Fly me to the Moon!

A Costly Venture

Getting things in space is an expensive affair. Flying the guitar that Chris Hadfield used in his music video up into space probably cost NASA around $75,000, given the launch costs at the time. And that's just a guitar. Imagine how much more it would take to get a satellite into orbit. Given that there is not much extra cash flow these days and that NASA's budget has been repeatedly cut we thought it would be relevant to figure out: What is the minimum amount of fuel required to get a satellite from Low Earth Orbit (LEO) to the Moon?

Depending on the rocket that is used, launch costs can range anywhere from around $4,000 to $30,000 per kilogram. Yes. Per kilogram. And so, just by slightly reducing the amount of fuel, or in other words, the weight that the rocket needs to carry up into orbit, we can greatly reduce the cost of the whole mission.

The Simplified Solar System

Before we optimized, we needed a realistic environment in which to run tests i.e. we needed to create the system in which our satellite would be travelling. The initial system consisted of three bodies interacting gravitationally - the Sun, the Earth, and the Moon - in a three dimensional space. When setting up the system we assumed that the bodies were point masses and that they were only subjected to gravitational forces from the other objects.

The gravitational force one body exerts on another can be calculated using Newton's Law of Universal Gravitation:

$$F = \frac{G^* m_1 {}^* m_2}{r^2}$$

F : Magnitude of the force exerted by body 1 on body 2 (N)
G : Gravitational constant ($6.67\times10{-}11$ N·(m/kg)^2)
m_1 : Mass of body 1 (kg)
m_2 : Mass of body 2 (kg)
r : Distance between the two bodies (m)

This law only applies to the gravitational interactions of binary systems. However, by adding all the forces from pairs of objects, the acceleration on a single object due to several others can be approximated. For example, the net gravitational force acting on the Moon would be the sum of the gravitational force due to the Earth and the gravitational force due to the Sun.

Initial Conditions

In order to replicate the behavior of the actual Sun, Earth, and Moon, we had to give the bodies in our model appropriate initial conditions. The masses of the bodies in our model were found with a quick Google search, but the initial positions and velocities were a little bit more tricky.

To simplify things, we assumed that all three bodies had circular orbits when their initial velocities were given. The true velocity of an object as it orbits is not constant, but since the orbits of the Sun, Earth and Moon are all very close to circular we can use the average orbital velocity and still represent the system quite accurately. Using this assumption, the initial conditions of the Earth and the Moon can be determined relatively easily. Earth's average orbital velocity is 29,783 m/s. The Sun also has an initial velocity, though it is very small. It can be found using conservation of momentum - the velocity of the Earth/Sun centre of mass is assumed to be zero, so the sum of the momenta of the Earth and Sun must also be zero resulting in a velocity of 0.0894267 m/s in the negative y direction.

As for their positions in three dimensional space, they are given by the x,y,z coordinates of the objects. We simplified matters by assuming that all the objects start in the plane y = 0, i.e. in line with the x-axis (*Fig. 1*). This means the objects only had velocity in the y direction and so we did not have to worry about accurately splitting the velocities into their three dimensional components.

