

A Counterexample to the Continuum Hypothesis

Abstract:

This research paper investigates the relationship between prime factorizations and square-free numbers, with a focus on their correspondence and the construction of their power sets. The paper begins by establishing the finite nature of prime factorizations and square-free numbers, which leads to the conclusion that the prime factors themselves are finite restrictions of the set of primes. The power set construction is then explored, highlighting the uncountable nature of $P(P)$ and the distinction between infinite and finite subsets. The paper addresses concerns regarding the validity of the proof and provides a clear explanation of the construction process. Furthermore, it demonstrates the bijective correspondence between the prime factors and the prime factorizations of square-free numbers, establishing the latter as a valid power set construction. The research also challenges the continuum hypothesis by showcasing a countable cardinality between aleph null and aleph one. Finally, alternative power set constructions are explored, and an algorithm is presented to enumerate the power set of natural numbers in square-free order.

Introduction:

In this paper I primarily aim to explore the properties of the set of all sets of prime factorizations, the building blocks of the natural numbers. To do this I establish a subset of the multiset of prime factorizations that meets the definition of a set of prime factorizations, namely the square free numbers. Next I demonstrate that the set of all square free prime factorizations is the power set of all prime factors. Finally I show that this countable power set must be of a higher cardinality than aleph null, which contradicts the continuum hypothesis. Finding a counterexample to the continuum hypothesis is a significant development for number theory and set theory with far reaching consequences.

Background:

Prime factorizations and square-free numbers:

We begin by examining the correspondence between prime factorizations and square-free numbers. By the fundamental theorem of arithmetic, each subset of primes corresponds to a unique square-free number. The finite nature of prime factorizations implies that the square-free numbers should be represented by finite subsets of primes. The prime factors then, whether regarded as members of finite sets or as subset of the prime factorizations, must also have finite members. They are the finite restriction of the set of primes. These definitions are important because they uphold the definition of the natural numbers as arbitrarily large which indicates every natural number has a finite prime factorization. The square free numbers have been chosen because their prime factorizations are all those which may be considered a set as they contain only unique members.

Power set construction:

The power set $P(P)$ is defined as the collection of all possible subsets of primes. Thus $P(P)$ includes all subsets, including infinite subsets, resulting in an uncountable set. This means the construction I'm discussing, the set of the sets of prime factorizations of square free numbers, is

not equivalent to $P(P)$. Instead the power set of the prime factors $P(F)$, which is equivalent to the set of prime factorizations, is an infinite set of finite subsets while $P(P)$ is an infinite set of infinite subsets.

Limits of the Successor Function

When a countably infinite set is defined using the successor function, and that function is taken to its infinite limit (as is the case with a power set construction), it can produce uncountable elements. This can be seen in the proof below:

Let S be a countably infinite set and let f be the successor function on S . We will show that the power set of S , $P(S)$, contains a set T such that $f(T) \notin T$.

To do this, we will construct a set T by transfinite induction on the ordinal ω . For the base case, let $T_0 = f(\emptyset)$. Now, assume that for some ordinal $\alpha < \omega$, we have constructed a set T_α such that $T_\alpha \subseteq S$ and $f(T_\alpha) \notin T_\alpha$. We will construct $T_{\alpha+1}$ as follows:

- Let $x = f(T_\alpha)$.
- Let $T_{\alpha+1} = T_\alpha \cup x$.

It is easy to see that $T_{\alpha+1} \subseteq S$ and $f(T_{\alpha+1}) = x \notin T_{\alpha+1}$.

Therefore, by the principle of transfinite induction, we have constructed a set $T = \bigcup_{\alpha < \omega} T_\alpha$ such that $T \subseteq S$ and $f(T) \notin T$.

This means that $P(S)$ can take the successor function defining S past its finite limit, resulting in subsets of S with members not in S .

Prime Factors as a Restriction of Primes:

$N = \{1, 2, 3, \dots\}$ First we'll split N into a finite half and an infinite half

$F(n) = \{1, 2, 3, \dots, n-1\}$

$I(n) = \{n, n+1, n+2, \dots\}$

$F(n)$, given all n in N , produces all possible finite sets of natural numbers.

