MATH2001 Cheat Sheet

Contents

ODE's 2

Linear Algebra 5

Orthogonal Projection: 6

Construction of an Orthonormal Basis: 7

Least squares approximation: 8

The Determinant 11

Eigenvalues and Eigenvector 11

Diagonalisation 12

Orthogonalization 13

Orthogonal Diagonalisation 13

Quadratic Forms 14

Complex Matrices 15

Multivariable Taylor Series 17

Critical Points in n-dimensions 17

Calculus 19

Double Integrals 19

Double integrals in polar coordinates 19

Mass, centre of mass and moments 20

Triple Integrals 20

Cylindrical Coordinates 20

Spherical Coordinates 20

Moments of Inertia (second moments) 20

Vector Fields 21

Conservative Vector Fields 21

Gradient of a scalar field, conservative vector fields 21

The fundamental theorem for line integrals 22

Test for conservative fields 22

Green's Theorem 22

Flux of a vector field 23

Outward flux across a closed curve in the plane 25

Divergence of a vector field 25

Parameterisation in R^3 25

Tangent planes 25

Surface Integrals 26

Variable transformations in double integrals 26

Flux integrals and Gauss' divergence theorem 27

Curl of a vector field 27

Curl of a conservative vector field 27

Stokes' Theorem 27

ODE's

<u>If ODE in the form:</u>

$$Q(y)y' = P(x)$$

$$Q(y)\frac{dy}{dx} = P(x)$$

Directly solve

$$\int Q(y)dy = \int P(x)dx$$

<u>If ODE in the form:</u>

$$y' = P(x),\ directly\ integrate$$

<u>If ODE in the form:</u>

$$P(x,\ y) + Q(x,\ y)\frac{dy}{dx} = 0$$

ODE is exact if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Integrate Q with respect to x and then differentiate with respect to y and equate with P to find integration constant.

<u>If ODE in the form:</u>

$$y' + P(x)y + Q(x) = 0$$

Integrating factor:

$$I = e^{\int P(x)dx}$$

$$\therefore Iy = \int IQdx$$

<u>If ODE in the form:</u>

$$y'' + P(x)y' + Q(x)y = R(x)$$

If R(x) = 0, ODE is homogeneous, if R(x) != 0, ODE is nonhomogeneous.

If homogeneous:

1. Find characteristic polynomial:

$$\left(\lambda^2 + a\lambda + b\right) = 0,\ a = P(x)\ and\ b = Q(x)$$

Three possible solutions:

1. 2 real and distinct roots, $\lambda_1\ and\ \lambda_2$:

$$y_H = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

2. 2 real indistinct roots, λ:

$$y_H = Ae^{\lambda x} + Bxe^{\lambda x}$$

3. 2 real complex roots, $\lambda_1\ and\ \lambda_2$

$$y_H = Acos(x) + Bsin(x)$$

If R(X) doesn't equal 0:

Two possible methods:

1. Undetermined coefficients:
   If $r(x) = e^x + \cos cos\ (x)$, $y_p = g_1(x) + \ldots + g_n(x)$
   e.g for cos $cos\ (x)$, $g(x) = (x) + bcos(x)\ or\ if\ r(x) = 2,\ g(x) = ax$
   Differentiate and double differentiate g(x) and substitute into original ODE and solve and substitute IVP if present

1. (*10 marks*) Solve the initial-value problem

$$y'' - y' = 2, \qquad y(0) = 0, \qquad y'(0) = 1.$$

For y_H: $\lambda^2 - \lambda = 0 \Rightarrow y_H = A + Be^x$

For y_p: Guess $y_p = ax \Rightarrow y_p' = a$, $y_p'' = 0$

$\Rightarrow y_p'' - y_p' = -a = 2 \qquad \Rightarrow a = -2$

$\Rightarrow$ general solution is $y = A + Be^x - 2x$

$y' = Be^x - 2$

Initial conditions

$y(0) = 0 \Rightarrow A + B = 0$

$y'(0) = 1 \Rightarrow B - 2 = 1 \Rightarrow B = 3,\ A = -3$.

$\Rightarrow y(x) = 3e^x - 3 - 2x$

2. Variation of parameters:

   Given the general solution, e.g. $y_H = Ae^{-2x} + Be^{-x}$, set $y_1 = e^{-2x}$ and $y_2 = e^{-x}$

$$W = y_1 y_2' - y_1' y_2$$

$$y_P = uy_1 + vy_2$$

$$u(x) = -\int \frac{y_2 r(x)}{W} dx \text{ and } v(x) = \int \frac{y_1 r(x)}{W} dx$$

   Thus, $y = y_H + y_P$, $solve\ IVP\ if\ needed$

(MATH2000 Q1)

1. **(8 marks)** Find the general solution of the ODE

$$y'' + 3y' + 2y = \frac{1}{1+e^x}.$$

General solution is of the form $y = y_H + y_P$.

For y_H: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda+2)(\lambda+1) = 0$

$\Rightarrow y_H = Ae^{-2x} + Be^{-x}$.