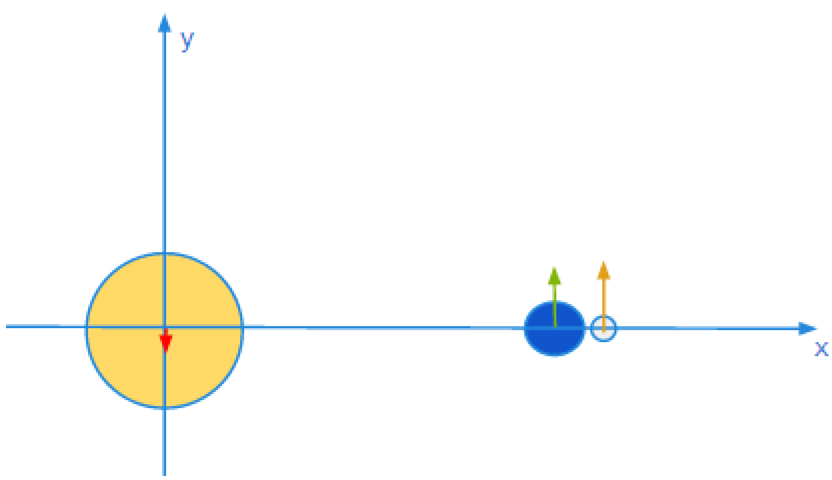

Fig.1: Projection of initial positions of objects on x-y plane

We treated the Sun's initial position as the origin and the Sun's orbital plane as the x-y plane. Because the Sun and the Earth move in essentially the same plane, the Earth's initial x-position is quite simply the average distance from the the Earth to the Sun: 1.49×10^{11} m.

Determining the Moon's initial conditions was a little bit more complicated because its orbital plane is tilted by five degrees in relation to the Earth-Sun orbital plane (*Fig.2*).

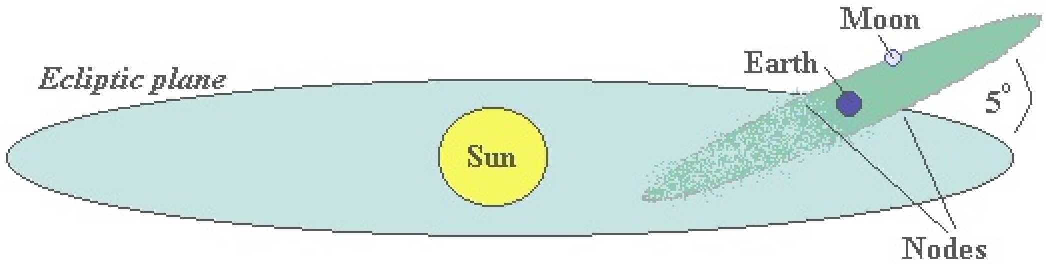

Fig.2: Tilt of Moon's orbit with respect to the Earth and the Sun's orbital plane

The average distance from the Earth to the Moon is 3.84×10^8 m. Assuming the moon starts at its maximum x-position and maximum z-position (*Fig.3*), we can write the initial x-position of the moon as: the Earth's initial x-position (1.49×10^{11}) plus the x-component of the distance ($3.84 \times 10^8 \times \cos(5)$). In this system, the Moon is the only object that initially has a non-zero z-position. This is given by: $3.84 \times 10^8 \times \sin(5)$.

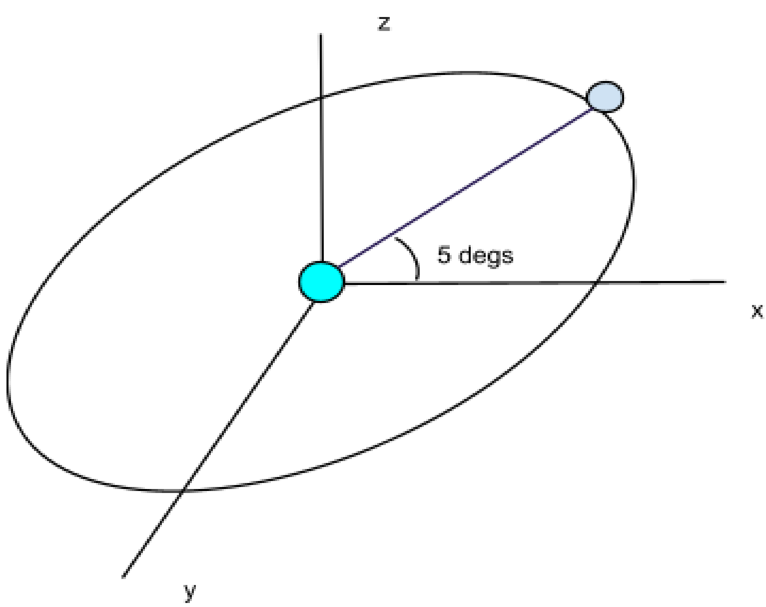

Fig.3: Close-up of the Moon's position relative to the Earth

When the moon is at its maximum x-position and maximum z-position, its velocity in the x and z directions is zero. And so at this point, its entire orbital velocity around the Earth can be expressed as a y-component. The moon is moving along with the Earth as it is orbiting, so the moon's initial velocity ($\mathbf{v_m}$) must be the vector sum of the Earth's orbital velocity ($\mathbf{v_1}$) in the Earth-Sun system and the moon's orbital velocity ($\mathbf{v_{om}}$) .

$$\mathbf{v_m} = \mathbf{v_1} + \mathbf{v_{om}}$$

An approximate three dimensional representation of the initial conditions of the system can be seen below (*Fig.4*).