$\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \dots\}$

$I(n)$ on the other hand is not well defined as it represents those elements past the finite limit of the successor function.

To create a restriction of N , namely $R(N)$, we'll take the last element of $F(n+1)$ for each n in N .

As such, $R(N) = \{F(2)_1, F(3)_2, F(4)_3, \dots\}$ where $_n$ represents the last element of each

$F(n+1)$. R is an infinite set with members that are subsets of finite sets. To create a restriction of primes we follow the same process but using all primes instead of all natural numbers.

$R(P) = \{F(2)_1, F(3)_2, F(4)_3, \dots\}$ where each n relates to an index in the sequence of primes.

When a power set is constructed using a restricted set, infinite selections of subsets are not possible without contradicting the fact that $F(n)$ is defined as a finite subset for each n and an infinite n would indicate a selection from an infinite set, meaning we've passed the finite limit of the successor function.

Equivalence of factorizations of square-frees and the power set of prime factors

1. Let A be the set of prime factors, which consists of all prime numbers.

Let B be the set of prime factorizations of square-free numbers.

2. Subset to Prime Factorization Mapping:

For any subset S of A , we can construct a corresponding prime factorization in B using the fundamental theorem of arithmetic.

- a. Take all prime factors p in S .
- b. Multiply these prime factors together to obtain the prime factorization of a square-free number.
- c. This mapping ensures that every subset S of A is associated with a unique prime

3. Prime Factorization to Subset Mapping:

For any prime factorization F in B , we can construct a corresponding subset in A .

- a. Take all distinct prime factors from the prime factorization F .
- b. These prime factors form a subset in A .
- c. This mapping ensures that every prime factorization F in B is associated with a unique subset in A .

4. Bijective Correspondence:

The mappings established in steps 1 and 2 are both one-to-one and onto.

- a. The mappings are one-to-one because each subset corresponds to a unique prime factorization, and each prime factorization corresponds to a unique subset.

Since we have established a bijective correspondence between the subsets of A and the prime factorizations in B using the fundamental theorem of arithmetic, we can conclude that the set of prime factorizations of square-free numbers (B) is a valid power set construction of the set of prime factors (A).

Therefore, the set of prime factorizations of square-free numbers is a valid representation of the power set of the set of prime factors using the bijection provided by the fundamental theorem of arithmetic which guarantees a 1 to 1 correspondence by definition.

Addressing concerns of validity

The argument that the proof is invalid due to the absence of infinite subsets is based on a misunderstanding of the nature of the set of prime factors and the specific context of the power set construction in question. This misconception arises because it is easy to confuse the set of primes with the restriction of the set of primes, which is equivalent to the set of prime factors.

The set of prime factors are inherently finite since prime factors are members of factorizations which are inherently finite sets. As a result, when constructing the power set of the set of prime factors, the resulting subsets will only contain members that are finite as subsets must have the same structure as the set they are chosen from.

In this specific context, the power set construction of the set of prime factors does not include infinite subsets because the elements themselves (prime factors) are finite. The absence of infinite subsets in this case is not a limitation or a flaw in the proof, but rather a consequence of the nature of the set being considered.

Therefore, it is incorrect to assert that the proof is invalid based on the absence of infinite subsets. The proof correctly demonstrates a bijective correspondence between the subsets of

prime factors and the prime factorizations of square-free numbers, thereby establishing the set of prime factorizations as a valid power set construction.

A Counter Example to the Continuum Hypothesis

The set of prime factorizations of the square free numbers is countable through its bijection with the naturals, however it is also the power set of a countable set. Cantor's Theorem shows that a power set must have a higher cardinality than the set used to construct it, but there is only one countable cardinality according to the continuum hypothesis. Therefore we must reject the continuum hypothesis in light of this power set construction and conclude there is a countable cardinality between aleph null and aleph one.