For y_P: Use variation of parameters, with $y_1 = e^{-2x}$, $y_2 = e^{-x}$

$W = y_1 y_2' - y_1' y_2 = e^{-2x}(-e^{-x}) - (-2e^{-2x})e^{-x} = e^{-3x}$

$y_P = uy_1 + vy_2$, where $u = -\int \frac{y_2 r}{W} dx$, $v = \int \frac{y_1 r}{W} dx$ (with $r = \frac{1}{1+e^x}$)

$u = -\int \frac{e^{-x} \cdot e^{3x}}{1+e^x} dx = -\int \frac{e^{2x}}{1+e^x} dx$ (Set $t = 1+e^x \Rightarrow dt = e^x$)

$= -\int \frac{t-1}{t} dt$

$= -t + \ln t$

$= -(1+e^x) + \ln(1+e^x)$

$v = \int \frac{e^{-2x} e^{3x}}{1+e^x} dx = \ln(1+e^x)$

$\Rightarrow y = Ae^{-2x} + Be^{-x} - (1+e^x)e^{-2x} + e^{-2x}\ln(1+e^x) + e^{-x}\ln(1+e^x)$

(Note: can ignore this term [$-(1+e^x)e^{-2x}$] since it can be absorbed into y_H).

Linear Algebra

Basis: Let B be a set of vectors in the vector space V. B is a basis for V if B is linearly independent and B spans V (every vector in V can be expressed as a linear combination of basis vectors).

If an element of the span is linearly dependant on the others, it can be removed from span.

Transition Matrix: In general, $P_{B\to B'} = ([v_1]_{B'} | [v_2]_{B'} | \ldots | [v_n]_{B'})$

$B = \{1, x\}$, $B' = \{1+x, 2x\}$

let, $v(x) = A(1+x) + B(2x)$

$1 = A(1+x) + B(2x)$ | $x = A(1+x) + B(2x)$

$A = 1$, $B = \frac{-1}{2}$ | $A = 0$, $B = \frac{1}{2}$

$[1]_{B'} = \begin{pmatrix} 1 \\ \frac{-1}{2} \end{pmatrix}$ $\quad [x]_{B'} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$

$$P_{B \to B'} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

Inner product space: A vector space with an associated inner product is called an inner product space. As we are assuming all vectors are real, we look at real inner product spaces.

Orthogonality:

The norm of a vector is defined as:

$$||v|| = \sqrt{\langle v, v \rangle}$$

The distance between two vectors is defined as:

$$d(u, v) = ||u - v||$$

Two vectors are orthogonal if:

$$\langle u, v \rangle = 0$$

Angle between two vectors:

$$\theta = \arccos\ arccos\left(\frac{\langle u, v \rangle}{||u||||v||}\right),\ \theta \in [0, \pi]$$

Let U be a subset of the real inner product space of V. The orthogonal complement of U, $U^{\perp}$, is the set of all vectors in V that are orthogonal to every vector in U. That is:

$$U^{\perp} = \{v \in V \mid \langle v, u \rangle = 0 \ \forall\ u \in V$$

Let V be a real inner product space. A non-empty set of vectors in V is orthogonal if each vector in the set is orthogonal to all the other vectors in the set. That is, the set $\{v_1, ..., v_n\} \subseteq V$ is orthogonal if $\langle v_i, v_j \rangle = 0,\ i \neq j$. If V is orthogonal, V is linearly independent.

An orthogonal set of vectors in V is called orthonormal if all the vectors in the set are unit vectors. That is, the set $\{e_1, ..., e_n\} \subset V$ is orthonormal if $\langle e_i, e_j \rangle = 0,\ i \neq j\ or\ \langle e_i, e_j \rangle = 1,\ i = j$. An orthonormal basis for V is a basis for V that is also an orthonormal set.

Orthogonal Projection:

Let U be a finite-dimensional subspace of the real inner product space V. Then, each $v \in V$ can be written in a unique way as:

$$v = u + w,\ u \in V,\ w \in U^{\perp}$$

The vector $u \in U$ is called the orthogonal projection of v onto U and is given by:

$$Proj_U(v) = \langle v,\ e_1 \rangle e_1 + \ldots + \langle v,\ e_k \rangle e_k$$

Likewise, the vector $u \in U^{\perp}$ is called the orthogonal projection of v onto $U^{\perp}$ and is given by:

$$Proj_{U^{\perp}}(v) = v - Proj_U(v)$$

One can show that:

$$dimV = dimU + dimU^{\perp}$$

Suppose you have found dimV – dimU linearly independent vectors that are all orthogonal to U. Then, these vectors will, in fact, form a basis for $U^{\perp}$.

Construction of an Orthonormal Basis:

My method:

1. Identify if anything thing in span is a linear combination of the other, if so, remove from set.
2. Set a new vector, matrix etc to be in orthonormal basis with variables. Solve inner product with every element in set to find the values of the variables (inner product = 0 when solving)
3. Verify that new elements are orthogonal
4. Normalise by taking 1/sqrt(inner product).

Let $\{v_1, \ldots, v_n\}$ be a linearly independent set of vectors in the real inner product space V. The corresponding Gram-Schmidt process is the following algorithm.

1. Set $e_1 = \frac{v_1}{\|v_1\|}$

   i + 1: Let $U_i = span\{e_1, \ldots, e_i\}$

   set $w_{i+1} = v_{i+1} - Proj_{U_i}(v_{i+1}) = Proj_{U^{\perp}}(v_{i+1})$

$$\Rightarrow w_{i+1} \in U_i^{\perp}\ \&\ w_{i+1} \neq 0$$

$$set\ e_{i+1} = \frac{w_{i+1}}{\|w_{i+1}\|}$$

   Result:

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$w_2 = v_2 - \langle v_2,\ e_1 \rangle e_1\ \&\ e_2 = \frac{w_2}{\|w_2\|}$$

$$w_3 = v_3 - \left(\langle v_3, e_1\rangle e_1 + \langle v_3, e_2\rangle e_2\right) \ \& \ e_3 = \frac{w_3}{\|w_3\|}$$

The Gram–Schmidt process [edit]

We define the projection operator by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u},$$

where $\langle \mathbf{u}, \mathbf{v}\rangle$ denotes the inner product of the vectors **u** and **v**. This operator projects the vector **v** orthogonally onto the line spanned by vector **u**. If **u** = **0**, we define $\mathrm{proj}_{\mathbf{0}}(\mathbf{v}) := \mathbf{0}$, i.e., the projection map $\mathrm{proj}_{\mathbf{0}}$ is the zero map, sending every vector to the zero vector.