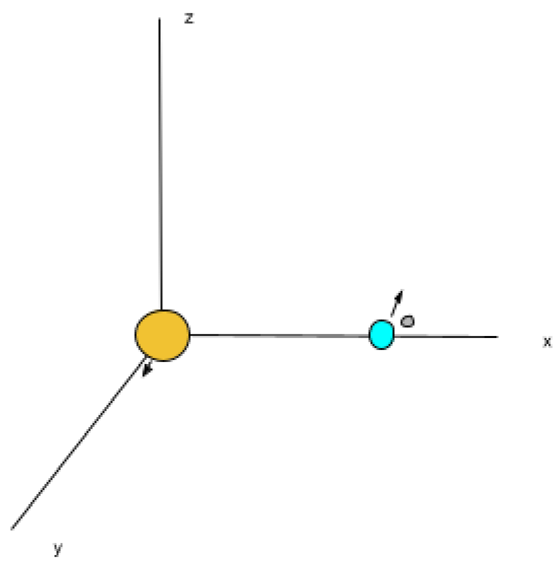

Fig.4: Representation of bodies in 3-dimensions

Now that we had the initial positions and velocity of the Sun-Earth-Moon system, our model had to represent how these velocities and positions change with time. We knew the magnitude and direction of the forces that act on each of the bodies (from Newton's Law of Universal Gravitation), and that we could divide these forces into their x-y-z components (*Fig.5*).

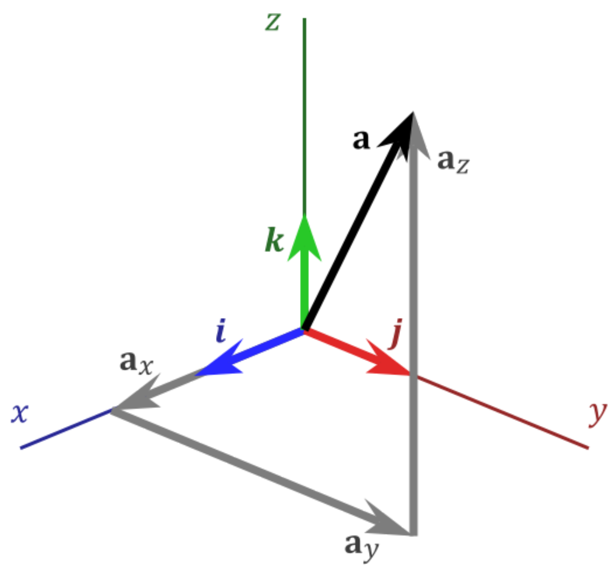

Fig.5: Decomposition force vector ***a*** *into its components*

Given F_{12}, the force exerted on body 2 by body 1 (*Fig.6*), we determined the x-y-z components of this force with simple ratios.

