AB Calculus Free Response 5.2

1. Let f be the function given by $f(x) = 3xe^{3x}$.

(a) Find $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$.

1: $\lim_{x \to -\infty} f(x)$

1: $\lim_{x \to \infty} f(x)$

(b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

1: solves $f'(x) = 0$

1: evaluates max/min at critical point

1: justification

(c) What is the range of f?

1: answer

(d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

1: sets y' equal to appropriate value

1: solves y'

1: evaluates y at critical point and gets answer independent of b

Solutions 1

(a) $\lim\limits_{x \to -\infty} f(x)\ 3xe^{3x} = 0$	1: 0 as $x \to -\infty$
$\lim\limits_{x \to \infty} f(x)\ 3xe^{3x} = \infty$	1: ∞ as $x \to \infty$
(b) $f'(x) = 3e^{3x} + 9xe^{3x} = 3e^{3x}(1 + 3x) = 0$	
$x = -\frac{1}{3}$	1: solves $f'(x) = 0$
$f(-\frac{1}{3}) = -\frac{1}{e}$	1: evaluates f at critical point
$-\frac{1}{e}$ is an absolute minimum because:	1: justification
$f'(x) < 0$ for all $x < -\frac{1}{3}$	
$f'(x) > 0$ for all $x > -\frac{1}{3}$	
and $x = -\frac{1}{3}$ is the only critical number	
(c) Range of $f = [-\frac{1}{e}, \infty)$	1: answer
or $-\frac{1}{e} < f(x) < \infty$	
(d) $y' = be^{bx} + b^2xe^{bx} = be^{bx}(1 + bx) = 0$	1: sets $y' = 0$
$x = -\frac{1}{b}$	1: solves for x
at $x = -\frac{1}{b},\ y = -\frac{1}{e}$	1: y-value independent of b
y has an absolute minimum value of $-\frac{1}{e}$ for all nonzero b.	

2. Let f be the function defined by $f(x) = e^x \sin x$.

(a) Find the average rate of change of f on the interval $0 \leq x \leq \frac{3\pi}{2}$.

1: answer

(b) What is the slope of the line tangent to the graph of f at $x = \pi$?

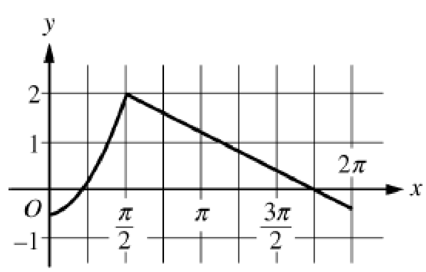
1: f'(x)

1: slope

(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.
1: sets f'(x) to appropriate value
1: identifies interior candidates
1: answer with justification

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$ and let $h(x) = e^x \cos x$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{h(x)}{g(x)}$ or state that it does not exist.

Justify your answer.
1: identifies limits situation
1: applies correct rule
1: answer



Graph of g'

Solutions 2

(a) The average rate of change on the interval $0 \le x \le \frac{3\pi}{2}$ is

$$\frac{f(\frac{3\pi}{2}) - f(0)}{\frac{3\pi}{2} - 0} = \frac{-e^{3\pi/2} - 0}{\frac{3\pi}{2}} = -\frac{2}{3\pi}e^{3\pi/2}$$ 1: answer

(b) $f'(x) = e^x \sin(x) + e^x \cos(x)$ 1: $f'(x)$

$f'(\pi) = e^{\pi}(0) + e^{\pi}(-1) = -e^{\pi}$ 1: slope

(c) $f'(x) = 0$ 1: sets $f'(x) = 0$

$e^x \sin(x) + e^x \cos(x) = 0$

$e^x(\sin(x) + \cos(x)) = 0$

$\sin(x) + \cos(x) = 0$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ 1: identifies $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$f(0) = 0$ as candidates

$f(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}e^{3\pi/4}$

$f(\frac{7\pi}{4}) = -\frac{\sqrt{2}}{2}e^{7\pi/4}$

$f(2\pi) = 0$

The absolute minimum value of f on $0 \le x \le 2\pi$ is $-\frac{\sqrt{2}}{2}e^{7\pi/4}$ 1: answer with justification

(d) $\lim_{x \to \pi/2} h(x) = 0$

Because g is differentiable, g is continuous. 1: g is continuous at $x = \frac{\pi}{2}$

$\lim_{x \to \pi/2} g(x) = g(\frac{\pi}{2}) = 0$ and limits equal 0

By L'Hôspital's Rule, 1: applies L'Hôspital's Rule

$$\lim_{x \to \pi/2} \frac{h(x)}{g(x)} = \lim_{x \to \pi/2} \frac{h'(x)}{g'(x)} = \frac{-e^{\pi/2}}{2}$$ 1: answer