as

$E\{\Psi(X)\} = \sum_i \Psi(X_i) f_i$, for a discrete distribution.

$= \int_{-\infty}^{\infty} \Psi(x) f(x) dx$, for a continuous distribution

Illustration. (i) Suppose a die is rolled. Let X be the number of points on the die. Then its values are 1, 2, 3, 4, 5, 6.

So, P(X=i) = 1/6 for i = 1 ,2 , 3 ,4 ,5 ,6.

So the distribution X is

$$X \quad : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$f_i \quad : \quad \frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}$$

Therefore its expectation

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{7}{2}$$

And

$$E\left(X^{2+}1\right) = \left(1^2 + 1\right) \bullet \frac{1}{6} + \left(2^2 + 1\right) \bullet \frac{1}{6} + \left(3^2 + 1\right) \bullet \frac{1}{6} + \left(4^2 + 1\right) \bullet \frac{1}{6} + \left(5^2 + 1\right) \bullet \frac{1}{6} +$$

(iii) Let the *pdf* of a continuous random variable X is

$$f(x) = \frac{1}{2} \quad in \; -1 < x < 1$$

= 0, elsewhere

The mean or expectation of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^{1} x \frac{1}{2} dx = 0$$

And also, $E\left(2X^3\right) = \int_{-\infty}^{\infty} 2x^3 f(x) dx = \int_{-1}^{1} 2x^3 \bullet \frac{1}{2} dx = \left[\frac{x^4}{4}\right] 1 \; - 1 \; = 0$

LECTURE 32:

Properties of Expectation

(i) *E(a) = a* , where *a* is being a constant
(ii) *E(aX) = a E(X)* where *a* being a constant
(iii) *E(X±Y) = E(X)* $\pm$ *E(Y)* , *X* , *Y* are two r.v.
(iv) *E(XY) = E(X) E(Y)* if the two r.v X and Y are independent.

Variance and S.D :

The variance of a r.v. X , denoted by $Var(X)$ is defined as $Var\ (X) = E((X - m^2))$, where $m = E(X)$

The positive squre root of $Var\ (X)$ is called the *standard deviation*of X and is denoted by $\sigma(X)$ or σ_x or simply . Thus $\sigma = + \sqrt{var(X)}$.

Remarks:

(i) The variance describes how widely the probability masses are spread about the mean i.e. it gives an inverse measure of concentration of the probability masses about the mean which is called the measure of dispersion.
(ii) As $Var(X) = 0$ only when $X - m = 0$ i.e $= m$, so in that case whole mass is concentrated at the mean.

Theorem:

(i) $Var(X) = E\left(X^2\right) - m^2 = E\left(X^2\right) - \{E(X)\}^2$
(ii) $Var(aX + b) = a^2 Var(X)$
(iii) $Var(k) = 0$ where *k* is constant.
(iv) $Var(X) = E\{X(X - 1)\} - m(m - 1)$ where *m* is mean of X

Illustration: Consider the following distribution of a random variable X:

$$f(x) = \frac{1}{2}x\ , \quad 0 \leq x \leq 2$$

$$= 0\ , \quad \text{elsewhere}$$

The expectation of X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{2} x\frac{1}{2}xdx = \left[\frac{x^3}{6}\right]2\ 0 = \frac{4}{3}$$

Now,

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{0}^{2} x^2\frac{1}{2}xdx = \left[\frac{x^4}{8}\right]2\ 0 = 2$$

Therefore , $Var(X) = E\left(X^2\right) - \{E(X)\}^2 = 2 - \frac{16}{9} = \frac{2}{9}$

So, $s.d = \frac{\sqrt{2}}{3}$

Ex. 1. Find the probability distribution of the number of heads when a fair coin is tossed repeatedly until the first tail appears.

The sample space corresponding to the random experiment of the tossing of the fair coin is *S= { T, HT, HHT, HHHT, …}*

Let the random variable X denotes “the number of heads in the experiment until the first tail appears”.

Then the spectrum of X is {0, 1, 2, 3, …}

Now $P(X = 0) = P(T) = \frac{1}{2}$

$P(X = 1) = P(HT) = P(H)P(T)$ [since trails are independent]

$$= \frac{1}{2} \bullet \frac{1}{2} = \frac{1}{2^2}$$

$P(X = 2) = P(HHT) = P(H)P(H)P(T) = \frac{1}{2^3}$ and so on

Hence the probability distribution of X is

X : 0 1 2 3 …

f_i : $\frac{1}{2}\ \frac{1}{2^2}\ \frac{1}{2^3}\ \frac{1}{2^4}$ …

Ex. 2. A random variable X has the following probability mass function

X :	0	1	2	3	4	5	6	7
$P(X=k)=f(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i)determine the constant k.

