From Dean Ballard comes a pernicious pizza puzzle:

Dean ordered a personal pizza that was precisely 10 inches in diameter. He ate half of it, and he wants to save the remaining semicircle of pizza in his refrigerator. He has circular plates of all different sizes, so to save space in his fridge, he'll place the pizza on the smallest plate that holds the entire semicircle (i.e., with no pizza hanging off the plate).

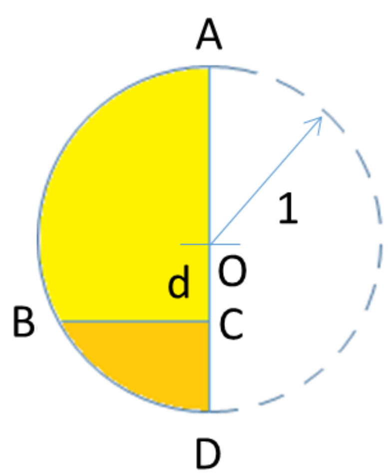
Unfortunately, the smallest plate that can hold half of a 10-inch pizza is just a circle with a 10-inch diameter. So much for saving space.

But Dean has a thought: If he cuts the pizza, he can squeeze both of the resulting pieces onto a smaller circular plate — again, with no pizza hanging off the plate and without the pieces lying on top of each other.

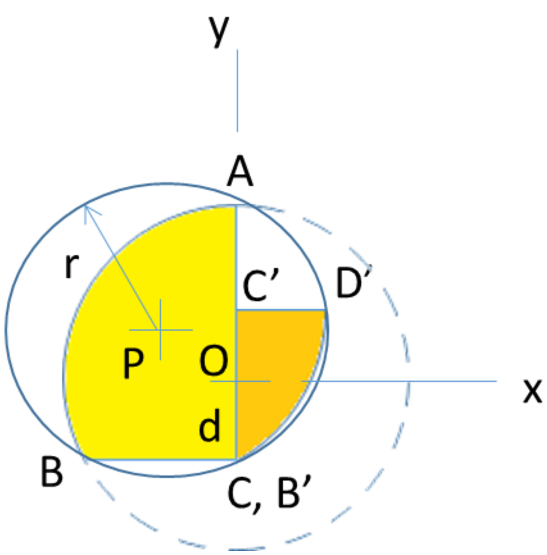
If Dean makes a single straight slice, what is the diameter of the smallest circular plate onto which he can fit the two resulting pieces?

Extra credit: Dean wants to save even more space in his fridge. So instead of one straight slice, he will now make two straight slices. First, he will cut the semicircular pizza into two pieces. Then, he will take one of those pieces and make his second slice, giving him a total of three pieces. What is the diameter of the smallest circular plate onto which he can fit all three pieces?

Let's normalize this problem to a pizza of radius 1. Everything will scale linearly. I will start by conjecturing without proof that the optimal cut is a line perpendicular to the diameter of the semicircle. Let's draw a diagram. We have our semicircle pizza, with center O, and the ends of the diameter points A and D. We cut the pizza on line BC, perpendicular to AD and at distance d from the center.



Now, points A, B, and C are on a circle centered at P. We want to choose d such that we can just fit the second piece, BCD, into the circle. Moving that second piece, we have new points B', C', and D'; D' is on the circle centered at P when the line B'C' is coincident with line AC. I will once again conjecture without proof that the optimum solution is when B' is also on the circle; that is, it is coincident with C. Note, because the pizza is a larger radius than the plate, the slice will be entirely on the plate. At this point, it becomes convenient to establish coordinates, with point O at (0, 0), x horizontal to the right, and y vertical up.



Because ACB is a right triangle, point P lies at the midpoint of line AB. Since BC = sqrt(1-d²), the x coordinate of P is –sqrt(1-d²)/2. P is at (1+d)/2 vertically from C, and thus the y coordinate of P is (1+d)/2 – d = (1-d)/2.

D' is at an x coordinate of 1-d. Vertically, it is B'C' = BC = sqrt(1-d²) above points C/B', and thus at a y coordinate of sqrt(1-d²) – d.

Thus, the distance between P and D' is equal to the radius of the circle at P,

$$r = \text{sqrt}[(D'_x - P_x)^2 + (D'_y - P_y)^2]$$

$$r = \text{sqrt}[(1-d + \text{sqrt}(1-d^2)/2)^2 + (\text{sqrt}(1-d^2) - d - (1-d)/2)^2]$$

$$r = \text{sqrt}[(1-d + \text{sqrt}(1-d^2)/2)^2 + (\text{sqrt}(1-d^2) - (1+d)/2)^2]$$

$$r = \text{sqrt}[(1-d)^2 + (1-d)\ \text{sqrt}(1-d^2) + (1-d^2)/4 + (1-d^2) - (1+d)\ \text{sqrt}(1-d^2) + (1+d)^2/4]$$

$$r = \text{sqrt}[(1-d)^2 - 2d\ \text{sqrt}(1-d^2) + 5(1-d^2)/4 + (1+d)^2/4]$$

$$r = \text{sqrt}[1 - 2d + d^2 - 2d\ \text{sqrt}(1-d^2) + 5/4 - 5d^2/4 + 1/4 + d/2 + d^2/4]$$

$$r = \text{sqrt}[5/2 - 3d/2 - 2d\ \text{sqrt}(1-d^2)]$$

Also, we have the position of point C/B', at an x coordinate of 0, and a y coordinate of -d. Thus, the distance between P and C, which is also equal to the radius of the circle at P, is

$$r = \text{sqrt}[(C_x - P_x)^2 + (C_y - P_y)^2]$$

$$r = \text{sqrt}[(0 + \text{sqrt}(1-d^2)/2)^2 + (-d - (1-d)/2)^2]$$

$$r = \text{sqrt}[(\text{sqrt}(1-d^2)/2)^2 + (-(1+d)/2)^2]$$

$$r = \text{sqrt}[(1-d^2)/4 + (1 + 2d + d^2)/4]$$

$$r = \text{sqrt}[(1 + d)/2]$$

Now, we can solve for d.

$$\text{sqrt}[5/2 – 3d/2 – 2d\ \text{sqrt}(1-d^2)] = \text{sqrt}[(1 + d)/2]$$

$$5/2 – 3d/2 – 2d\ \text{sqrt}(1-d^2) = (1 + d)/2$$

$$1 = d + d\ \text{sqrt}(1-d^2)$$

This has two solutions, $d = 1$ (where the square root term is 0, corresponding to not cutting the pizza at all), and $d \approx 0.544$. This fits on a plate of radius

$$r = \text{sqrt}[(1 + d)/2] \approx 0.879$$

Scaling up to a pizza with a diameter of 10 inches, and a radius of 5 inches, we find that the optimal place to cut the pizza is at a distance of $d * 5$ in ≈ 2.718 in from the center, again, on a line perpendicular to the diameter. This will fit on a plate of diameter $2r * 5$ in ≈ 8.785 in.

I’m going to leave my conjectures – that the optimum is cutting the pizza normal to the diameter, and arranging the smaller slice such that the corners are coincident. I suspect they are correct, but this is not proven.

I’m not attempting the extra credit.