The Gram–Schmidt process then works as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}\\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}\\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}\\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|}\\
&\ \vdots & &\ \vdots\\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1}\mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

Example:

Euclidean space [edit]

Consider the following set of vectors in $\mathbf{R}^2$ (with the conventional inner product)

$$S = \left\{\mathbf{v}_1 = \begin{bmatrix}3\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}2\\2\end{bmatrix}\right\}.$$

Now, perform Gram–Schmidt, to obtain an orthogonal set of vectors:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix}3\\1\end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \begin{bmatrix}2\\2\end{bmatrix} - \mathrm{proj}_{\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} - \frac{8}{10}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-2/5\\6/5\end{bmatrix}.$$

We check that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are indeed orthogonal:

$$\langle \mathbf{u}_1, \mathbf{u}_2\rangle = \left\langle \begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-2/5\\6/5\end{bmatrix}\right\rangle = -\frac{6}{5} + \frac{6}{5} = 0,$$

noting that if the dot product of two vectors is 0 then they are orthogonal.

For non-zero vectors, we can then normalize the vectors by dividing out their sizes as shown above:

$$\mathbf{e}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}$$

$$\mathbf{e}_2 = \frac{1}{\sqrt{\frac{40}{25}}}\begin{bmatrix}-2/5\\6/5\end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix}-1\\3\end{bmatrix}.$$

Least squares approximation:

From the best approximation theorem, $u = Proj_U(v)$ is the best approximation to $v \in V$ in a finite-dimensional subspace U of V.

Solution 1: Use gram-schmidt to construct an orthonormal basis $\{e_1,\ldots,e_n\}$. Then:

$$Proj_U(v) = \langle v, e_1\rangle e_1 + \ldots + \langle v, e_n\rangle e_n$$

Solution 2:

Let $\gamma = \{u_1, u_2, ..., u_n\}$ be a basis of the subspace U. Then any $u \in U$ can be written as $u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$. We seek coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ that minimise $\|v - u\|$, or equivalently, minimise $\|v - u\|^2 = \|v - \left(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n\right)\|^2$ (same outcome, avoid square root).

$$\|v - \left(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n\right)\|^2 = \langle v - \left(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n\right), v - (\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)\rangle$$

$$= \langle v, v\rangle - 2\alpha_1\langle v, u_1\rangle - 2\alpha_2\langle v, u_2\rangle - ... - 2\alpha_n\langle v, u_n\rangle + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\langle u_i, u_j\rangle$$

$$= E(\alpha_1, \alpha_2, ..., \alpha_n)$$

Set $\nabla E = 0$

$$\frac{\partial E}{\partial \alpha_k} = -2\langle v, u_k\rangle + 2\sum_{l=1}^{n} a_l\langle u_k, u_l\rangle = 0,\ k = 1, 2, ..., n$$

This is a system of n equations with n unknowns, which may be expressed in the matrix form:

$$\begin{pmatrix} \langle \mathbf{u}_1, \mathbf{u}_1\rangle & \langle \mathbf{u}_1, \mathbf{u}_2\rangle & \dots & \langle \mathbf{u}_1, \mathbf{u}_n\rangle \\ \langle \mathbf{u}_2, \mathbf{u}_1\rangle & \langle \mathbf{u}_2, \mathbf{u}_2\rangle & \dots & \langle \mathbf{u}_2, \mathbf{u}_n\rangle \\ \vdots & \vdots & & \\ \langle \mathbf{u}_n, \mathbf{u}_1\rangle & \langle \mathbf{u}_n, \mathbf{u}_2\rangle & \dots & \langle \mathbf{u}_n, \mathbf{u}_n\rangle \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \langle \mathbf{v}, \mathbf{u}_1\rangle \\ \langle \mathbf{v}, \mathbf{u}_2\rangle \\ \vdots \\ \langle \mathbf{v}, \mathbf{u}_n\rangle \end{pmatrix}.$$

Solve this system for $\alpha_1, \alpha_2, ..., \alpha_n$ to obtain:

$$Proj_U(v) = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$$

Solution 3:

For least squares solutions of linear systems, we have a more direct (and simpler) method. Shown below (quadratic fit):

4 Data Points in the form (t, p(t)):

(1, 5)

(2, 2)

(4, 7)

(5, 10)

$$let\ p(t) = a_0 + a_1 t + a_2 t^2$$

Linear System:

$$a_0 + a_1 + a_2 = 5$$

$$a_0 + 2a_1 + 4a_2 = 2$$

$$a_0 + 4a_1 + 16a_2 = 7$$

$$a_0 + 5a_1 + 25a_2 = 10$$

$$\therefore A = (1\ 1\ 1\ 1\ 2\ 4\ 1\ 4\ 16\ 1\ 5\ 25\ ),\ B = (5\ 2\ 7\ 10\ )$$

Overdetermined (equations outnumber the unknowns) and inconsistent but the columns of a are linearly independent and thus, there exist a least squares solution.