(ii)evaluate P(X<6), P(X≥6), P(3<X≤6), and P(3< X/X ≤ 6)

(iii)find the minimum value of x so that P(X≤x)>1/2

(iv)obtain the distribution function F(x)

(i) Since *f(x)* is a p.m.f, $\sum_x f(x) = 1$

$$\therefore \sum_{x=0}^{7} f(x) = 1 \rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$= 10k^2 + 9k - 1 = 0$$

$$k = -1, \frac{1}{10}$$

$$\therefore k = \frac{1}{10} \quad [since\ f(x) \geq 0,\ \forall x = 0, 1, 2 \cdots 7\ and\ so\ k \neq -1]$$

(ii) $$P(X < 6) = 1 - P(X \geq 6) = 1 - \{P(X = 6) + P(x = 7)\}$$
$$= 1 - \left\{2\left(\frac{1}{10}\right)^2 + 7\left(\frac{1}{10}\right)^2 + \frac{1}{10}\right\} = \frac{81}{100}$$

$$\therefore P(X < 6) = 1 - P(X < 6) = 1 - \frac{81}{100} = \frac{19}{100}$$

$$P(3 < X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6) = \frac{33}{100}$$

$$P < \frac{X}{X} \leq 6 = \frac{P\{(3<X) \cap (X \leq 6)\}}{P(X \leq 6)} = \frac{P(3<X \leq 6)}{P(X \leq 6)} = \frac{\frac{33}{100}}{\frac{83}{100}} = \frac{33}{83}$$

(iii) $$P(\leq 3) = P\left(\frac{1}{10} < \frac{1}{2}\right),\ P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$
$$= \frac{1}{10} + 2 \cdot \frac{1}{10} = \frac{3}{10} < \frac{1}{2}$$

$$P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{1}{10} + 2 \cdot \frac{1}{10} + 2 \cdot \frac{1}{10} = \frac{1}{2}$$

$$P(X \leq 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(x = 4)$$

$$= \frac{8}{10} = \frac{4}{5} > \frac{1}{2}$$

Thus minimum value of x so that $P(X \leq x) > \frac{1}{2}$ is 4.

(iv) The distribution function F(x) is given below

$$F(x) = 0, \quad -\infty < x < 1$$

$$\frac{1}{10} \quad 1 \leq x < 2$$

$$\frac{1}{10} + 2 \cdot \frac{1}{10} = \frac{3}{10} \quad 2 \leq x < 3$$

$$\frac{3}{10} + 2 \cdot \frac{1}{10} = \frac{1}{2} \quad 3 \leq x < 4$$

$$\frac{1}{2} + 3 \cdot \frac{1}{10} = \frac{4}{5} \quad 4 \leq x < 5$$

$$\frac{4}{5} + \left(\frac{1}{10}\right)^2 = \frac{81}{100} \quad 5 \leq x < 6$$

$$\frac{81}{100} + 2 \cdot \left(\frac{1}{10}\right)^2 = \frac{83}{100} \quad 6 \leq x < 7$$

$$\frac{83}{100} + 7\left(\frac{1}{10}\right)^2 + \frac{1}{10} = 1 \quad 7 \leq x < \infty$$

Ex.3. Let $F(x) = 0 \quad -\infty < x < 1$

$$= \frac{1}{5} \quad 1 \leq x \leq < 2$$

$$= \frac{3}{5} \quad 1 \leq x < 3$$

$$= 1 \quad 3 \leq x < \infty$$

Show that *F(x)* is a possible distribution function. Determine the spectrum and the probability mass of the distribution.

Clearly *F(x)* is a monotonic decreasing function and is continuous on the right at all points.

Also, $F(-\infty) = 0$ and $F(\infty) = 1$. Hence $F(x)$ is a possible distribution function.

Again $F(x)$ is a step function and step points are 0, 1, 3. So the spectrum is { 0, 1, 3}.

