ELEC 3004 / 7312 – Systems: Signals & Controls

# Solutions to Problem Set 1

**QUESTION 1-5 are "all or nothing" ...**
**While this strictly means "0 or 5 points", to compromise and be "a little forgiving", what we'll ask is that this be graded as 0, ½ or 1 (or 0, 3, or 5 points).**

**Where:**
**1** (or 5 points): The answer is correct (or might have a small numerical error)
½ (or ~~2.5 points~~ 3 points [as Platypus does not support floats in the answer field yet]): The answer is correct in form and shows understanding, but has errors
**0**: The answer is wrong (or mostly wrong).

**Q1. Even/Odd Signals** **[5 points]**

An aperiodic signal is defined as $x(t) = \sin(\pi t)u(t)$, where $u(t)$ is the continuous-time step function. Is the odd portion of this signal, $x_0(t)$, periodic? Justify your answer (include any definitions and diagrams you deem relevant to your explanations).

$x_0(t) = \frac{1}{2}\{x(t) - x(-t)\}$

$x(t) = 0$ for t<0
$x(t) = sin(\pi t)$ for t>0

$x(-t) = -sin(\pi t)$ for t<0
$x(-t) = 0$ for t>0

:. $x_0(t) = \frac{1}{2}\{sin(\pi t)\}$
$x_0(t+T) = \frac{1}{2}\{sin(\pi(t+T))\} = \frac{1}{2}\{sin(\pi t+\pi T))\}$ which equals $x_0(t) = \frac{1}{2}\{sin(\pi t)\}$ for all t if T is a multiple of 2. :. $T_0$=2

**Suggested marking criteria:**

**A full answer (5 points) :** would have the do most of the following: (1) acknowledges that x(-t) is u(-t)sin(-$\pi t$) either explicitly or implicitly and not x(t) for t is negative; (2) show that sin(-$\pi t$) = -$sin(\pi t)$; (3) Correctly applies previous stages to show $x_0(t)$ is periodic.

**A half answer (3 points):** might have incorrectly interpreted $x_0(t) = \frac{1}{2}\{x(t) - x(-t)\}$ as meaning $\frac{1}{2}$*{the value of x(t) for positive t - the value of x(t) for negative t}* will typically have arrived at the right answer despite not understanding odd and even parts of a function and should not be rewarded. Some students have seemingly random alternation between u(t) and u(-t). If the work implies that the students has understanding and this is a typo mark as if typed correctly and then deduct 1 to 2 marks depending on your level of confidence that this is a slip rather than an error.

**Q2. What's the Difference?** **[5 points]**

We may describe a system as being:

- linear [1]
- time-invariant [2]
- Causality

Determine which of these three properties hold (**yes**/**no**) for:

(a) Backward differencer (`diff` in MATLAB): $y[t] = x[t] - x[t-1]$

x[t]--> ax[t] => y'[t]=ax[t] - ax[t-1] = a(x[t] - x[t-1])= ay[t] so Its linear.

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delay the input: x[t]--> x[t-a] => y'[t]= x[t-a] - x[t-a-1] (1)

by delaying the output : y[t]--> y[t-a] => y[t-a]=x[t-a]-x[t-a-1] (2)

(1),(2) => y'[t]=y[t-a] we observed by shifting the time in input we get the relative shifted output that we expected hence it is time invariant. in time invariant systems, output in a specific time is a function of input for that specific time.

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the system is causal since the output only depends on the input of that time and before

(b) Forward differencer: $y[t] = x[t+1] - x[t]$

Linear/ time invariant / non causal

(c) Central differencer: $y(t) = x(t + \frac{1}{2}) - x(t - \frac{1}{2})$

linear/ time invarient/non causal

**Suggested marking criteria:**

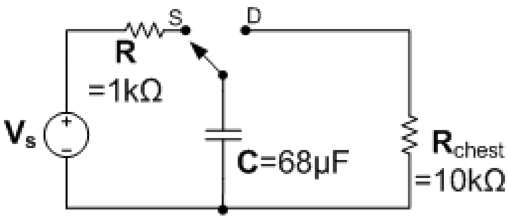
**A full answer (5 points) :** would have all 9 elements correct. **No explanation needed.**

**A half answer (3 points):** would have 3 elements incorrect.

**Extra Credit:** +1 point for exceptional answers (e.g., with proofs, discussion, etc.)

**Q3. I ♥ Systems** **[5 points]**

A defibrillator is used to deliver a strong shock across the chest of a person in cardiac arrest (i.e., fibrillation). A simple design for one may be constructed using the following circuit:




With the switch in the standby mode ("S"), the 68μF capacitor is charged using a controller (having a Thevenin equivalent of $\mathbf{V}_s$ and **R**=1kΩ). To defibrillate, the switch is thrown (to "D") and the capacitor discharges across the patient's chest, which can be approximated as a 10kΩ resistor.

Determine Vs so that the dose is 150J (Assume the capacitor is fully charged when the switch is thrown)? Using this value, specify how long it takes (in seconds) to deliver 95J?

E = integral {v.i} dt
i = C dv/dt

=> E = intergral {v.C.dv/dt}dt = integral {C.v}dv = ½ C.v^2

Students may have simply looked up this equation rather than deriving it.

v = sqrt{2E/C} = sqrt{2*150/68*10^-6} = 2100.42V ≈ 2.1kV
=> $\mathbf{V_s}$ = 2100.42V ≈ 2.1kV

After 95J delivered 55J is remaining. Therefore Voltage across capacitor is:
v = sqrt{2E/C} = sqrt{2*55/68*10^-6} = 1271.87V

When the switch is at D current i(t) flows in the right hand loop, through both the capacitor and the chest.

$$i(t) \;=\; C\frac{dv_C(t)}{dt} = \frac{v_R(t)}{R}$$

But since voltages in a closed loop sum to zero $v_C = -\,v_R$
so
$$C\frac{dv(t)}{dt} \;=\; -\,\frac{v(t)}{R}$$

Which can be solved in a number of ways.

**Method 1 (direct solution by separation of variables)**
$$C\frac{dv(t)}{dt} \;=\; -\,\frac{v(t)}{R}$$

$$\int_{v_0}^{v_t} v.dv \;=\; -\int_0^t \frac{dt}{CR}$$

$ln(v_t) - ln(v_0) \;=\; -\frac{t}{CR}$ We can solve for v but we don't need to because it is t we are looking for.
$$ln(\frac{v_t}{v_0}) \;=\; -\frac{t}{CR}$$
$$t \;=\; -\,CRln(\frac{v_t}{v_0})$$
**Method 2 (assume solution of exponential form)**

Assume: $v = Ae^{-\lambda t}$
$\Rightarrow v' \;=\; -\,\lambda Ae^{-\lambda t}$
Substituting in gives:
$$-\,CA\lambda e^{-\lambda t} \;=\; -\,\frac{Ae^{-\lambda t}}{R}$$
$$\Rightarrow \lambda C \;=\; \frac{1}{R}$$
$$\Rightarrow \lambda = \frac{1}{CR}$$

so $v = Ae^{-\frac{t}{CR}}$ and $v(0) \;=\; v_0 = Ae^{-\frac{0}{CR}} \;=\; A$

$\Rightarrow v = v_0 e^{-\frac{t}{CR}}$ (Again students may have looked up this equation rather than deriving it)

**Method 3 (using Laplace transform)**

$C\frac{dv(t)}{dt} = -\frac{v(t)}{R}$

Taking laplace transform:

$C[sV - v_0] = -\frac{V}{R}$

$CR[sV - v_0] = -V$

$CRsV + V = CRv_0$

$V[CRs + 1] = CRv_0$

$V = \frac{CRv_0}{CRs+1} = \frac{v_0}{s+\frac{1}{CR}}$

Returning to time domain:

$v = v_0 e^{-\frac{t}{CR}}$ as before.