$$\hat{x} = \left(A^T A\right)^{-1} A^T B = \left(8\ -\tfrac{9}{2}\ 1\right)$$

$$\therefore p(t) = 8 - \tfrac{9}{2}t + t^2$$

If $A^T A\ is\ invertible,\ \hat{x} = \left(A^T A\right)^{-1} A^T B$

$A^T A\ is\ invertible\ if\ the\ columns\ of\ A\ are\ linearly\ independant.$

Another Example:

Find the least squares approximation for sin(x) in the subspace of $C[0, \pi]$ spanned by $\beta = \{1, x, x^2\}$.
Use the following inner product:

$$\langle p, q\rangle = \int_0^\pi p(x)q(x)dx$$

Using "Solution 2" from above:

$$let\ y = \alpha_1 1 + \alpha_2 x + \alpha_3 x^2$$

Solve:

$$\left(\langle 1, 1\rangle\, \langle 1, x\rangle\, \langle 1, x^2\rangle\, \langle x, 1\rangle\, \langle x, x\rangle\, \langle x, x^2\rangle\, \langle x^2, 1\rangle\, \langle x^2, x\rangle\, \langle x^2, x^2\rangle\right)\left(\alpha_1\, \alpha_2\, \alpha_3\right) = \left(\langle sin(x), 1\rangle\, \langle sin(x), x\rangle\, \langle \sin sin\, (x)\right.$$

$$Note: \int_0^\pi x^n dx = \frac{\pi^{n+1}}{n+1}$$

$$\left(\pi\, \frac{\pi^2}{2}\, \frac{\pi^3}{3}\, \frac{\pi^2}{2}\, \frac{\pi^3}{3}\, \frac{\pi^4}{4}\, \frac{\pi^3}{3}\, \frac{\pi^4}{4}\, \frac{\pi^4}{5}\right)\left(\alpha_1\, \alpha_2\, \alpha_3\right) = \left(2\, \pi\, \pi^2 - 4\right)$$

Solving:

$$\alpha_1 = \frac{12(\pi^2 - 10)}{\pi^3},\ \alpha_2 = \frac{-60(\pi^2 - 12)}{\pi^4},\ \alpha_3 = \frac{60(\pi^2 - 12)}{\pi^5}$$

The Determinant

Under a linear transformation A, the area of any region in the x-y plane scales by the same amount. This amount (up to a sign) is called the determinant of A. If det(A) < 0, this implies the region has undergone a "flip" or change in orientation.

Eigenvalues and Eigenvector

To get eigenvalues, solve det $det\,(A - \lambda I) = 0$.

To then get the eigenvectors, substitute an eigenvalue into $(A + \lambda I)v = 0$.

$Where\ v\ is\ the\ matrix\ (a\ b\ c\ )\ if\ a\ 3x3\ matrix\ or\ (a\ b\ c\ d\ )\ if\ a\ 4x4\ matrix\ etc\ and\ solve\ for\ v.$

Repeat for all eigenvalues. See example below:

Now find the eigenvectors. Solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5I)\boldsymbol{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5.$$

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Diagonalisation

Steps to diagonalising an n x n matrix, A:

1. Find eigenvalues and eigenvectors
2. Check if A has n linearly independent eigenvectors
3. If no, A is not diagonalisable. If yes, A is diagonalisable. In this case, form matrix P = $(v_1|v_2|\ldots|v_n)$ where v are the eigenvectors. Then, the diagonalised matrix D = $P^{-1}AP$

Two matrices A and B are similar if there is a non-singular matrix P such that B = $P^{-1}AP$

The two statements "A is diagonalisable" and "A is similar to a diagonal matrix" are equivalent.

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \lambda \text{ are the eigenvalues}$$

The question remains, if A has fewer than n distinct eigenvalues, how do we know if A is diagonalisable?

16.3.1 Example

Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Easy to see the characteristic equation of both A and B is $(2-\lambda)(1-\lambda)^2 = 0$, so $\lambda = 2, 1, 1$. Solve $(A - \lambda I)\underline{v} = \underline{0}$

A: $\lambda = 2$: $\begin{pmatrix} 0 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$R_2, R_3 \to b = c = 0$, a free $\Rightarrow v_1 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 1$: $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow R_1$: $a + b + 3c = 0 \Rightarrow a = -b - 3c$

$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b-3c \\ b \\ c \end{pmatrix} = b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

two linearly independent eigenvectors for $\lambda = 1$.

$\Rightarrow$ A is diagonalisable.

B: $\lambda = 2$: $\begin{pmatrix} 0 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$R_2, R_3 \to b = c = 0 \to v_1 = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 1$: $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$R_2 \to c = 0$, $R_1 \to a + b = 0$

$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b \\ b \\ 0 \end{pmatrix} = b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

Only have two linearly independent eigenvectors

$\Rightarrow$ B is NOT diagonalisable.

- The geometric multiplicity of the eigenvalue λ_i is the dimension of the eigenspace (number of eigenvectors for this eigenvalue) corresponding to λ_i
- The algebraic multiplicity of the eigenvalue λ_i is the number of times $(\lambda - \lambda_i)$ appears as a factor in the characteristic polynomial.

A square matric is diagonalisable if and only if the geometric and algebraic multiplicities are equal for every eigenvalue.

If A is diagonalisable and the result is D, then $A^n = PD^nP^{-1}$ where D is the diagonalised matrix of A.

Orthogonalization

A square matrix, Q, is orthogonal if it is invertible and $Q^{-1} = Q^T$.

If $(v_1|...|v_n)$ is orthogonal $\Leftrightarrow\{v_1, ..., v_n\}$ is an orthonormal set.

Orthogonal Diagonalisation

Given an n x n matrix A, we call A orthogonally diagonalisable if there exists an orthogonal matrix, P such that $P^{-1}AP = P^TAP$ is diagonal.