$$F_{x12} = \frac{(x1 - x2)}{Distance(1\ to\ 2)} F_{12}$$

$$F_{y12} = \frac{(y1 - y2)}{Distance(1\ to\ 2)} F_{12}$$

$$F_{z12} = \frac{(z1 - z2)}{Distance(1\ to\ 2)} F_{12}$$

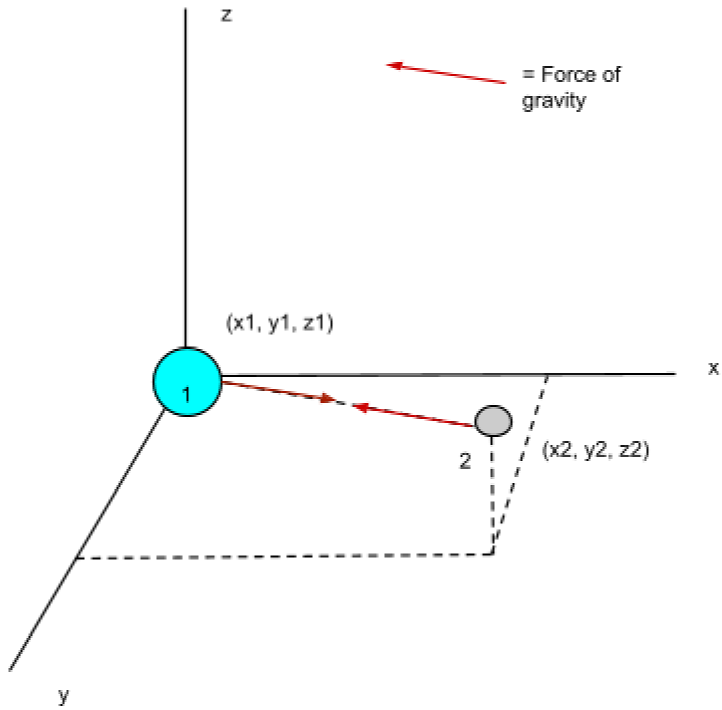

Fig.6: Gravitational interaction of two bodies

In this manner, we can decompose all the forces acting on body 2, and then we could get the net force in the x-direction, y-direction, and z-direction. The forces due to the other objects can also be determined in this manner. By adding the components of the forces acting on a single object and dividing by the object's mass we can determine the net acceleration of that body at a certain time.

$$a_x = \frac{Fxnet1}{m_1}$$

$$a_y = \frac{Fynet1}{m_1}$$

$$a_z = \frac{Fznet1}{m_1}$$

Considering we have the initial positions and velocities of the bodies, we can perform one step of Euler to get the positions of the bodies after a timestep DT.

(x1, y1, z1) = (x0, y0, z0) + (vx, vy, vz)*DT

With two positions for each of the bodies, verlet's method can be used to track the positions of the Sun/Earth/Moon as time elapses. This use of Verlet's method makes the assumption that the acceleration of the bodies will remain constant for DT.

$$\mathbf{(x2, y2, z2)} = 2*(x1, y1, z1) - (x0, y0, z0) + (ax, ay, az)*(DT)^2$$
$$(x0, y0, z0) = (x1, y1, z1)$$
$$(x1, y1, z1) = (x2, y2, z2)$$

In this method, the acceleration is recalculated every iteration with respect to the new positions of the bodies.

To see a video of our model at this stage click here. The moon appears to be directly on top of the Earth because of the large distance scales and the comparatively large images.

Validating the Code

To validate our simulation, we compared the properties of the Sun-Earth-Moon system in our model to the properties of the actual Sun-Earth-Moon system.

After plotting the motion of the Earth over a certain amount of time it was shown that our simulation reproduced the motion of the actual Earth (*Fig.7*)