In cases of methods 2 and 3 rearrange to find t:

$v = v_0 e^{-\frac{t}{CR}}$

$\frac{v}{v_0} = e^{-\frac{t}{CR}}$

$ln(\frac{v}{v_0}) = -\frac{t}{CR}$

$t = -CRln(\frac{v_t}{v_0})$

Alterately students may have:

$v = v_0 e^{-\frac{t}{CR}}$

$e^{\frac{t}{CR}} = \frac{v_0}{v}$

$t = CRln(\frac{v_0}{v})$

Subbing in C=68×10^-6, R=10×10^3, v0 =2100.42, vt = 1271.87
gives t = 0.34 seconds

**Marking Scheme:**

**A full answer (5 points) :** To get the "5 Marks" it should:

- states formula E=1/2CV^2
- Have a procedure to rearrange and find
  (it would be nice, but not necessary, if they get it correct, v = 2100V)
- Recognises need to calculate v for E
  (again, it would be nice, but not necessary to do this correctly. $v_t$ = 1272V)
- Present a formula (e.g., $v = v_0 e^{-\frac{t}{CR}}$ ) to find t
- It might have tiny numerical errors, as long as the procedure is correct.

**A half answer (3 points):** It would have recognized the general nature of the problem -- as a system with two subsystems: The first as a charge-pump RC circuit with no initial value for C and a second one with a RC circuit for the patient where C has an initial charge.

It may contain errors in calculation and follow-through, but it should contain the core logic.

**Bonus marks**
**+ 1 mark** derives E=1/2CV^2

**+ 2 marks** derives formula $v = v_0 e^{-\frac{t}{CR}}$ or equivalent
(e.g., 1 mark for setting up ODE and 1 for solving)

## Q4. Signal Sampling: Touché! [5 points]

Touché is a Swept Frequency Capacitive Sensing technique that samples the return voltage many times of a capacitive touch surface whose operating frequency is being swept (changed) from a low frequency to a high frequency.

(a) If an AC base voltage applied to the touch surface ranges from **0.5 MHz** to **3.5 MHz**, what is the Nyquist frequency for sampling this in order to recover the base signal?

(b) If the signal also had high-frequency (**2.4 GHz**) noise, would the noise appear in the sampled signal? If so, what could be done to prevent this?

This problem is more nuanced than initially meets the eye...

**(a)** As noted in the newsgroups and in the class, Nyquist holds! This is a band-pass signal whose frequency exists over a band from 2-1.5 MHz < f < 2+1.5 MHz. It has a **bandwidth of 3 MHz** and can be **determined by 6M samples/sec**. However, the sampling process is more complex in this case -- it involves two interlaced sample trains each at 3M samples/sec (i.e., second-order sampling). Please refer to Lathe, page 770 for details.

Now, if one assumes first-order sampling as is default on most data acquisition systems, then the "bandwidth" of the signal has to be measured from DC, thus giving **7M samples/sec.**

**There seems to be some confusion about the phrase "Nyquist frequency" in the instruction "Nyquist frequency for sampling". Particularly coming from Wikipedia's article.**
**If the solution is 3.5 MHz (7 MHz/2) and it is *explained*, then this is also acceptable.**

**(b)** Yes. Since there is no (anti-aliasing) filtering there would be aliasing of the 2.4 GHz noise into the signal. For example, if we were to use a first-order sampling strategy at 7MHz, then we would have aliasing at 1MHz from the 2.4GHz signal.

**Rubric:**
**A full answer (5 points) :** Should include the sample rate -- **either 6MHz or 7MHz** -- and an appropriate explanation. It should at least mention the need for filtering (but need not mention the aliased frequency).

**A half answer (3 points):** Just a value and no explanation. It should also mention the need for filtering (but need not mention the aliased frequency).

**Bonus marks**
**+ 1 mark** if they explain second-order sampling

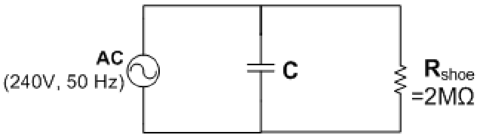
## Q5. Mains Touch [5 points]

Uncle Robert wants to work on the mains (AC, 240 V, 50 Hz) to fix a light switch in the house. Since he has rubber shoes, he conjectures that he should be safe. To prove his point, he pulls out his multimeter and measures the resistance as 2MΩ and points out (according to Ohm's law) that this is much less than the 75mA that would cause fibrillation. Is he correct? Justify briefly.
(**Hint**: Consider the equivalent circuit between him and the floor)

**Sadly, for Uncle Robert, NO**: The shoes have capacitance and there is a potential closed loop through is two arms.

This question is about system modelling. While there was some ambiguity about the shoe's resistance (as compared to both shoes' combined resistance), it just changes the equivalent value, but not the fundamental principle.

Let's consider the case where there is only one hand on the light switch. We need to factor the impedance of the capacitor into the analysis. Let's consider the worst case where we neglect the person's resistance. This is in parallel with the resistance of the shoe so the resulting circuit looks like:




Solving for the impedance, we get $\frac{1}{Z} = \frac{1}{2\times10^6} + 2 \times \pi \times 50 \times C$

Also the source is AC mains where 240V is the RMS value. Thus, peak is $\sqrt{2} \times 240V \approx 340V$ so we have: $1/Z_{eq}$=(1/2E6) + 2*π*(50Hz)*C.

Ohm's law gives: $I \le V(1/Z_{eq})$. Thus:
75E-3 ≤ (340) * (5E-7 + 100*π*C)
C≥705nF

Thus, if there is large, but not trivial capacitive coupling between the shoe, we might have a problem. As a thought experiment, what is the dielectric constant (or coupling) between the shoe and the ground for this to occur? (Given that two feet are about 1 $ft^2$ (or ~0.1 $m^2$) in area)

Also, consider that the 75mA "threshold" quoted is a 2-sigma level. Meaning that for 5% of the population, fibrillation will occur at less than 75mA, thus there is risk even if the "threshold" is not surpassed!

**So, don't touch mains. Bob's your uncle.**

**Rubric:**
**The problem is about the concepts and modelling. A solution may receive full marks either way -- it is about having the methodical process and explaining the answer.**

**Full answer (5 points):** For full marks, the solution should have the correct process (more so than specific values, which might change depending on which assumptions are taken). It should show (or explain why it's negligible): (1) a realisation that shoes offer two parallel resistances, (2) that the shoes have capacitive coupling, and (3) that Bob has some resistance.