An orthogonal matrix is a real square matrix Q such that the columns of Q are mutually orthogonal unit vectors with respect to the Euclidian inner product (i.e. $v_i \bullet v_j = 0\ if\ i \neq j,\ and\ ||v_i|| = 1$).

An immediate consequence of an orthogonal matrix is that det(Q) = ± 1

A matrix, A, is symmetric if $A = A^T$. Easy to identify as they are mirrored across the diagonal.

An n x n matric is orthogonally diagonalisable if and only if it is symmetric.

If A is symmetric:

1. All the eigenvalues of A are real
2. A has n linearly independent eigenvectors

Quadratic Forms

Two variable equation:

$$Q(x,\ y) = ax^2 + by^2 + cxy$$

$$Q(x,\ y) = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & \frac{c}{2} \\ \frac{c}{2} & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Three variable equation:

$$Q(x,\ y,\ z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

$$Q(x,\ y,\ z) = \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

19.2.1 Express $-3x^2 - 2y^2 - 3z^2 + 2xy + 2yz$ exclusively as a sum of square terms.

$$Q(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix}\underbrace{\begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{x}^T A \underline{x}.$$

From previous lectures, $A = PDP^T$ with

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}, \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$$\Rightarrow Q(x,y,z) = \underline{x}^T A \underline{x} = (\underline{x}^T P) D (P^T \underline{x}) = \underline{u}^T D \underline{u}$$

where

$$\underline{u} = P^T \underline{x} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x-z}{\sqrt{2}} \\ \frac{x+2y+z}{\sqrt{6}} \\ \frac{x-y+z}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\Rightarrow Q = -3u^2 - v^2 - 4w^2$$

or

$$Q(x,y,z) = -\frac{3}{2}(x-z)^2 - \frac{1}{6}(x+2y+z)^2 - \frac{4}{3}(x-y+z)^2$$

To identify a quadratic equation as a conic section:

1. Write the quadratic equation: $ax^2 + by^2 + cxy + dx + ey + f = 0$ in the matrix form $x^T Ax + Kx + f = 0$ where $x = (x\ y)$, $K = (d\ e)$.
2. Find a matrix P that orthogonally diagonalises A, so $A = PDP^T$. You may need to swap columns of P to ensure det(P) = 1 (and hence corresponds to a rotation, -1 corresponds to a reflection).
3. Define new variables u, v such that $v = (u\ v) = P^T x \Rightarrow x = Pv$
4. Substitute v into the matrix form of the equation giving $v^T Dv + KPv + f = 0$
5. Complete the square if required. This is necessary if u^2 and u are both present (or v^2 and v). This defines a new set of variables s, t by translating u, v. The translations will be in the form $s = \alpha u + \beta$ and $t = \gamma u + \delta$
6. If it is a non-degenerate conic, the final equation in s and t should be in a conic section standard form.

See A2 Question 2 for example.

Complex Matrices

Let a be a complex matrix (consists of complex numbers). The conjugate transpose of A, A* is given by $(\overline{A})^T$, where $\overline{A}$ is the matrix whose entries are complex conjugates of the corresponding entries of A. Note, that if A is real, $A^* = A^T$.

A complex matrix, A, is said to be unitary if $A^{-1} = A^*$

Complex inner product: $u \cdot v = u_1\overline{v}_1 + u_2\overline{v}_2 + \ldots + u_n\overline{v}_n$ where $\overline{v}$ is the complex conjugate of v.

In matrix notation: $u \cdot v = v^* u$

Hermitian (self-adjoint) matrices: A complex matrix, A, is called Hermitian if $A = A^*$. Can be identified similar to symmetric matrices:

$$\begin{pmatrix} a_{11} & a_{12}+ib_{12} \\ a_{12}-ib_{12} & a_{22} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12}+ib_{12} & a_{13}+ib_{13} \\ a_{12}-ib_{12} & a_{22} & a_{23}+ib_{23} \\ a_{13}-ib_{13} & a_{23}-ib_{23} & a_{33} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12}+ib_{12} & a_{13}+ib_{13} & a_{14}+ib_{14} \\ a_{12}-ib_{12} & a_{22} & a_{23}+ib_{23} & a_{24}+ib_{24} \\ a_{13}-ib_{13} & a_{23}-ib_{23} & a_{33} & a_{34}+ib_{34} \\ a_{14}-ib_{14} & a_{24}-ib_{24} & a_{34}-ib_{34} & a_{44} \end{pmatrix},$$

All symmetric matrices are Hermitian. All Hermitian matrices have real eigen values.

Unitary Diagonalisation: A square matrix, A, with complex entries is said to be unitarily diagonalisable if there is a unitary matrix, P, such that P^*AP is diagonal.

Normal matrices: A square, complex matrix is called normal if it commutes with its own conjugate transpose, i.e., if $AA^* = A^*A$. The following matrices are normal:

- Unitary
- Hermitian
- Real skew-symmetric ($A^T = -A$)
- Any diagonal matrix

Normal 2x2 matrices are either symmetric or of the form $(a\ b\ \ -b\ a)$

An n x n matrix is unitarily diagonalisable if and only if it is normal.

If possible, diagonalise the matrix $\begin{pmatrix} 6 & 2+2i \\ 2-2i & 4 \end{pmatrix} = A$.