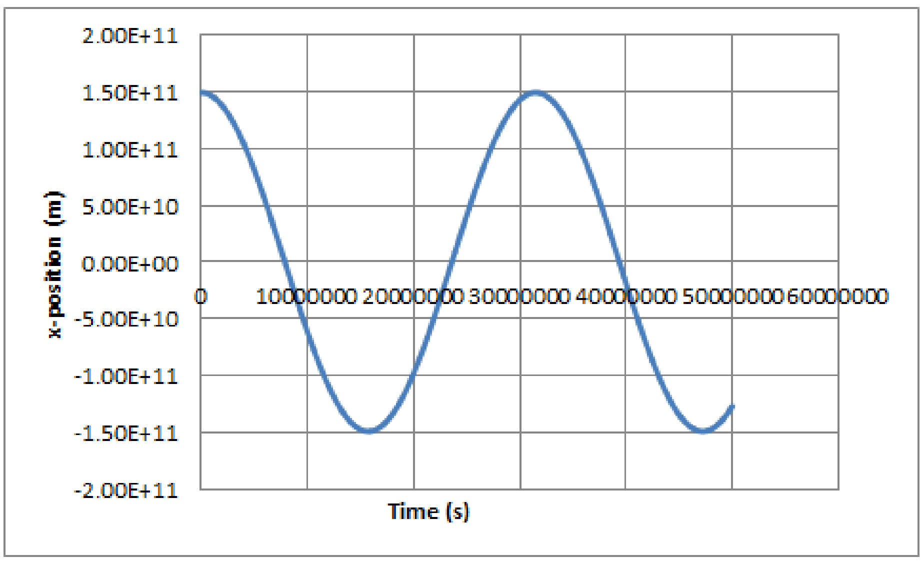

Fig.7: Change in Earth's x-position as time elapses

As mentioned, we assumed that the orbit of the Earth was circular. The average distance from the Earth to the Sun is around 1.49 * 10^{11}m, so it would make sense that the largest value for the x-position is around that number when the Earth all the way to the right or left. Also, we know that there is 365 days in a year and that a year is approximately one orbit. 365 days translates to ~32000000s, and if you look at the graph you can see that one rotation takes around that long.

Similarly after plotting the motion of the Moon over a certain amount of time it was shown that our simulation also reproduced the motion of the actual Moon (*Fig.8*).

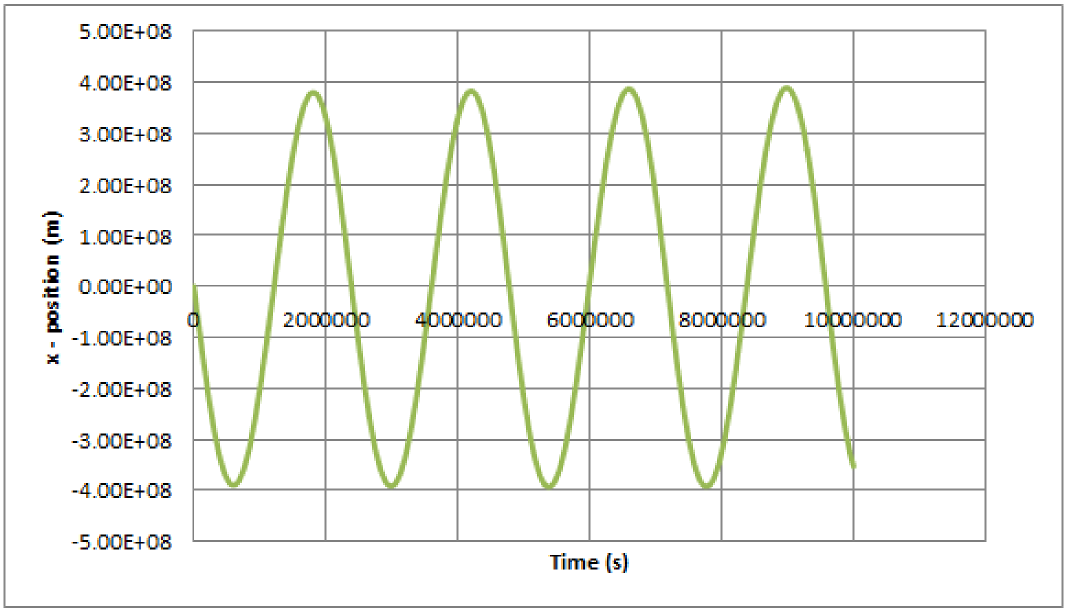

Fig.8: Change in the Moon's x-position relative to the Earth as time elapses

The moon takes around a month to orbit the Earth. 30 days corresponds to ~2.5 million seconds and that is approximately the period of our moon. The average radius $3.84*10^8$ m, and because we have circular orbits, the amplitudes of our curve makes sense as well.

Finally, we have a graph of the Sun's motion (*Fig.9*). It may look a bit suspicious because we don't normally think of the sun as something that has an orbit. But when two bodies are orbiting, they orbit around the system's centre of mass. The centre of mass of the Earth Sun system is around 450 km from the Sun, and if you cut the graph horizontally through the middle you can see that our Sun's orbit does have a radius that is that size.