**A half answer (3 points):**
If the answer neglects or assumes away the capacitive terms and/or provides very limited explanations or a cursory system model.

**Bonus marks**
+1 may be awarded for recognising or discussing the path of current and that if alternate path provided Bob gets a shock.

Incomplete or flippant answers (e.g., "Trick Question! Uncle Robert nudges his metal toolbox that is on the floor with his knee and died, ha ha ha!" ) do not get awarded a mark,

## Long Questions

**Q6. Signals with Discreteness:** **[20 points]**

Let x(t) be the continuous-time complex exponential signal:

$$x(t) = e^{jw_0 t}$$

With fundamental frequency, $w_0$, and fundamental period, $T_0 = 2\pi/w_0$. Consider the discrete-time signal obtained by taking equally spaced samples of x(t) = that is:

$$x[n] = x(nT) = e^{jw_0 nT}$$

(a) Show that *x[n]* is periodic if and only if $T/T_0$ is a rational number – that is, if and only if some multiple of the sampling interval exactly equals a multiple of the period of *x(t)*.

(b) Suppose that *x[n]* is periodic – that is, that:

$$\frac{T}{T_0} = \frac{p}{q}$$

where *p* and *q* are integers. What are the fundamental period and frequency of *x[n]*? Express the fundamental frequency as a fraction of *x[n]*.

(c) Again assuming that $T/T_0$ is rational, determine precisely how many periods of *x(t)* are needed to obtain the samples that form a single period.

(d) It has been proposed to arbitrarily space the samples of x(t) (asynchronous sampling). What might be (one) advantage and (one) disadvantage of this? Justify.
Shall we run to the patent office?

a) This is equivalent to saying that x[n] is periodic if and only if $qT = pT_0$, i.e., some multiple of the sampling interval T exactly equals a multiple of the period of x(t).

**(i)** For x[n] to be periodic, we need a value of $N_0$ such that: $x[n+N_0] = x[n]$ where $N_0$ is an integer value because x[n] can only take integer arguments.
Given that we know that $x(t) = e^{jw_0 t}$, substituting the value of $x[n] = x(nT) = e^{jw_0 nT}$ and hence $x[n+N_0] = x((n+N_0)T) = e^{jw_0(n+N_0)T}$,
therefore:

$$e^{jw_0(n+N_0)T} = e^{jw_0 nT}$$

$$e^{jw_0 nT} e^{jw_0 N_0 T} = e^{jw_0 nT}$$

**(ii)** For the above statement to be true, $e^{jw_0 N_0 T}$ must equal 1 meaning that:

$e^{jw_0 N_0 T} = e^{j2\pi k}$ where k is an integer value

Therefore:

$w_0 N_0 T = 2\pi k$ where $w_0 = 2\pi / T_0$

$$\frac{2\pi}{T_0} N_0 T = 2\pi k$$

So: $\frac{T}{T_0} = \frac{k}{N_0}$

**(iii)** Since $k/N_0$ must be a rational value since both k and N0 are integers, so too must $T/T_0$

**[5 marks]**
**Suggested Rubric:**
Full marks if the proof provided is about the same, all assumptions are stated clearly(parts i, ii, iii) and the process is well explained (something similar to the steps i, ii and iii) so that the final answer is justified. Give 4 marks if some justification is missing. If the answer gets to step ii with all the necessary bits in between give 3 marks.

b) **(i)** Since the fundamental period is the smallest value of $N_0$ such that $x[n+N_0] = x[n]$ and we are given that x[n] is periodic, i.e., for some integers p and q, we have:

$$\frac{T}{T_0} = \frac{p}{q}$$

$$T = \frac{p}{q}T_0 = \frac{p}{q}\frac{2\pi}{w_0}$$

**(ii)** Substituting T back into x[n] = x[n+N0], we get

$$x[n] = e^{jw_0nT} = e^{jp(2\pi/q)n}$$

and

$$x[n + N_0] = e^{jw_0nT}e^{jw_0N_0T} = e^{jp(2\pi/q)n}e^{jp(2\pi/q)N_0}$$

Again, as in Q6a)

$$p(2\pi/q)N_0 = 2\pi k$$

$$pN_0 = kq$$

**(iii)** Here, it can be seen that $pN_0$ is a multiple of q (multiplied k times) and $pN_0$ is also a multiple of p. Therefore minimising $N_0$, is the same as minimising $pN_0$, i.e. $pN_0$ must be the least common multiple (i.e. the smallest number that p and q can multiply together to produce) or p and $N_0$:

$pN_0$ = LCM(p, q)

Therefore fundamental period, $N_0 = \frac{LCM(p,q)}{p}$.

Also note that *k* can be represented as $\frac{LCM(p,q)}{q}$ since $k = \frac{p}{q}N_0$

**(iv)** Furthermore, just as $w_0 = \frac{2\pi}{T_0}$, the fundamental frequency can also be stated as:

$$\Omega_0 = \frac{2\pi}{N_0}$$

Substituting $N_0 = \frac{q}{p}k = \frac{T_0}{T}k$ gives:

$$\Omega_0 = \frac{2\pi T}{kT_0} = \frac{1}{k}\frac{2\pi}{w_0}T = \frac{1}{k}w_0T = \frac{q}{LCM(p,q)}w_0T$$

Also acceptable is:

$\Omega_0 = \frac{GCD(p,q)}{p}w_0T$ since $N_0$ can also be represented as $\frac{q}{GCD(q,p)}$

based on *GCD(q, p)LCM(q, p) = pq*, therefore: $\frac{LCM(p,q)}{p} = \frac{q}{GCD(p,q)}$ where GCD means the greatest common divisor (i.e. the biggest number that integrally divides each number, e.g. 8 and 12 have GCD of 4)

**[5 marks]**
**Suggested rubric:**

Same as Part a basically. Answer doesn't need to follow this order explicitly, but should have similar parts and explanations as well as both fundamental frequency and period. If it gets part way or leaves off one of the fundamentals give 3 marks. As mentioned at the bottom of the answer, writing in terms of GCD instead of LCM is acceptable.

**** Update **** Also, if they note that for simplicity, p and q can be considered as one to one, i.e. GCD = 1 and LCM = pq (e.g. 3 and 7, not 2 and 4), this is also acceptable to derive an answer from as long as this assumption is stated.

c) Note that the units of $T_0$ and $N_0$ are different. For clarity, let's assume (without loss of generality) that time is measured in units of seconds. Then x(t) has a period of $T_0$ seconds, x[n] has a period of $N_0$ samples, and the sampling period T has units of samples/second. To translate $N_0$ back into units of time, we need to multiply by the sampling period. In this case, $N_0T$ seconds need to pass before we get enough samples for one period of x[n].
However, the fundamental period of x(t) is $T_0$ seconds, and the question is asking how many periods of x(t) fit within the $N_0T$ seconds needed to get enough samples for one period of x[n]. In other words, we need to find some integer *k* such that:

$$kT_0 = N_0T$$

$$k = \frac{N_0T}{T_0} = \frac{p}{q}N_0 = \frac{q}{GCD(p,q)}\frac{p}{q} = \frac{p}{GCD(p,q)} \text{ OR } \frac{LCM(p,q)}{p}\frac{p}{q} = \frac{LCM(p,q)}{q}$$

**[5 marks]**
**Suggested Rubric:**
Full marks for equations like these or close enough (i.e. they are demonstrating the same concept and arrive at the same answer or one that could be derived from this answer) and a similar explanation as the one here, or something similar that explains the math at the bottom appropriately. Award 3 marks for partial working, if some explanations are missing or generally unclear.