$A^* = \begin{pmatrix} 6 & 2+2i \\ 2-2i & 4 \end{pmatrix} = A \Rightarrow A$ is Hermitian

$\Rightarrow A$ is normal $\Rightarrow A$ is unitarily diagonalisable

$\det\begin{pmatrix} 6-\lambda & 2+2i \\ 2-2i & 4-\lambda \end{pmatrix} = 24 - 10\lambda + \lambda^2 - 8$

$= (\lambda-8)(\lambda-2) \Rightarrow \lambda = 8, 2.$

$\lambda = 8$: $\begin{pmatrix} -2 & 2+2i \\ 2-2i & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$R_1 \Rightarrow -2a + (2+2i)b = 0 \Rightarrow \boxed{a = (1+i)b}$

$\Rightarrow \underline{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} (1+i)b \\ b \end{pmatrix} = b\begin{pmatrix} 1+i \\ 1 \end{pmatrix} \longleftarrow \underline{v}$

$a = (1+i)b \Rightarrow \frac{a}{1+i} = b \Rightarrow \frac{a}{(1+i)} \cdot \frac{1-i}{(1-i)} = b$

$\Rightarrow \frac{a}{2}(1-i) = b$

which is the equation hidden in R_2.

Similarly $\lambda = 2$: $\underline{v}_2 = a\begin{pmatrix} 1 \\ -1+i \end{pmatrix}$

Note that $\underline{v}_1^* \underline{v}_1 = (1-i)(1+i) + 1 = 3$
(set $b=1$) $\Rightarrow \|\underline{v}_1\| = \sqrt{3}$.

& $\|\underline{v}_2\| = \sqrt{1 + (-1-i)(-1+i)} = \sqrt{3}$.

Form the unitary matrix

$P = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1+i}{\sqrt{3}} \end{pmatrix}$. (check $PP^* = I = P^*P$).

then $P^*AP = D = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$.

Multivariable Taylor Series

Hessian matrix is defined as:

$$H = \left(f_{x_1 x_1} \ \cdots \ f_{x_1 x_n} \ \vdots \ \ \vdots \ f_{x_n x_1} \ \cdots \ f_{x_n x_n} \right)$$

e.g. $f(x,\ y) = x^3 y + 2y$

$$H = \left(f_{xx}\, f_{xy}\, f_{xy}\, f_{yy} \right) = \left(6xy\ 3x^2\ 3x^2\ 0 \right)$$

Multivariable Taylor series for f(x):

$$f(x,\ y) \approx f(a,\ b) + f_x(a,\ b)(x - a) + f_y(a,\ b)(y - b) + \frac{1}{2}\Big[f_{xx}(a,\ b)(x - a)^2 + 2f_{xy}(a,\ b)(x - a)(y - b) +$$

Critical Points in n-dimensions

Recall Taylor series of a multivariable function in n variables about a point $\mathbf{x} = \mathbf{x}_0$ is given by;

$$f(\mathbf{x}) = f(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0))^T(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T H(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \langle \text{ higher order terms} \rangle,$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $H(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}_0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}_0) \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{x}_0) \end{pmatrix} = H(\mathbf{x}_0)^T$

i.e. $H(\mathbf{x}_0)$ is a real symmetric matrix.

22.1 Classification of critical points in n dimensions

Definition 1. Critical points (extrema and saddle points) occur when

$$\nabla f(\mathbf{x}) = \mathbf{0} \qquad \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

or is undefined.

In this lecture, we will only be considering the first kind of critical point.

Definition 2. A critical point $\mathbf{x}_0$ is a *local maximum (local minimum)* if there exists some $\epsilon > 0$ such that
$f(\mathbf{x}_0) \geq f(\mathbf{x})$ $(f(\mathbf{x}_0) \leq f(\mathbf{x}))$ for all $\mathbf{x}$ such that $||\mathbf{x} - \mathbf{x}_0|| < \epsilon$

Definition 3. A critical point $\mathbf{x}_0$ is a *saddle point* if it is neither a maximum or a minimum, i.e. for all $\epsilon > 0$, there exists $\mathbf{x}_1, \mathbf{x}_2$ such that

$$||\mathbf{x}_1 - \mathbf{x}_0|| \leq \epsilon, \quad ||\mathbf{x}_2 - \mathbf{x}_0|| \leq \epsilon$$
$$\text{and } f(\mathbf{x}_1) < f(\mathbf{x}_0) < f(\mathbf{x}_2)$$

Since H is real symmetric, H is orthogonally diagonalisable. There exists a matrix P such that, $P^T HP = D$ with some diagonal matrix D. Since H is symmetric, all eigen values of H are real.

It follows that

$$\mathbf{x}^T H \mathbf{x} = (\mathbf{x}^T P) D (P^T \mathbf{x}) = \mathbf{y}^T D \mathbf{y}.$$

(i.e. diagonalisation suggests set $P^T\mathbf{x} = \mathbf{y}$).

The critical point is still at $\mathbf{y} = \mathbf{0}$, because $P^T\mathbf{0} = \mathbf{0}$.

Let F denote the function f expressed in this new coordinate system i.e. $F(\mathbf{y}) = f(\mathbf{x}(\mathbf{y}))$.

$$\begin{aligned}\implies F(\mathbf{y}) &= f(\mathbf{0}) + \frac{1}{2}\mathbf{y}^T D \mathbf{y} + \langle \text{ higher order terms } \rangle \\ &= f(\mathbf{0}) + \frac{1}{2}\left(\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2\right) \\ &\quad + \langle \text{ higher order terms } \rangle,\end{aligned}$$

where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$.

Four cases to consider, let λ_i be an eigenvalue of H:

1. $\lambda_i > 0\ \forall i = 1, 2, \ldots, n$ then local min.
2. $\lambda_i < 0\ \forall i = 1, 2, \ldots, n$ then local max.
3. If $\exists\ i, j\ (i \neq j) s.t\ \lambda_i, \lambda_j$ have opposite sides, then saddle
4. If all non-zero λ_i have same sign but there are some $\lambda_k = 0$, we can't identify critical point.