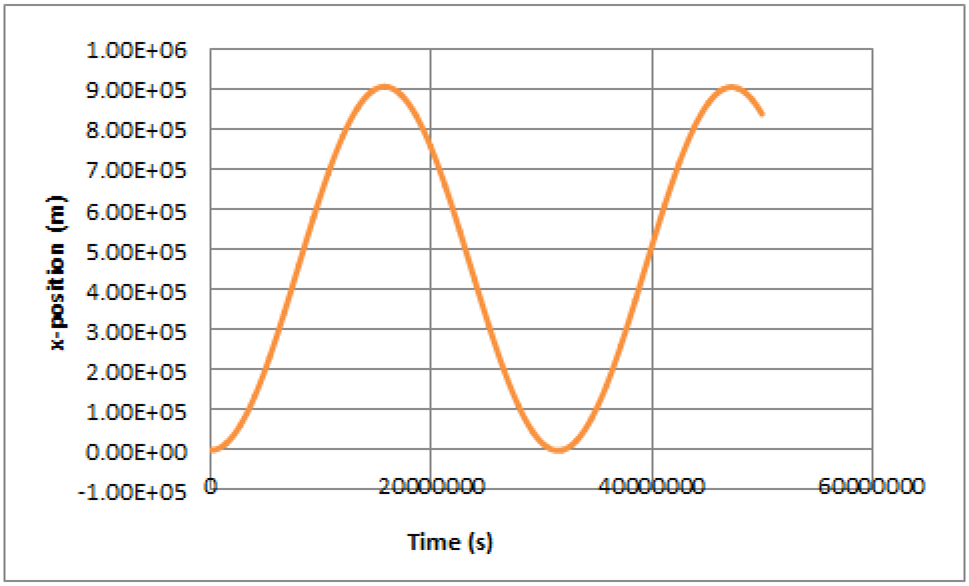

Fig.9: Change in the Sun's x-position as time elapses

From these graphs we could see that our model seems to be a fairly accurate representation of the actual Earth/Moon/Sun system.

Putting our Satellite into Orbit

The satellite we added, that is initially orbiting the Earth, is just another body in this system. Considering the satellite has a mass of only 1500 kg (extremely small compared to the masses of the other bodies), it will not have a significant effect on the model, and the model will remain fair representation of the actual system.

For the mass of our satellite, we just browsed through a list of typical satellite masses and selected one of the heavier ones. For the initial conditions of the satellite, we knew we wanted it to start in low Earth orbit, which is around 200 km to 2000 km from the Earth's surface. Of course, model considers the Earth to be a point-mass, so we have to add the altitude to the radius of the Earth to get the satellite's initial distance from the Earth. Once we had its distance from the Earth, we determined its orbital velocity with the following equation:

$$v_o \approx \sqrt{\frac{GM}{r}}$$

v_o : Orbital velocity of one body orbiting a larger body
G : Gravitational constant
M : Mass of the larger body
r : Distance between the centres of the two bodies

This equation is an approximation, and can only be used if the following conditions are met:

- The orbit has a small eccentricity (we are assuming a circular orbit, so...check!)
- The mass of one of the bodies must be negligible compared to the other (Mass Earth >> 1500 kg...check!)

To see a video of our satellite orbiting the Earth click here.

Getting it to the Moon

We decided to minimize the fuel required to send the satellite to the moon by using a Hohmann transfer. The Hohmann transfer is a manoeuvre used to transfer between two circular orbits in the same plane (*Fig.10*). It minimizes fuel by minimizing the energy required to change the orbit size and direction of the satellite.