Now $P(X = 0) = F(0) - F(x) = \frac{1}{5} - 0 = \frac{1}{5}$

$$P(X = 1) = F(1) - F(x) = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}$$

$$P(X = 3) = F(3) - F(x) = 1 - \frac{3}{5} = \frac{2}{5}$$

So the required probability mass of the distribution is

$$X \quad : \quad 0 \quad 1 \quad 3$$

$$f_i \quad : \quad \frac{1}{5} \; \frac{2}{5} \; \frac{2}{5}$$

Ex. 4. The distribution function *F(x)* of a variate X is defined as follows

$$F(x) = A \qquad -\infty < x < -1$$

$$= B \qquad -1 \leq x < 0$$

$$= C \qquad 0 \leq x < 2$$

$$= D \qquad 2 \leq x < \infty$$

Where A,B,C,D are constants. Determine the values of A, B, C, D given that P(X=0)= 1/6 and P(X>1)= 2/3.

We have $F(-\infty) = 0$

$$\therefore F(x) = 0 \quad i.e \quad A = 0 \qquad \therefore A = 0$$

Again $F(\infty) = 1 \qquad \therefore F(x) = 1$

i.e. $(D) = 1 \qquad \therefore D = 1$

Now, $\frac{1}{6} = P(X = 0) = F(0) - F(x) \qquad [P(X = a) = F(a) - F(x)\,]$

$$= C - F(x) = C - B$$

$$\therefore C - B = \frac{1}{6} \qquad \cdots(1)$$

$$again\ P(-\infty < X < \infty) = P(-\infty < X \leq 1) + P(1 < X < \infty)$$

$$\therefore 1 = P(-\infty < X \leq 1) + P(X > 1) = P(-\infty < X \leq 1) + \frac{2}{3}$$

$$\therefore P(-\infty < X \leq 1) = 1 - \frac{2}{3}$$

$$i.e.\ F(1) = \frac{1}{3}[F(x) = P(-\infty < X \leq x)]$$

$$\therefore C = \frac{1}{3}$$

From (1)$B = C - \frac{1}{6} = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

So, A=0, B=1/6, C=1/3, D=1

LECTURE 33:

Correlation Coefficient and Regression Lines

Correlation Coefficient $r(or\ \rho) = \dfrac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\,\bar{y}}{\sigma_x \sigma_y} = \dfrac{\mathrm{cov}(x, y)}{\sigma_x \sigma_y}$

Regression line Y on X is $Y - \bar{y} = r\dfrac{\sigma_y}{\sigma_x}(X - \bar{x})$

Regression line X on Y is $X - \bar{x} = r\dfrac{\sigma_x}{\sigma_y}(Y - \bar{y})$

Example : If $r = 0.4, \mathrm{cov}(x, y) = 10, \sigma_y = 5, find\ \sigma_x.$

Ans.

$$r = \frac{\mathrm{cov}(x, y)}{\sigma_x \sigma_y}$$

$$0.4 = \frac{10}{\sigma_x X 5}$$

$$\sigma_x = 5$$

Example: Calculate the correlation coefficient and determine the regression lines of Y on X and X on Y for the sample

X	8	10	5	8	9
Y	1	3	1	2	3

Ans:

$$\bar{x} = \frac{8+10+5+8+9}{5} = 8$$

$$\bar{y} = \frac{1+3+1+2+3}{5} = 2$$

$$\sigma_x^{\,2} = \frac{8^2+10^2+5^2+8^2+9^2}{5} - 8^2 = 2.8$$

$$\sigma_y^{\,2} = \frac{1^2+3^2+1^2+2^2+3^2}{5} - 2^2 = 0.8$$

$$\frac{1}{5}\sum_{i=1}^{5} x_i y_i = \frac{1}{5}(8X1+10X3+5X1+8X2+9X3) = 17.2$$

$$r = \frac{\frac{1}{5}\sum_{i=1}^{5} x_i y_i - \bar{x}\,\bar{y}}{\sigma_x \sigma_y} = \frac{17.2-8X2}{\sqrt{2.8}\sqrt{0.8}} = 0.802$$

Therefore correlation coefficient =0.802

Regression line Y on X is

$$Y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(X - \bar{x})$$

$$Y - 2 = 0.802\ x\ \sqrt{\frac{0.8}{2.8}}(X - 8)$$

$$Y = 0.429X - 1.432$$

Regression line X on Y is

$$X - \bar{x} = r\frac{\sigma_x}{\sigma_y}(Y - \bar{y})$$

$$or,\ X - 8 = 0.802\ x\ \sqrt{\frac{2.8}{0.8}}(Y - 2)$$

$$or,\ X = 1.5Y + 5$$

Linear Curve Fitting

Example: Fit a linear equation to the following data

X	2	4	6	8	10
Y	2.3	2.6	2.9	2.10	2.12

Ans:

x_i	y_i	$x_i X\ y_i$	x_i^2
2	2.3	4.6	4
4	2.6	10.6	16
6	2.9	17.4	36
8	2.10	16.80	64
10	2.12	21.20	100
$\sum x_i = 30$	$\sum y_i = 12.02$	$\sum x_i X\ y_i = 70.4$	$\sum x_i^2 = 220$

Let the linear equation be $y = a + bx$ where a, b are constants.