**** Update **** If the answer has the fundamental frequency in rad/s instead of the discrete counterpart (which won't be rad/s since seconds are the continuous time measurement), that's fine too, mostly because a) we weren't explicit in the question, b) the question had an error in the requested answer format. In future though, when dealing with discrete systems, if the question asks for frequency or period, assume the discrete counterpart.

d) This list is not exhaustive, compare answers to these and see if the gist is the same.
Some advantages:
- Would require a lower sampling rate in sections of the signal where the frequency is lower and can increase the sampling rate for other sections where the frequency increases (called adaptive sampling)
- doing this with uniform sampling means oversampling in periods of low information and potentially undersampling sections with high information content, i.e. used to more accurately depict information in a signal.
- Can reduce energy consumption, due to decreased sampling rate as described above.

Some disadvantages:
- More difficult and complex to implement than uniform sampling because the nature of the signal is not known beforehand - can lead to errors, distortion, aliasing etc

- More difficult to detect whether a signal is periodic because the samples will not necessarily be taken at rational multiples of the period (the whole T/T0 thing), hence a periodic signal may appear aperiodic.

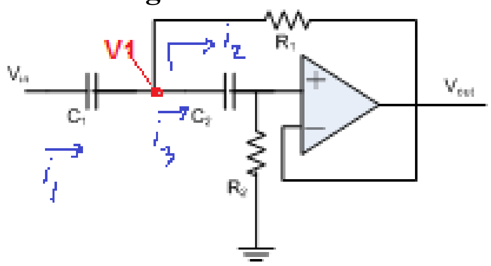
Not necessarily:
More samples are needed to acquire the same accuracy as synchronous sampling --> imagine a signal that is mostly low frequency but has bursts of high frequency content, e.g. accelerometer data from a wild animal. Uniform sampling would require far more points to capture the useful data than asynchronous sampling if the latter's sampling interval adapts depending on the signal information and is not purely random.

**Suggested Marking Rubric:**
**Full answer (4-5 marks):**
Should include a correct advantage and disadvantage, with a justification of how it is advantageous or disadvantageous. Ideally should give a very brief explanation of the theory behind the advantage/disadvantage. Answers don't have to be exact to the wording above, but should indicate a similar understanding, e.g. that AS can improve information gain in signal sampling. Minus a mark if some justification missing.

**Half answer (3 marks)**
At least one advantage OR one disadvantage with justification provided, as above, but the other half is missing. Also, could be that an advantage and disadvantage are given, but neither are explained properly, i.e. if you find yourself asking 'but why?' for both parts, award a 3.

**Not so great (2 or less)**
As above, advantage or disadvantage missing and justification missing too. The less there is, the less marks should be awarded.

**Q7. Linear Time-Invariant Modelling: Filtered and Distilled** **[25 points]**




Consider the circuit shown in Figure above with input voltage $V_{in}$ and output voltage $V_{out}$

(a) Derive the model for this circuit? (**Hint**: is it low-pass or high-pass?)

(b) Draw the Root-Locus of this circuit (i.e., plot its zeros and poles on the s-plane).

(If you can not upload the figure, upload your **commented** MATLAB code for generating it)

(c) For $C_1$=0.01 µF and $R_1$=10 kΩ, for what values of $C_1$ and $R_2$ is the circuit stable?

**Clearly something seems to have gone wrong in the uploading of the solution and rubric. It seems something went to the wrong place and we did not realize that.**

**We apologize for this. Concrete steps are in place to make sure that this does not happen again. A couple of tutors have sent in solutions. Thanks for that. We have presented these below. A rubric follows.**

**Thanks -- SpnS**

**a)**

**-Ideal Op-Amp Characteristic: $V_+ = V_-$**

**KCL for node $V_1$:**

**$i_1 = i_2 + i_3$ => $(Vi-V_1) / (1/C_1S) = (V_1-Vo) / (R_1) + (V_1-Vo) / (1/C_2S)$**

**=> $(Vi-V_1) = (V_1-Vo)(1+R_1C_2S) / R_1C_1S$ => (1)**

**$V_+ = Vo$ , $i_3 = (V_1-V_+) / (1/C_2S)$ , $i_3 = (V_1-Vo)(C_2S)$ (2)**

**Likewise: $i_3 = V_+/R_2 = Vo/R_2$ (3)**

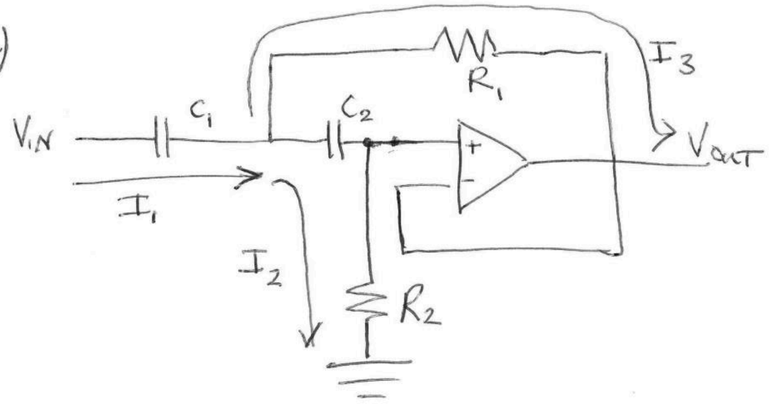
**2,3 => $V_1 = Vo(1+R_2C_2S)/R_2C_2$ (4)**

**Substitute (4) in (1) Vo/Vi = ...**

**[R1.R2.C1.C2.s^2]/[R1.R2.C1.C2.s^2 + R1(C1+C2)s + 1]**

**Solution from Kate: Thanks Kate :-)**

a)




KCL:

$$I_1 = I_2 + I_3 \Rightarrow I_3 = I_1 - I_2$$

Considering path from $V_{IN}$ to ground

$$V_{IN} = \frac{I_1}{sC_1} + \frac{I_2}{sC_2} + I_2 R_2$$

Realising that $V_+ = V_{OUT}$

$$\frac{I_2}{sC_2} = I_3 R_1 = (I_1 - I_2)R_1 \quad \text{also} \quad V_{OUT} = I_2 R_2$$

$$\Rightarrow I_2 = \frac{V_{OUT}}{R_2}$$

$$= I_1 R_1 - \frac{V_{OUT} R_1}{R_2}$$

$$I_3 = \frac{I_2}{R_1 s C_2} = \frac{V_{OUT}}{s R_1 R_2 C_2}$$

$$I_1 = \frac{V_{OUT}(sR_1C_2 + 1)}{sR_1R_2C_2}$$

so now we have the currents $I_1, I_2, I_3$ and we can sub in to our equation for $V_{IN}$:

$$V_{IN} = \frac{(sR_1C_2 + 1)V_{OUT}}{s^2 R_1 R_2 C_1 C_2} + \frac{V_{OUT}}{sC_2R_2} + V_{OUT}$$

$$= V_{OUT}\left[\frac{(sR_1C_2 + 1) + sR_1C_1 + s^2 R_1 R_2 C_1 C_2}{s^2 R_1 R_2 C_1 C_2}\right]$$