Semester Two Final Examination, 2019 MATH2001 Advanced Calculus and Linear Algebra II

6. (*10 marks*) Find all critical points of the function $f(x, y, z) = x^3 + z^2 - xy$ and classify them as local maxima, local minima or saddle points.

$\nabla f = \begin{pmatrix} 3x^2 - y \\ -x \\ 2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ Only critical point is $(0,0,0)$

Look at quadratic part, or Hessian matrix

$H = \begin{pmatrix} 6x & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ at $(x,y,z) = (0,0,0) \Rightarrow H = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\det \begin{pmatrix} -\lambda & -1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)(\lambda^2 - 1) \Rightarrow \lambda = 2, 1, -1.$

(We have $f(x,y,z) = \frac{1}{2}(x\ y\ z)\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + x^3$)

Since the eigenvalues are all non-zero & of opposite sign, critical point must be a saddle.

Calculus

Double Integrals

Fubini's Theorem:

$$If\ D = \{(x,\ y)|\ a \leq x \leq b\ ,\ c \leq y \leq d\}$$

$$\iint f(x,\ y)dA = \int_c^d \int_a^b f(x,\ y)dxdy$$

Special Case:

$$\iint f(x,\ y)dA = \int_c^d \int_a^b g(x)h(y)dxdy = \int_a^b g(x)dx \int_c^d h(y)dy$$

Type 1 Region – generally bounded by two constant values for x and two functions of x for y.

$$D = \{a \leq x \leq b,\ g(x) \leq y \leq h(x)\}$$

$$\iint f(x,\ y)dA = \int_a^b \int_{g(x)}^{h(x)} f(x,\ y)dydx$$

Type 2 Region – generally bounded by two constant values for y and two functions of y for x.

$$D = \{a \leq y \leq b,\ g(y) \leq x \leq h(y)\}$$

$$\iint f(x,\ y)dA = \int_a^b \int_{g(y)}^{h(y)} f(x,\ y)dxdy$$

If a region is made up of multiple type 1 and type 2 regions, split up integral and domain and then add together.

Interchanging order of integration: Can't just swap dx and dy, need to change the limits of integration as well. Can do this by graphing in the x-y plane and going from there.

Double integrals in polar coordinates

$$x = rcos(\theta),\ y = rsin(\theta)$$

$$\iint f(x,\ y)dxdy = \iint f(rcos(\theta),\ rsin(\theta))r\ dr\ d\theta$$

In other words:

$$dxdy \rightarrow rdrd\theta$$

Polar coordinates are useful when integrating regions of circles or ellipses as hard to parameterise. Use whenever x^2+y^2 is present.

Mass, centre of mass and moments

The centre of mass is located at coordinates $(\bar{x}, \bar{y})$, where:

$$\bar{x} = \frac{M_y}{m} = \frac{\iint xp(x, y)dA}{\iint p(x, y)dA}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint yp(x, y)dA}{\iint p(x, y)dA}$$

Where p(x, y) is the density function.

Triple Integrals

If $D = \{(x, y, z) | a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$

$$\iiint f(x, y, z)dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dxdydz$$

Cylindrical Coordinates

$$x = rcos(\theta),\ y = rsin(\theta),\ z = z$$

$$\iiint f(x, y, z)\, dxdydz = \iiint f(rcos(\theta), rsin(\theta), z) r\, dr\, d\theta\, dz$$

Useful for when working with cylinders

Spherical Coordinates

$$x = rcos(\theta) \sin sin (\phi),\ y = rsin(\theta) \sin sin (\phi),\ z = rcos(\phi)$$

$$\iiint f(x, y, z)\, dxdydz = \iiint f(rcos(\theta)sin(\phi), rsin(\theta)sin(\phi), rcos(\phi))\, r^2 sin(\phi)\, dr\, d\theta\, d\phi$$

Useful when working with spheres

Moments of Inertia (second moments)

The mass of a solid with a density p(x, y, z) occupying a region R in R^3 is given by:

$$m = \iiint p(x, y, z)\, dV$$

The moments about each of the three coordinate planes are:

$$M_{yz} = \iiint xp(x, y, z)dV$$

$$M_{xz} = \iiint yp(x, y, z)dV$$

$$M_{xy} = \iiint zp(x, y, z)dV$$

The centre of mass is located at the point $(\bar{x}, \bar{y}, \bar{z})$ where:

$$\bar{x} = \frac{M_{yz}}{m}, \bar{y} = \frac{M_{xz}}{m}, \bar{z} = \frac{M_{xy}}{m}$$

The moments of inertia about each of the three coordinate axes work out to be:

$$I_x = \iiint \left(y^2 + z^2\right)p(x, y, z)dV$$

$$I_y = \iiint \left(x^2 + z^2\right)p(x, y, z)dV$$

$$I_z = \iiint \left(x^2 + y^2\right)p(x, y, z)dV$$

Vector Fields

Conservative Vector Fields

Notation: $r = xi + yj + zk$

$$F(r) = F(x, y, z) = F_1(x, y, z)i + F_2(x, y, z)j + F_3(x, y, z)k$$

Gradient of a scalar field, conservative vector fields

For a differentiable scalar function, f(x, y, z) we define:

$$gradf = \nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

Thus

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$

If given a vector field, F(x, y) and asked to determine a potential function, integrate i component with respect to x and then partially derive with respect to y. Compare the partial derivative with respect to y with the j component to determine the integration constant. Do the same if three variables, F(x, y, z).