We get the normal equations as

$$\sum y_i = na + b\sum x_i$$
$$\sum x_i X\ y_i = a\sum x_i + b\sum x_i^2$$

Where n is the number of elements.

Here n=5

From the table we get

$$12.02 = 5a + 30b$$
$$70.4 = 30a + 220b$$

Solving these two equations we get

$a = 10.73$

$b = 0.043$

Therefore the fitted linear curve is $y = 10.73 + 0.043x$.

Example: Fit a parabola $y = a + bx + cx^2$ using least square method for the following data

X	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Y	1.1	1.3	1.6	2.0	2.7	3.3	4.1

Ans:

x_i	y_i	x_i^2	$x_i X\ y_i$	x_i^3	$x_i^2 y_i$	x_i^4
1.0	1.1	1.0	1.1	1.0	1.1	1.0
1.5	1.3	2.25	1.95	3.375	2.925	5.0625
2.0	1.6	4.0	3.2	8	6.4	16.0
2.5	2.0	6.25	5.0	15.625	12.5	39.0625
3.0	2.7	9.0	8.1	27	24.3	81
3.5	3.3	12.25	11.55	42.875	40.425	150.0625
4.0	4.1	16.0	16.4	64	65.6	256
$\sum x_i = 17.$	$\sum y_i = 16.1$	$\sum x_i^2 = 50.75$	$\sum x_i X\ y_i = 47.3$	$\sum x_i^3 = 161.875$	$\sum x_i^2 y_i = 153.25$	$\sum x_i^4 = 548.1875$

Using least square method, we get the normal equations as

$$\sum y_i = na + b\sum x_i + c\sum x_i^2$$
$$\sum x_i X\ y_i = a\sum x_i + b\sum x_i^2 + c\sum x_i^3$$
$$\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4$$

Where n is the number of observations.

Here n=7

From the above table we get,

$$16.1 = 7a + 17.5b + 50.75c$$
$$47.3 = 17.5a + 50.75b + 161.875c$$
$$153.25 = 50.75a + 161.875b + 548.1875c$$

Solving the above equations, using Gauss Elimination method, we get

$a = 1.0571$

$b = -0.20714$

$c = 0.24286$

Therefore the fitted parabola is $y = 1.0571 - 0.20714x + 0.24286x^2$

MCQ

1. Four coins are tossed. Expectation of number of heads is
   (a) 1 (b)2
   (c) 3 (d)4
2. A card is drawn at random from a well shuffled pack of 52 cards. The probability of getting a heart or a diamond is
   (a) 1 **(b)1/2**
   (c) 1/26 (d)3/13

3. Let A, B be two events and P(□A) = 0.3, P(B) =0.4, P(A□B) = 0.5, then P(A+□B) =
   (a) 0.5 **(b)0.8**
   (c) 1 (d) None of these

4. The probability that a leap year, selected at random, will contain 53 Sunday is
   (a) 1/7 (b)2/365
   (c) 2/7 (d) None of these

5. If A and B are two mutually exclusive events, then P(A+B) =
   (a) P(A)+P(B) (b) P(A) P(B)
   (c) P(A) –P(B) (d) None of these

6. If P(A) = 1/2, P(B) = 1/3, P(AB) = ¼, then the value of P(A ∪ B) is
   (a) 6/7 (b) 3/7
   (c) 1 **(d) 7/12**

7. " Two mutually exclusive events are always independent ". This statement is
   (a) true **(b) false**

8. If □A is the complementary event of A, then
   (a) P(□A)=1-P(A) (b) P(□A)=P(A)
   (c) P(□A)=1+P(A) (d) None of these

9. An unbiased die is thrown. The probability that either an even number or a number greater than 2 will turn up is

(a) 1/6 (b) 2/3
(c) 5/6 (d) None of these

10. A man draw at random three balls from a bag containing 6 red and 5 green balls. The probability of getting the balls all red is
    (a) 6/11 (b) 3/22
    (c) 4/33 (d) 1/6

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Shishir Gupta & S.Dey, Numerical Methods, TMH
C.Xavier: C Language and Numerical Methods.
Dutta & Jana: Introductory Numerical Analysis.
J.B.Scarborough: Numerical Mathematical Analysis.
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