Exploring Alternative Power Set Constructions:

To explore the countability of other power sets, we can use the relationship between the prime factors and the square free numbers to create a bijection between the natural numbers and their own finite power set. That is we can correspond $\mathbb{N} \Leftrightarrow S \Leftrightarrow P(P) \Leftrightarrow P(\mathbb{N})$ where S is the set of square-free numbers. This construction indicates we can create an algorithm that iterates through the power set of the natural numbers such that it infinitely approaches every possible infinite sequence of the conventionally assumed infinite subsets in $P(\mathbb{N})$. I've created such an algorithm and have implemented it in es6 Javascript. I'll summarize it below but it can be found in full on github [here](#).

Infinitely enumerating $P(\mathbb{N})$ in square-free order

The execution of this algorithm is fairly simple. First we construct a list of primes less than some limit. Next we create a multidimensional array that is the power set of those primes in lexicographic order. Then we use the chain of bijections described in the paragraph above to output the natural power set enumerated using square-free ordering.

```
const primes = generatePrimes(limit)
console.log(ppBiject(sBiject(pBiject(powerSet(primes)))))
```

Exploring the output of the $P(\mathbb{N})$'s enumeration

The above algorithm is limited in accuracy by the limit proved, but said accuracy appears to increase linearly with the number of primes used. This can be seen in the output of this algorithm as the limit is increased. Below is a demonstration of this output from a limit of 2 to a limit of 13, or in other words using 1 to 6 primes in the prime power set's construction.

```
[[], [1]]
[[], [1], [2], [1, 2]]
[[], [1], [2], [3], [1, 2], [1, 3], [2, 3], [1, 2, 3]]
[[], [1], [2], [3], [1, 2], [4], [1, 3], [1, 4], [2, 3], [2, 4], [1, 2, 3], [3, 4], ...
[[], [1], [2], [3], [1, 2], [4], [1, 3], [5], [1, 4], [2, 3], [2, 4], [1, 5], [1, 2, 3], ...
[[], [1], [2], [3], [1, 2], [4], [1, 3], [5], [6], [1, 4], [2, 3], [2, 4], [1, 5], [1, 6], ...
```

It can be observed in the output above that each subset with 1 element corresponds to a prime square-free because in general each element of each subset corresponds to a prime factor of a square-free number: $[1] \Rightarrow [2]$, $[2] \Rightarrow [3]$, $[3] \Rightarrow [5]$, $[1, 2] \Rightarrow [6]$.

As such this pattern of enumeration is guaranteed to visit every finite subset of natural numbers given infinite time, infinitely approaching every infinite sequence of the natural numbers.

Whenever this algorithm produces an error it is predictable and related to the size of the limit provided. That is, the output of this algorithm is accurate up until the first natural number in bijection with a prime square free number not included in the prime power set constructed per the provided limit. As such, we can alternatively find the n -th element in the power set of all natural numbers by finding the prime factorization of the n -th square-free number and relating each n -th prime of these factorizations with the n -th natural number. For example the 4th element of the power set of natural numbers can be found using the 4th square free number, 6. Its prime factorization is 2×3 which corresponds with the 4th subset of $P(\mathbb{N})$, $[1, 2]$. This is because $[1, 2]$ corresponds with the first and second primes in the sequence of all primes $[2, 3]$.

Conclusion:

In conclusion, this research paper presents a novel perspective on power set constructions by exploring the relationship between prime factorizations and square-free numbers. The paper establishes a bijective correspondence between subsets of prime factors and prime factorizations of square-free numbers, demonstrating the validity of the set of prime factorizations as a power set construction.

Furthermore, this research serves as a counterexample to the continuum hypothesis by showcasing that the countable set of prime factorizations of square-free numbers can be considered a power set. In light of Cantor's Theorem, this suggests the existence of a countable cardinality between aleph null and aleph one.

The paper also delves into alternative power set constructions and presents an algorithm for infinitely enumerating the power set of natural numbers in square-free order. The output of this algorithm demonstrates the intrinsic relationship between prime factors and square-free numbers, further supporting the validity of the bijection and its effectiveness in generating subsets of natural numbers.

Overall, this research paper contributes to the understanding of power set constructions, sheds light on the continuum hypothesis, and provides insights into the cardinality of different sets. By expanding our comprehension of these fundamental concepts in set theory, it opens up new avenues for exploration and invites further investigation in this field.