∴ TRANSFER FUNCTION –

$$\frac{V_{OUT}}{V_{IN}} = \frac{s^2 R_1 R_2 C_1 C_2}{R_1 R_2 C_1 C_2 s^2 + R_1(C_1 + C_2)s + 1}$$

We can see straight away that there is a double pole at 0.

b) If we consider factorising the denominator to find the poles we would have.

$$\frac{V_{OUT}}{V_{IN}} = \frac{s^2 R_1 R_2 C_1 C_2}{(As+1)(Bs+1)}$$

Where $AB = R_1 R_2 C_1 C_2$

$A + B = R_1 C(C_1 + C_2)$

The poles are $-1/A$ and $-1/B$. These would only become +ve making the system unstable is A or B were negative which since they both multiply & sum to a positive value is impossible.

∴ The system is stable for all values of $R_1$, $R_2$, $C_1$ & $C_2$.

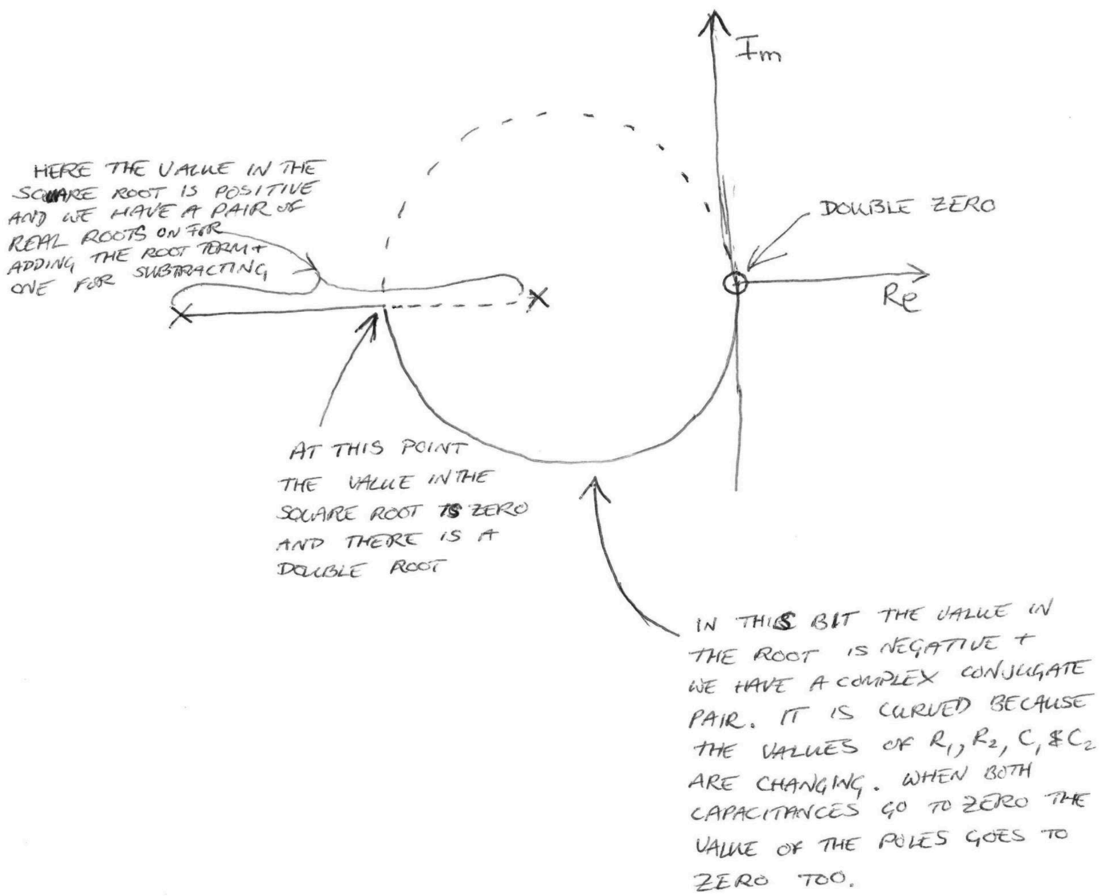
c)
We can apply the quadratic formula to find the poles:

$$P_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-R_1(C_1+C_2) \pm \sqrt{R_1^2(C_1+C_2)^2 - 4R_1R_2C_1C_2}}{2R_1R_2C_1C_2}$$

$$= \frac{-(C_1+C_2)}{2R_2C_1C_2} \pm \sqrt{\frac{R_1^2(C_1+C_2)^2 - 4R_1R_2C_1C_2}{4R_1^2R_2^2C_1^2C_2^2}}$$

There are basically 3 cases:
Either the bit in the square root is positive and we have a pair of real roots, or it goes to zero and we have a real double root, or it is negative and we have a pair of complex conjugate roots.




**Rubric: Please interpolate between the grading levels as appropriate**

(a) Derive the model for this circuit? (**Hint**: is it low-pass or high-pass?)

**[15 Points]**

- **15:** for a full derivation of the model with appropriate working.
- **10:** Working with small errors -- still gets a second-order response
- **5:** Partial working and/or that starts correct but has the wrong order, still gets that this is a high-pass filter (a DC signal will not transmit past the capacitors)

(b) Draw the Root-Locus of this circuit (i.e., plot its zeros and poles on the s-plane).
(If you can not upload the figure, upload your **commented** MATLAB code for generating it)

**[5 points]**

- **5:** Full root locus and/or code for this.
  Or a pole-zero diagram and an explain that the effects of the op-amps feedback are ignored.
- **3:** Get's the intuition correct, but graph as minor errors.
- **1**: Anything resembling a graph with poles and zeros.

(c) For $C_1$=0.01 μF and $R_1$=10 kΩ, for what values of $C_1$ and $R_2$ is the circuit stable?

