Mathematics as an Area of Knowledge

Scope: what does mathematics study?

- most areas of knowledge have several features in common
  - they aim to give a true description of the world
  - identify generalized relationships and causal connections
- they vary in their methods and the scope of their conclusions because they study different aspects of the shared world
- Math has both similarities and difference
  - Similarities*: how do we know? What methods can we use to investigate, and what justifications can we offer for our knowledge claims?*
  - Differences: it is completely divorced from the physical world, and deals solely with abstraction. It is an area of clarity and certainty

Mathematics as the study of pattern

- Abstract patterns that places concepts in a systematized relationship to one another, expressed in a symbolic system that we can manipulate using reason alone
- What types of patterns?
  - Real or imagined
  - Visual or mental
  - Static or dynamic
  - Qualitative or quantitative
  - Ultilitarian or abstract
- Different kinds of patterns are studied in different maths
  - Number theory: patterns of number and counting
  - Geometry: patterns of shape
  - Calculus: patterns of motion
  - Logic: patterns of reasoning
  - Probability: patterns of chance
  - Topology: patterns of closeness and position

Mathematics and the real world

- One of the big issues is *do we discover mathematics or invent it?*
  - Galileo and and G.H. Hardy believed that we discover it
  - Einstein questioned the connection between human thought and the world
- What we know for certain is that there is an uncanny relationship between mathematical knowledge and the world and its structures
- Examples
  - Pi and Euler's constant
    - Pi: ratio of a circle's circumference to its diameter
    - Euler:: describes radioactive decay, spread of epidemics, compound interest, and population growth
  - Also, Euler's equation: unites mathematical thinking in a way that is economical and beautiful.
- All of these imply a connection between the world and mathematical calculations

Pure and Applied mathematics

- What the preceding examples show is that simultaneous with its movement toward abstractions of the mind, it also establishes connection with the world.
- One way of dealing with the "real" or "invented" question is to divide math into the categories of "Pure" and "Applied"
    - Pure: algebra, geometry, number theory, and topology, all which do not deal with the physical world or have any direct application
    - Applied: scientific computing, mathematical physics, information theory, control theory, actuarial science, all of which involve practical application of maths to the world.
- However, this is a somewhat artificial and imprecise demarcation
    - So often once abstract maths are eventually used to explain events in the physical world
    - Also, many discoveries in the physical world are not empirical, but the result of deductions from mathematics, using Lorenz curve
    - Conclusion: both pure and applied mathematics deal solely with ideas at a level of extreme abstraction.

Methods: shared knowledge

- Mathematics is shared knowledge, meaning their individual work is contributed to a communal pool of knowledge.
    - Other mathematicians use a peer review, scrutinizing individual efforts for errors.
- However, it differs from other forms of shared knowledge in several ways:
    - Mathematicians work individually, unlike scientists, for example
    - Once the knowledge is established it is certain and unchangeable, and it is a basis for all later math
- Still, while it is done individually, it does not imply static research
    - Ideas are borrowed from many cultures
    - The areas of mathematics is ever expanding
- It works less like science then a creative enterprise like poets or novelists

Ways of Knowing

- It is artificial to separate the ways of knowing
- What we are looking at is the balance of way of knowing in mathematics, beginning with sense perception.
    - Dantzig argues that there is no tangible connection between math and sense perception
    - He uses the example of the garment maker who has no conception of who will wear the clothes

WOK: Sense Perception

- Whitehead: math is certain precisely because mathematics distances itself from the sensory world, which allows it to be free of all the uncertainty of observations
- It allows maths to produce results true in all cases, without exception.
- Of course, uncertainty is a precondition of sensory observations; while we may use our senses, our justifications do not rest on observation

WOK: Language

- The justification for mathematical conclusion does not rest on language
- We do use language to learn and teach concepts
- Mathematical symbolism is a subset of the ideas developed in language
- As it has grown, it has taken on many of the features of a language itself
- As a symbolic system, it allows ideas to be manipulated in the mind and communicated with others

Characteristics of mathematical "language"