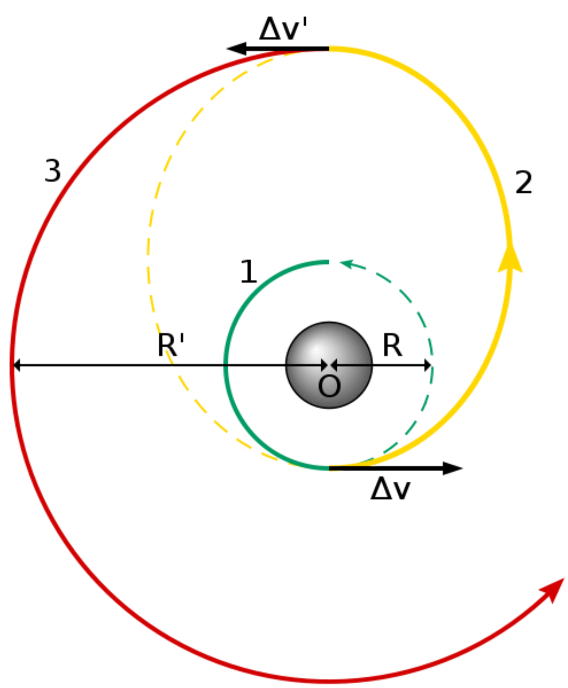

Fig 10: Outline of Hohmann Transfer

As the diagram shows, essentially a spacecraft in a circular orbit accelerates so that it follows an elliptical path. And then when the spacecraft reaches the furthest point of its elliptical orbit, it accelerates once more to enter a larger circular orbit. The problem is that we don't want our satellite to enter a larger circular orbit, we want our satellite to orbit the moon. So we just used elliptical transfer orbit to get the satellite close to the moon, and then tried to get the moon to capture the satellite. If we start in a circular orbit with radius r_1, and we wish to reach an orbit with radius r_2, then we must provide our satellite with a delta-v given by the following equation (derived from the conservation of angular momentum and the conservation of energy).

$$\Delta v_1 = \sqrt{\frac{2GMr_2}{r_1}\left(\frac{1}{r_2+r_1}\right)} - \sqrt{\frac{GM}{r_1}}$$

Δv_1: Delta-v required to put the satellite into an elliptical orbit
G : Gravitational constant
M : Mass of the larger body
r_1 : Radius of the initial circular orbit
r_2 : Radius of the orbit that needs to to be reached

We chose the r_2 to be 3.82×10^8 so the satellite could get very close to the moon. We added this delta-v when the satellite was at its furthest point right (relative to the Earth) so that it could reach the moon when the moon was at the leftmost point in its orbit. To make sure Moon and the satellite came close together, we played with the initial position of the moon. We were making quite the assumption that the slight incline of the moon's orbit wouldn't affect the transfer. It was a fairly foolish assumption, but we couldn't figure out how to account for it. Unfortunately, we could not get the moon to capture the satellite after a single delta-v, so we decided to give the satellite a second delta-v when its in proximity of the moon so that it has an appropriate velocity needed to be captured by the moon. To that end, we decided to complete the Hohmann transfer (provide Δv_2shown in Fig 9) the so that the satellite gets a velocity similar to that of the moon. We assumed that if they begin moving together, it will be more likely that the moon will capture the satellite. The Δv_2(or the $\Delta x'$ in Fig 9) can be obtained from the following equation:

$$\Delta v_2 = \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_2+r_1}}\right)$$

Δv_2: Delta-v required to change the elliptical orbit of the satellite
G : Gravitational constant
M : Mass of the larger body
r_1 : Radius of the initial circular orbit
r_2 : Radius of the orbit that needs to to be reached

After this second delta-v, our satellite was captured by the moon. To see a video showing the satellite launch and capture click here. In this run we also used a third Delta-v to get the orbit around the moon a bit smaller.

Calculating the fuel used by the Hohmann Transfer

The amount of fuel used depends quite a bit on the thrust specific fuel consumption consumption (TSFC) of the rocket used in satellite. TSFC is fuel efficiency of the rocket with respect to thrust output, in other words, it is the rate of fuel consumption per unit thrust ($\frac{kg/s}{N}$). Admittedly, we didn't spend too much time deciding the TSFC of the rocket used in our satellite. We read that the TSFC of rockets used in satellites are much lower than the TSFC of rockets used in shuttle, and so, after finding a list of rockets and their TSFC, we selected one of the lowest ones.

The delta-v in the Hohmann transfer is an instantaneous change in velocity. We implemented this delta-v into our code by giving a sudden acceleration to our satellite at certain, calculated times. This sudden acceleration would act over the timestep DT (the 2s that is in our code in our Verlet's method). This acceleration which would act over 2s would give the satellite the required delta-v.