**[5 points]**

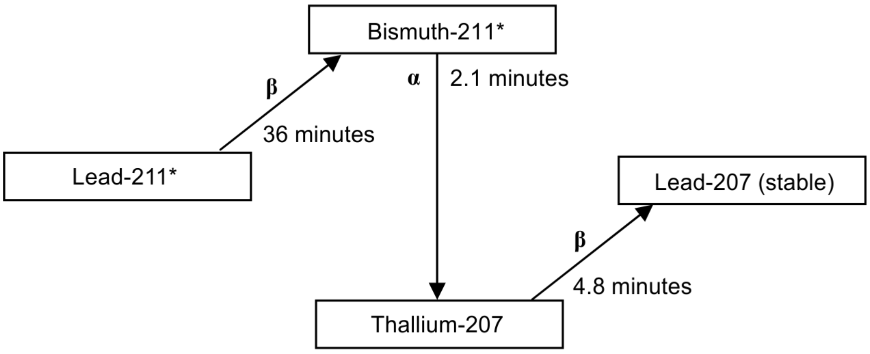
- **5:** Correct value and justification
- **3:** Correct value without justification OR incorrect value with justification
- **1**: Incorrect value without justification

## Q8. Systems Alchemy **[30 points]**

Nuclear decay is given by a first order linear differential model (the Bateman equation). This is given as: $\frac{dN}{dt} = -\lambda N$, where N is the number of atoms and $\lambda$ is the decay constant. (n.b.,$\tau = \frac{1}{\lambda}, t_{\frac{1}{2}} = \tau \cdot ln(2)$)

(a) How much (percent) is left after one lifetime ($\tau$)?

(b) How long will it take for concentrations to get to 10%




Lead-211 Decay Series [1]

It has long been the goal to turn lead into gold. Thus, we consider the decay if Lead-211 (the "parent") to Bismuth-207 (the "child"). These are the latter parts of the Actinium decay series (which includes $^{235}$U).

(c) Generate a plot of the concentrations (as a function of atom count) of both isotopes as a function of time starting with one normalized quantity of Lead-211.

(d) Sadly, this is not a gilded pathway (it leads to Lead-207). How long will it take for 1 kilogram of Lead-211 to turn into 900 grams of Lead-207 (you can neglect the mass loss associated with the alpha and beta emissions)? (hint: assume you are starting with 100% pure $^{211}$Pb)

(e) Determine and plot the ensemble radiation emission as a function of time for the duration determined above in part (d). (i.e., you may treat the $\alpha$ and $\beta$ as equivalent "particles of radiation").


[1] Argonne National Laboratory, "Natural Decay Series: Uranium, Radium, and Thorium", Human Health Fact Sheet, August 2005. [http://www.ead.anl.gov/pub/doc/natural-decay-series.pdf]


### Solution and Rubric:

The solution of the the Bateman equation is central to this entire problem. This can be rewritten as a homogeneous, first-order, ODE for N(t): $\frac{dN}{dt} + \lambda N = 0$, the solution of which is :

$N(t) = N(0)exp(-\lambda t)$

a) [5 points]

At one lifetime, t = $\tau$, $\tau = 1/\lambda$, thus:

N(t) = N(0)*exp(-$\lambda/\lambda$) = N(0)*exp(-1) = N(0)/e ≈ 0.368*N(0).

Since we are looking for a percentage, we normalize by N(0), to get 36.8%.

**Rubric:** Full credit if correct (and a logical explanation is provided). If the problem has the correct solution to the ODE, but the wrong values, then it gets 3 points. If the solution to the ODE is wrong, but the rest of the part seems as if it would have been correct, then it receives 2 points.

b) [5 points]

Recall that: $N(t) = N(0)exp(-\lambda t)$

Here, N(t)/N(0) = 0.1. Solving for t we get and recalling the definition of $\tau = t_{\frac{1}{2}} / ln(2)$

$$t = -ln(0.1) \cdot \tau$$
$$t = \frac{-ln(0.1)}{ln(2)} \cdot t_{\frac{1}{2}} \approx \mathbf{(3.32)} \cdot t_{\frac{1}{2}}$$

This intuits as three "half-times" would give 12.5% of the material.

As an illustrative example, for Pb-211 to Bi-211, the decay to 10% would be take t = 3.32 × 36 ≈ 119.6 mins or approx 2 hrs.

**Rubric:** Full credit if correct -- all that is needed is the request time ($t_{\frac{1}{2}} \approx \mathbf{(3.32)} \cdot t_{\frac{1}{2}}$) and a logical explanation of how the value was generated.

The next 3 parts -- (c) to (e) -- relate to the Pb-211 to Pb-207 portions of the Actinium decay series. This is a three-decay chain that expands on the single step decay model from above. While this is discussed in the Bateman paper, it can be analyzed by considering the rate dynamics of the decay and the generation. To follow along with nuclear science/chemistry references, we adopt the notation of the first item in the chain being the parent and the second item being the child.

Thus we have:

N1(t): Pb-211, N2(t): Bi-311, N3(t): Th-207, N4(t): Pb-207

For an initial number of parent and child atms we have: N1(0), N2(0), N3(0), and N4(0)

The first step in the decay is given by the single step decay model : $\frac{dN}{dt} + \lambda N = 0$ whose solution is $N1(t) = N1(0)exp(-\lambda_1 t)$. The relation for the child needs to include a factor for the decay **and** one for the generation by the parent. Thus:

$$\frac{dN2(t)}{dt} = -\lambda_2 N2(t) + \lambda_1 N1(t)$$

Similarly for N3(t) and N4(t) we get:

$$\frac{dN3(t)}{dt} = -\lambda_3 N3(t) + \lambda_2 N2(t)$$
$$\frac{dN4(t)}{dt} = \lambda_3 N3(t)$$

Remember $\lambda_4$ is zero because Pb-207 is stable. We assume that we start with 100% Pure Pb-211 (let's not worry about the practicalities of enriching a sample to this level for now -- we leave that to chemical engineering or your favorite despot). Thus: N1(0) = 1, N2(0)=0, N3(0)=0, and N4(0)=0.

To solve the ODE, we could factoring the solution from the previous decay and solve the simultaneous ODE directly. For example for $\frac{dN2(t)}{dt}$, we substitute in the solution to N1(t) (from above) to get:

$$\frac{dN2(t)}{dt} = -\lambda_2 N2(t) + \lambda_1 N1(0)exp(-\lambda_1 t)$$

Solving this gives:

$$H(s) = \frac{(1+)}{}$$

$$N2(t) = N2(0)exp(-\lambda_2 t) - N1(0)\frac{\lambda_1}{\lambda_2-\lambda_1}(exp(-\lambda_2 t) - exp(-\lambda_1 t))$$

However, this approach could get more complicated moving on to N3(t) and N4(t). Instead we employ the Laplace transform:

$$sN1(s) - N1(0) = -\lambda_1 N1(s)$$

$$sN2(s) - N2(0) = -\lambda_2 N2(s) + \lambda_1 N1(s)$$

$$sN3(s) - N3(0) = -\lambda_3 N3(s) + \lambda_2 N2(s)$$

$$sN4(s) - N4(0) = \lambda_3 N3(s)$$

Bateman gives us a solution pathway for an N-series chain. For brevity in this solution, we will go straight to the solution of N3(t) and N4(t) (as N1(t) and N2(t) are presented above). For those of you following Nucleonica's (above linked) discussion of Bateman's equation, note that the Branching Ratio for this series is 1 and we are interested in equations 1,2,and 3.

$$N3(t) = \lambda_1\lambda_2 N1(0)\left[\frac{exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} + \frac{exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right]$$

and for N4, there are two ways of making this "easier," which are to either go back to the definition of N4 and integrate or to use Bateman's solution with $\lambda_4 = 0$. Thus:

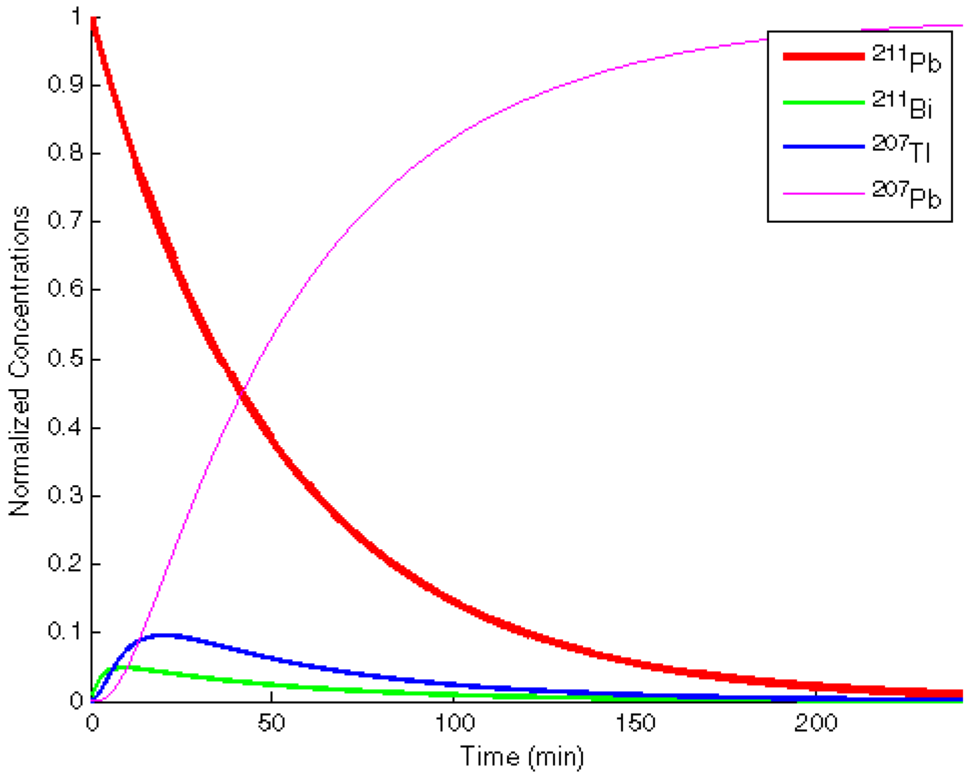
$$N4(t) = \lambda_1\lambda_2\lambda_3 N1(0)\left[\frac{exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(-\lambda_1)} + \frac{exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(-\lambda_2)} + \frac{exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(-\lambda_3)} + \frac{1}{(\lambda_1\lambda_2\lambda_3)}\right]$$

or a little more simply:

$$N4(t) = N1(0)\left[1 - \left[\frac{\lambda_2\lambda_3 exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} - \frac{\lambda_1\lambda_3 exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} - \frac{-\lambda_1\lambda_2 exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right]\right]$$

c) [10 points]

With the appropriate discussion (such as above) and with appropriate fiddling in MATLAB:




**Rubric:** Full credit if correct -- Extra Credit (+3) if there is a derivation of the Bateman equation or good, documented Matlab code provided.

At this point the problem (sadly) became far trickier than I had hoped because we have to **assume an activity rate.** because not all atoms in a material are going to "emit" radiation. An "ideal model" is to assume that sample is fully active and thus depends on its **mass m** (number of radioactive atoms) and the **decay rate**. Recalling (or referencing) a little chemistry, we can get the number of atoms via the molar mass (μ) and Avogadro's number (Na). Thus:

$$A = \lambda N = \lambda \frac{m}{\mu}(Na) = \left(\frac{Na}{\mu}\right)\lambda(m)$$

Recall that the unit for activity is 1 Becquerel [Bq] which is 1 decay per second
(The media often quotes "Curies" which are 3.7E10 [Bq] and are a sizable unit). So we have to be careful with units here -- Na is in mol and molar mass is in g/mol and we need lamba in seconds for the radiation [Bq].

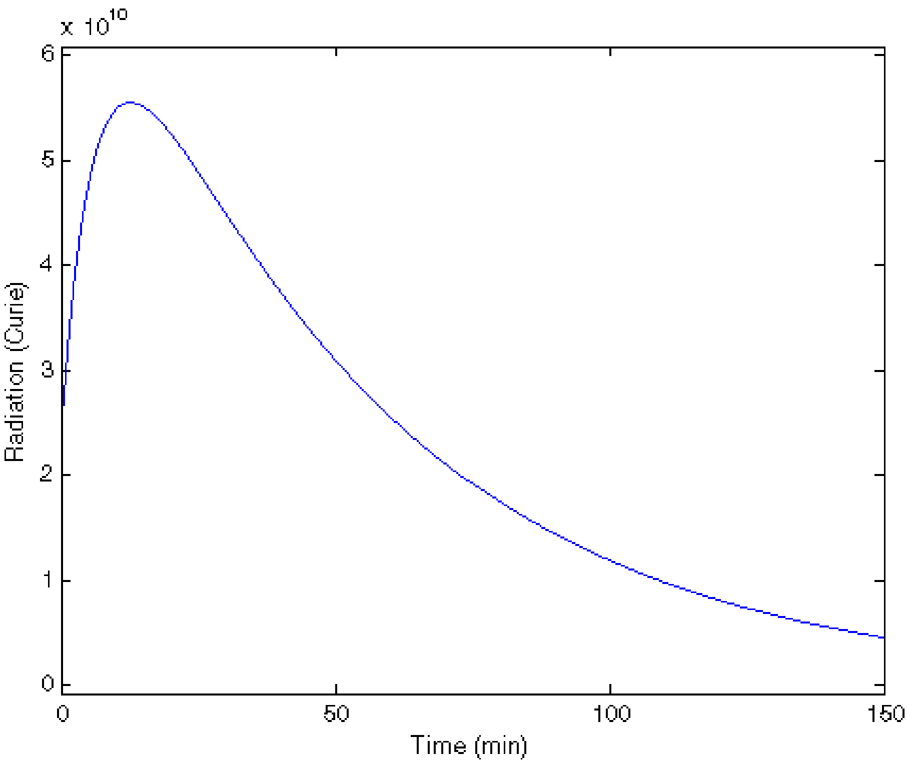
d) [ 5 points ]
The molar mass of Pb-211 is 211. Na=6.022E23
Thus, 1kg or Pb-211 has 2.854E24 atoms or N1(0)=2.85E24
We want 900 g of Pb-207, which is 2.618E24 atoms.
Thus we are looking to solve for t (let's call this t*) such that N4(t)=2.618E24
Note that N4(t*)/N1(0)=0.917

Using Matlab's solve (something like `solve(N4(t) == 2.618E24`, t) we get:
t=140 min

**Rubric:** If the problem is setup correctly and involves finding t* -- this is worth full credit. It does not matter if it is not exactly 140 min per se. If it factors the change in isotopes, then +1 extra credit point. If it has a good description, then also +2 points. If the solution points out that it could be “infinite” because their activity model assumes not all nuclei are active, then this okay, but it has to be justified.

e) [5 points]

We now want to calculate the Activity of all this (i.e., what we would see with a geiger counter). Recall that the definition of decay rate is given by $A = \lambda N$. Thus the ensemble activity is given by: $A(t) = \lambda_l N1(t) + \lambda_2 N2(t) + \lambda_3 N3(t)$




Note that the values are large so as the problem makes a number of idealised assumption -- note that a reactor upon shutdown would have activity in the range of giga Curie.