- Mathematical symbols have very precise meaning
- Likewise, the relationships between the symbols are rule-defined
- It is far more precise than natural language because it is a symbolic system for abstract, rational argument
    - It is precise and explicit
    - It is compact, as the example of the Pythagorean theorem shows
    - It is transferable without the loss of reason
        - Commutation,
        - Reducing fractions
        - Factoring a polynomial (compare this to translating from one language to another)
    - It is completely abstract and conceptual, manipulating its statements using only its own rules
    - It can lead to conclusions that may not be new, but present knowledge that is new to us
- How does the language of math work?
    - It is a purely self-referential and abstract system, and this makes it certain
        - It focuses on relationships (variables) between terms, relationships such as symmetry, proportion, sequence
        - It also provides the vocabulary and grammar that enables us to talk about such relationships
        - When we apply maths, we define specifically what the terms and the variables mean, and we can then investigate how these patterns operate in the world.
    - Miles Davenport highlights a crucial function of math as the explainer of patterns
        - Most sciences "split" or divide knowledge into separate categories in order to better understand them
        - Maths use patterns to lump or to connect what often appear
        - to be disparate elements into a single pattern.

WOK: Intuition

- On the surface it appears that maths, like other areas of knowledge use intuition and imagination to gain new insights
- The question is: does it operate in the same manner as other areas of knowledge?
- Intuition, as defined in the text as the fast and rough grasp of patterns is fraught with faults and is not particularly useful in mathematical knowledge

    - The solutions of math are often counter-intuitive
- Imagination—defined as the capacity to reassemble familiar concepts into new ones, or to project beyond them into fresh conceptualization
- It uses imagination in a different manner than other areas of knowledge
    - Although mathematicians speak as though their mathematical objects and concepts are real, the subject matter of mathematics is already in a world of "imagination" even before they manipulate it creatively
    - Mathematician use *propositional imagining*—they imagine that a statement is true and trace the implications of considering it to be so, much as scientists do with a hypothesis
    - Beyond this, not only is imagination used in the generating of the proposed statement but the testing of it demands that the imagination be engaged, because the implication are played out in the realm of the imagination.
- Cantor example:
    - Started with the abstract concept of infinity, and then constructed a proof for the existence of multiple infinities
    - Also used proof by contradiction or *reductio ad absurdum* to complete his proof.
    - This is both a creative and abstract approach, but here imagination may be used in a different manner than in other areas of knowledge.

WOK: Reason

- Reason is the primary way of knowing, as mathematics operates independent of sense perception, so it is the primary investigative tool
- How exactly does reason function in mathematics?
    - Mathematics uses deductive reasoning, which builds new logically derived conclusions from its initial premises
    - Deduction uses strict rules to perform operations that transform initial premises into new forms that preserve the truth values of the original premises
    - The premises then provide the content or subject matter to which reasoning is applied
- A key issue in math and deductive logic is the difference between validity and truth
    - <u>Validity</u>: is the exclusive domain of deductive logic—valid means that the conclusions follow directly from the premises by applying the rules of logic. Math (especially pure math) is based solely on logical consistency
    - <u>Truth</u>: is not a mathematical concept. Math done correctly shows an internally consistent pattern. It guarantees that if the premises are true, then we are certain the conclusion is true. Its validity guarantees the certainty of true premises.
- This distinction has significant implications for the issue of certainty in mathematics

Methods: building on foundations

- If mathematics is primarily about structure, about patterns, about validity, then the issue becomes *how do we establish certainty?*
- This in turn requires an understanding of how the concept of justification has changed over the course of the history of mathematics
- Since we know that math builds in cumulative and building block (Lego) fashion, the issue becomes *how do we justify the premises of mathematical systems?*

Building on Euclid's axioms: validity and truth

- It all started 2,300 years ago when Euclid identified 10 axioms or postulates from which he claimed to be able to deduce all the propositions of geometry
- He claimed that they did not require justification because they were "self-evident" and for 2,000 years this was considered sufficient justification to establish the validity of the system.
- It was certainly valid (self-consistent), but was it true? It would seem that if they were true, they must be the only possible correct axioms
- Furthermore, they were *useful*, readily applicable to the everyday world
- Investigating the axioms
    - The first four appeared to be self-evident because they could be illustrated
    - The fifth axiom—the parallel postulate—was impossible to illustrate because the line could continue forever. It was also impossible to prove it as if it were a theoreum
    - Fortunately, this axiom was only used in one proof, and Gauss in the 1800s showed that geometry could be built without it.
- However, Gauss' work got mathematicians thinking about whether there could be alternate sets of axioms, and paved the way for non-Euclidean geometries, which exposed the flaw in the Euclidean system, and called the *truth* (but not the *validity*) of Euclidean geometry into question

Mathematical truth

- After Gauss, Euclid's system was considered *valid*, but not true in the sense that there could be other consistent systems
- This shifted the mathematical understanding of truth away from the self-evident nature of axioms (correspondence theory of truth) to the idea that a true mathematical system was true because its was *internally consistent* (coherence theory of truth).
    - Thus Euclidean, Lobachevsky and Riemann all produced true mathematical systems
- The criteria for truth shifted from self-evident to self-consistent
- However, this in turn demanded a new means establishing truth, and made the concept of "proof" of paramount importance.

