Projects for Differential Geometry

[[*first draft 1/15*. Point values subject to change before final posting on 1/22. Also, links to details coming]]

Curves

Project title or central idea	Points Awarded
Distinguish between the Tractrix and “the Shifrix”	10
Darboux’s kinematical framing of curve theory	40
Fundamental Theorem of Curves: curves of constant curvature	30
Fundamental Theorem of Curves: curves of constant torsion	30
Curiosity: Osculating Circles Don’t Touch!	30
The Locus of Centers	30
Differential Equation characterizing a helix	30
Pedal Curves	30
Envelopes, including Caustics	30
Deeper into Spherical Curves	40
Total Torsion	40
Bertrand Mates	30
Involutes and Evolutes	30
Bishop Framings and 3d printing a tube without twisting	40
Parallel Curves and 3d printing a tube with pinching	30
Beautiful Curve Splines - matching curvatures in piecewise curves	50
Whitney-Graustein Theorem	40
Animate TNB for a circular helix	30
Curves of Constant Width and Barbier’s theorem	30

The Tractrix and the “Shifrix”

The description of the tractrix that Shifrin gives is different from the usual one. The change in possible interpretation makes for a different curve. The project is to investigate the new curve and explore the differences between these two curves.

Darboux’s Kinematics

Gaston Darboux gave an interesting way to think about the the Frenet-Serret moving frame as if it were a rigid body rotating in space, like a tumbling asteroid. The project is to learn about how Darboux recast the FS equations as if they come from describing this motion.

Fundamental Theorem: Curves of Constant Curvature

You will explore the way that the existence and uniqueness theorem for ODEs applies to curve theory, and find a way to describe *all* space curves with a given constant curvature.

Fundamental Theorem: Curves of Constant Torsion

You will explore the way that the existence and uniqueness theorem for ODEs applies to curve theory, and find a way to describe *all* space curves with a given constant torsion.

Kiss but don’t touch

You will investigate how nearby osculating circles have the following curious property: “consecutive” ones don’t meet!

The Locus of Centers

For any given point on a space curve, one can find the center of the osculating circle to the curve at that point. This point is called the “center of curvature.” Suppose that as you move along the curve, you decide to trace the changing location of the center of curvature. This makes a curve called the locus of centers. What does that look like?

Characterizing a Helix

Find a differential equation which is satisfied by a curve if and only if that curve is a “generalized helix.”

Pedal Curves

Suppose you have a curve α in the plane, and some point P that does not lie on the curve. The "pedal curve" of α with respect to P is made as follows: for each point Q on α, draw the tangent line L to α at Q, and then find the foot P' of the perpendicular to L through P.
You will find a way to describe a pedal curve and investigate its geometry.

Envelopes, Including Caustics

Given a family of lines, one can sometimes make a curve so that those lines are the tangent lines of the curve. These are called "envelopes." You will investigate envelopes and their geometry. You may also explain why the way light reflects into your coffee cup makes a cardioid.

Deeper into Spherical Curves

You will investigate the special geometry of space curves which are constrained to lie on a sphere. There is a differential equation characterizing them, and some really beautiful properties that can help study general curves.

Total Torsion

We will study a few "global" theorems for curves involving the total curvature $\int \kappa(s)\, ds$. What can we say about the total torsion? Work of Fenchel might be of interest here.

Involutes and Evolutes

Suppose we lay a long string along a curve, and then begin to unwrap that string by pulling it directly away from the curve. This builds the "involute" of the original curve. Passing in the other direction, we call the original the "evolute." You will explore the geometry of involute/evolute pairs.

Bertrand Mates

Two space curves are called Bertrand Mates when they can be brought into a correspondence so that corresponding points have the same normal lines. You will investigate when this is possible and study the geometry of Bertrand mates.

Bishop Framing

The Frenet-Serret framing is not the only reasonable way to put a moving frame on a curve. Another set of options was investigated by Richard Bishop, and involves finding "vector fields parallel along the curve." I have hope this would be useful in 3d printing a curve, since the natural FS framing can sometimes have lots of twisting. (too much torsion!). You will investigate Bishop's idea, and see if it can make a more efficient 3d printing curve, with less twisting.

Parallel Curves

Two curves are parallel if they have the same tangents at corresponding points. You will investigate this idea and try to use it to make a small tube around a space curve. If you know the curvature, can you decide the proper width of a tube that won't get "pinched" by overlapping itself as the curve bends around?

Beautiful Splines

A spline is a way of approximating a curve with small bits of well-controlled curves of a special type (usually polynomials, because we can compute those simply). This is useful in all sorts of design problems, from architecture to font design. You will learn about a set-up for cubic splines with very smooth transitions between pieces made by matching curvatures! Maybe you will make some beautiful designer curves (knots or ...)?

The Whitney-Graustein Theorem

This is a pretty theorem which helps us decide when one curve can be deformed into another (through a "homotopy") by measuring their total curvatures or winding numbers.

Animating TNB for a Circular Helix

Make an animation of the FS framing along a circular helix, with enough interesting visuals to help people see the basic ideas.

Curves of Constant Width

It turns out that the circle is not the only shape in the plane with a constant diameter. There are many others, called "curves of constant width." The most famous is called the Releaux Triangle. There are many interesting properties of these curves.