Code for generating the graphs:

```
%% ELEC 3004
% Quick & Dirty Code for Question 8
% For a review of Bateman's Equation. See:
% -- http://www.nucleonica.com/wiki/index.php?title=Help:Decay_Engine#Radioactive_Decay_Chains
% -- http://chemistry.sfu.ca/assets/uploads/file/Course%20Materials%2012-1/NUSC%20342/L9.pdf
% -- http://en.wikipedia.org/wiki/Radioactive_decay

%% Have Matlab Solve the ODE
% The solution can be found from the Bateman's Equation
% But this is a "lazy" way to type in all the equations
syms N1(t) N2(t) N3(t) N4(t) l1 l2 l3 l4 N0

q1=dsolve(diff(N1)==-l1*N1, N1(0) == N0);
q2=dsolve(diff(N2)==-l2*N2+l1*q1, N2(0) == 0);
q3=dsolve(diff(N3)==-l3*N3+l2*q2, N3(0) == 0);
q4=dsolve(diff(N4)==l3*q3, N4(0)==0);


%% Handle Normalized Case (part c)
% Input lamba values
k1=log(2)/36;
k2=log(2)/2.1;
```

```
k3=log(2)/(4.8);
k4=0;
N0val = 1;  % value for N1(0)

% Substitutions
eq1=subs(q1, [l1,l2,l3,l4, N0], [k1,k2,k3,k4, N0val])
eq2=subs(q2, [l1,l2,l3,l4, N0], [k1,k2,k3,k4, N0val])
eq3=subs(q3, [l1,l2,l3,l4, N0], [k1,k2,k3,k4, N0val])
eq4=subs(q4, [l1,l2,l3,l4, N0], [k1,k2,k3,k4,N0val])

% Output in Latex
% (speaking of which --
% also check out "publish('elec3004q8graph.m','latex')"
texeq1=latex(simplify(eq1));
texeq2=latex(simplify(eq2));
texeq3=latex(simplify(eq3));
texeq4=latex(simplify(eq4));

% Set graph limits
tmin=0;
tmax=240;
ymin=0;
ymax=1;

% Reset figure
figure(123)
close(123)
figure(123)

% Plot easily via ezplot
hold on;
h1=ezplot(char(eq1), [tmin,tmax,ymin,ymax] );
set(h1, 'Color', 'r', 'LineWidth', 3);

h2=ezplot(char(eq2), [tmin,tmax,ymin,ymax]);
set(h2, 'Color', 'g', 'LineWidth', 2);

h3=ezplot(char(eq3), [tmin,tmax,ymin,ymax]);
set(h3, 'Color', 'b', 'LineWidth', 2);

h4=ezplot(char(eq4), [tmin,tmax,ymin,ymax]);
set(h4, 'Color', 'm', 'LineWidth', 1);

hold off;

legend('^{211}Pb', '^{211}Bi', '^{207}Tl', '^{207}Pb');

title('')
xlabel('Time (min)')
ylabel('Normalized Concentrations')

%% Handle case with 1 kg of atoms

% N1(0) for 1 kg of Pb-211:
N0val = (1000/211)*(6.022E23);  % (1000 g/211 g/mol)*(Avogadro's Constant)

% Solve for New Concentration
eq4_1kg=subs(q4, [l1,l2,l3,l4, N0], [k1,k2,k3,k4,N0val]);
solve(eq4_1kg == ((900/207)*(6.022E23)), t)

% New lambda terms in seconds
k1sec=log(2)/(36*60);
k2sec=log(2)/(2.1*60);
k3sec=log(2)/((4.8)*60);
k4sec=0;

% New Substitutions (Assuming all nuclei are radioactive -- the curve will
```

```
% have the same shape even with an appropriate fraction of them)
radiation=l1*q1+l2*q2+l3*q3;
eq_rad=subs(radiation, [l1,l2,l3,l4, N0], [k1sec,k2sec,k3sec,k4sec, N0val]);

syms tm   % time in minutes
eqradmin=subs(eq_rad, t, tm*60)
eradcurie=eqradmin*(1/3.7E10);

figure(234);
ezplot(eradcurie, [0 150])
title('');
xlabel('Time (min)')
ylabel('Radiation (Curie)')
```

## For ELEC 7312 Only:

**Q9. A Flood of Signals [* for ELEC 7312 Only *]** **[20 points]**

Consider an idealised hydrology model for river height based on past rainfall in the region. Let u(t) denote the rainfall rate (in mm/hour) in a region at time t and y(t) denote the flood level (m), above a reference (non-flood) level. A simplified analysis gives the transfer function as

Y(s) = F(s) U, $Y(s) = F(s)U(s),\ F(s) = [\frac{10}{(7s+1)(49s+1)}]$

(**FYI**: The fast pole is from surface runoff, which is relatively, but quick. The slow pole is from tributaries and groundwater, which contribute more water into the river over a much longer time scale)

Consider an intense rain after a long dry spell in which it which it rains 100 mm-per-hour for 30 minutes. This causes the river height to rise and then recede. Then:

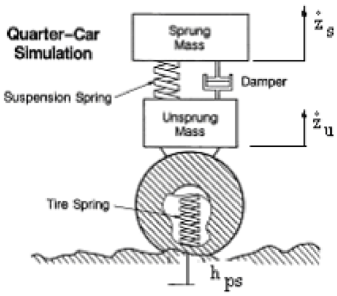
(a) How long does it take, after the beginning of the downpour, for the river to reach its maximum height? (Tidal effects can be neglected, consider positive time (t>0) measured in hours)

(b) Determine and plot the height of the river (in meters) as a function of time (in hours).

**Q10. An Efficient Ride [* for ELEC 7312 Only *]** **[30 points]**

Henry wants better fuel economy, so he increases the pressure in his tyres.

Note that a car's suspension can be viewed as a spring (for the tyre), a mass for the wheel (called the "unsprung mass"), the suspension spring and damper and mass for the car (typically ¼ the vehicle's mass). This is called the **quarter-car model**.




Quarter-Car Model (from Google Images c/o US DOT http://goo.gl/wOY5m)

(a) What is the overall order of the system as seen from the a passenger sitting in the car?

(b) Sketch (or plot) the expected signal for running over a pothole as seen by an accelerometer on the floor of the car and by an accelerometer on the seat. Consider the pothole as an impulse and the car's damper as overdamped (shocks are new).

(c) Is this wise? Discuss using your own research. (1 page maximum)