Methods: proof and peer review

- Proofs are the rigorous application of logical rules that extend the knowledge contained in the axioms.
- The new knowledge or "theorems" are simply the logical implication contained in the premises, but are new to us.

- While mathematic work is individual, it becomes shared knowledge when it is checked by other mathematicians through peer review
- The best proofs are not only rigorous, they are also economical, which lends them a degree of elegance and beauty.

Peer Review

- An example of peer review is Andrew Wiles' of Fermat's last theorem
  - Submitted an initial "traditional mathematical proof" that omits logical steps
  - The initial proof turned out to have flaws, exposed via peer review
  - In the end he solved it and realized a dream from childhood

Development across time

- Thus far we have examined how proofs work within mathematics
  - It builds upon the shared knowledge confirmed through peer review
  - Thus far we have rejected the correspondence theory of truth in favor of a coherence theory of truth
  - However, this came under attack in the 20th century, and was not something addressed by Andrew Wile's proof
  - Thus we return to the idea of how we justify the axioms or underlying assumptions of mathematical systems
- Bertrand Russell showed an *internal* contradiction within all mathematical systems
  - This meant that a purely internally coherent system, a requirement of the coherence theory, is impossible
  - It focuses on the effort to build a set of all sets that are not members of themselves
    - If I make a catalog of all the library catalogs and that catalog is not included, the set is incomplete
    - However, if it is included, then it is no longer a catalog of all catalogs, as it includes itself
    - So, if it does include itself it shouldn't, and if it doesn't, it should
  - To use an analogy from language, consider the Liar's paradox: *All Cretans are liars*
    - If he is telling the truth does that mean he is lying?
    - If he is lying does that mean he is telling the truth?
- The attempt to explain the internal contradictions led to Godel's incompleteness theorem
  - There is no quarantte that there is not a contradiction within any axiomatic system
  - He found this out by trying to refute Russell by grounding mathematics more firmly in logic, but encountered Russell's contradiction

- This did not affect the work of mathematicians, as they were perfectly willing to accept that there is no absolutely certainty about axioms
    - However, from a TOK point of view, it means that we must abandon the coherence theory of truth, and need to seek another method of justifying the starting point.
    - Godel leaves us a clue about how we might proceed: *every mathematical system rests upon an assumption or set of assumptions that are themselves <u>outside</u> of the system*.
    - This would seem the way forward, as we will discover when we read the Kemeny article.

Mathematics in its social context

- To review
    - Mathematics is created by individuals and confirmed through peer review
    - While it is very abstract, it nevertheless has an intimate relationship with the natural world
    - However, what is its relationship with the social context within which it is generated?

Universal or cultural?

- Maths would appear to be universal as it is rational, depersonalized—it has nothing to do with any of us yet has do do with all of us
- There are six forms of recurring mathematical ideas spanning cultures
    - Counting
    - Locating
    - Measuring
    - Designing
    - Playing
    - Explaining
- As for the cultural aspect, it is true that there are innumerable different counting systems, each containing a subtly different concept of space, and having a different social context (what the math is used for)

Cultural dominance of "western" mathematics

- While the so-called "western" mathematics draws from diverse traditions, it has imposed a certain viewpoint on mathematics
    - It is rational and objective, a view of the world as a series of discrete objects
    - It is based on the concepts of universality and cultural neutrality
- Ethnomathematics shows us that there are cultures that use math in a more integrated context
- In conclusion, math seems to be universal in its thought process, but cultural in the specific forms that it takes

Social Attitudes toward mathematics

- What role should math play in education and society?
- The controversy is between basic practical mathematical calculation or math as discipline that teaching thinking processes

- Invariably, math is used as an entrance barrier, a test to sort out better students, rather than giving them useful information
- The crucial issue here is the same one we began with—pure math for the discovery of new ideas, or things that are immediately practical?
- Of course, we need both