With this in mind,, we could calculate the necessary thrust by using the acceleration that corresponds to each delta-v. The thrust required for a certain acceleration can be found using Newton's Second Law,

$$F = ma$$

F : Thrust (N)
m : Mass of the satellite (kg)
a : Magnitude of the sudden acceleration of the satellite (m/s^2)

Multiplying the necessary thrust by the thrust specific fuel consumption would give the rate of fuel consumption(kg/s) necessary to produce such a thrust. In our code, the acceleration acts over two seconds to produce the necessary change in satellite velocity, so logically, the thrust also acts over two seconds and the rate of fuel consumption is maintained over two seconds.

So the fuel consumed at each use of our satellite's rocket can be calculated as follows,

Fuel Consumed = a*m*TSFC*DT

m : Mass of the satellite (kg)
a : Magnitude of the sudden acceleration of the satellite (m/s^2)
TSFC : Thrust specific fuel consumption of the satellite rocket ($\frac{kg/s}{N}$)
DT : Timestep over which the satellite is subject to the rocket's acceleration (s)

The total amount of fuel consumed is just the sum of the fuel consumed at each "boost".

Minimizing fuel with some manual fixes…

From this you could see that to minimize the fuel consumed, the delta-v used for the transfers must be minimized. It would have been nice to be able to write some algorithm that evaluates a bunch of situations with the moving bodies, and then finds the optimal delta-v that acts on the satellite at the best timel time and gets it to orbit the moon. Unfortunately, we were not able to think of such a code. Instead, our method was much more crude. As mentioned previously (in the Hohmann transfer section), with a little analysis of the motion of the bodies, we found a delta-v (and an an appropriate time to apply this delta-v) that would get our satellite to orbit the moon (we actually needed two delta-v because we weren't able to figure out a way to get it to orbit with only one…).

And then once our satellite was orbiting the moon, we just played with the delta-v (just as we messed with the parameters in the Earthquake lab to minimize the resonance curve) so that the delta-v would be lowered while the orbit was maintained.

Our embedded video shows one of our older runs in which the first two delta-v got the satellite to orbit the moon, and then the third delta-v was just there to make the orbit of the satellite around moon smaller. Since then, we have manage to remove the third delta-v by making it so that the orbit was acceptable after the second delta-v (we don't have a video of this because of software problems, but if you just run our code you should be able to see the animation). This came about because we noticed that even if we decrease the second delta-v somewhat, the satellite still gets captured by the moon. When the second delta-v was decreased far enough, the orbit the satellite became more circular than it is after the second delta-v in the v (though if we decreased the second delta-v too much the moon would simply not capture the satellite).

In the run shown in the video, after the second delta-v approximately 93 kg of fuel is used and after the third delta-v, the fuel consumed goes up to roughly 102 kg. After the bit of tinkering mentioned in the above paragraph, the fuel consumed after the second delta-v is 88 kg and there is no third delta-v.

Admittedly, it was a fairly crude method, but we did find a solution that maybe does not minimize but at least lowers the amount of fuel needed to get a satellite to orbit the moon.

So if it ever happens that you have met these conditions:

- You have a 1500 kg satellite orbiting 8000 km from the centre of the Earth with a velocity of 7056.425 m/s
- The Earth, Moon, Sun, and Satellite are at the following positions relative to one another (in m):
    - Sun (-1.49×10^{11}, 0, 0)
    - Earth (0, 0, 0)
    - Moon (0, $3.84 \times 10^{8} \times \cos(5)$, $3.84 \times 10^{8} \times \sin(5)$)

- Satellite (8×10^6, 0, 0)

Then wait 100,000 s, and then provide an acceleration of 1,410 m/s^2 over two seconds in a direction perpendicular to the line connecting the Earth and the Satellite (in the xy-plane). After this acceleration, wait another 427,998 s, and provide an acceleration of 300.09 m/s^2 over two seconds, once again in the direction perpendicular to the line connecting the Earth and the satellite (in the xy-plane). If it all goes well, your satellite should be orbiting the moon. And if your satellite happened to have a thrust specific fuel consumption of 17×10^{-6}kg/(N*s), then you will have only consumed around 88 kg of fuel.