The fundamental theorem for line integrals

Work done by F along curve C:

$$W = \int F(r)\cdot dr = \int F_1(x, y)dx + F_2(x, y)dy$$

Steps to evaluate $\int F(r)\cdot dr$:

1. Parameterise C by finding a $r(t) = x(t)i + y(t)j,\ t\in[a, b]$
2. Write F® restricted to C as F(r(t)) = F(x(t), y(t))
3. Write $dr = r'(t)dt$
4. Convert the line integral into an ordinary integral in terms of the parameter, t:

$$\int F(r)\cdot dr = \int_a^b F(r(t)) \bullet r'(t)\, dt$$

If a vector field is conservative, $F = \nabla f \therefore \int F(r)\cdot dr = \int_a^b \nabla f\cdot dr = f(r(b)) - f(r(a))$

If a vector field is conservative, $\int F\cdot dr$ is path independent

Test for conservative fields

If $F = F_1 i + F_2 j$ is a conservative vector field, then:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

Green's Theorem

Let D be a region in the xy plane bounded by a piece-wise smooth, simple closed curve C, which is traversed with D always on the left (anti-clockwise). Let $F_1(x, y),\ F_2(x, y), \frac{\partial F_1}{\partial y}\ and \frac{\partial F_2}{\partial x}$ be continuous in D. Then:

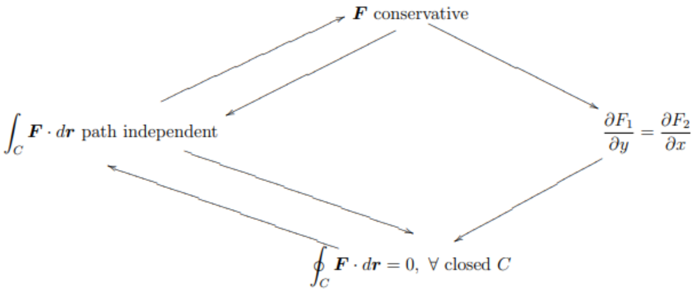
$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \oint_C F_1 dx + F_2 dy$$

If

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

Then

$$\oint_C F \cdot dr = 0$$

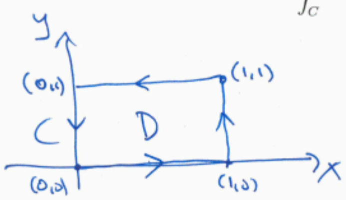


In particular, we now have a test to determine whether or not a given two dimensional vector field is conservative:

The vector field $\boldsymbol{F}$ is conservative if and only if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

7. (*10 marks*) Let C be the closed square path in the x-y plane connecting the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ traversed anti-clockwise, viewed from above. Evaluate the line integral

$$\oint_C \sin(x^3)dx + (xy^2 + x^2)dy.$$



By Green's Theorem

$$\oint_C F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy$$

$$= \int_0^1 \int_0^1 (y^2 + 2x)\, dx\, dy$$

$$= \int_0^1 (y^2 + 1)\, dy = \frac{1}{3} + 1 = 4/3$$

Flux of a vector field

In three dimensions, the flux of a vector field across a given surface is defined to be the "flow rate" of the vector field. Consider the velocity vector of a fluid. In three dimensions, the flux of a fluid across a surface is given in units of volume per unit time. In other words, the flux tells us how much fluid (volume) passes through a given surface in one second.

Flux integral (in 2D):

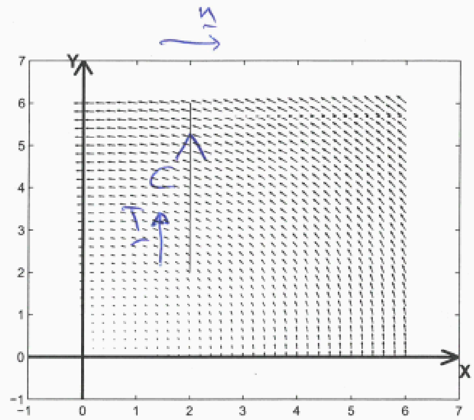
$$\int_C v \cdot n \, dS$$

To evaluate the flux integral:

1. Parameterise C by finding a $r(t) = x(t)i + y(t)j \; with \; t \in [a, b]$ that describes C
2. Write v(x, y) restricted to C as $v(r(t)) = v(x(t), y(t))$
3. Compute a unit tangent vector to C by using $T(x, y) = \frac{r'(t)}{||r'(t)||}$, where $r'(t) = x'(t)i + y'(t)j$ is a tangent vector to C
4. Be careful of the direction of n. By the definition of the cross product, and since k is a unit vector normal to the x-y plane, we can take n = T x k. We could also take n = k x T depending if asking positive or negative flux and the direction of n.
5. Write $dS = ||r'(t)||dt$
6. Evaluate the 2D flux integral as a definite integral in terms of the parameter t:

$$\int_C v \cdot n \, dS = \int_a^b v(r(t)) \bullet \left(r'(t) \times k\right) dt \; or \; \int_a^b v(r(t)) \bullet \left(k \times r'(t)\right) dt$$

Depending on the direction of n.



direction of positive flux, i.e. n

38.1.2 Calculate the flux of $v = -yi + xj$ (in the positive x direction) across the line $x = 2$ (for $2 \le y \le 6$).

Strategy: Flux $= \int_C v \cdot n \, ds$

C: $r(t) = 2i + tj$, $2 \le t \le 6$.

flux $= \int_2^6 v(r(t)) \cdot (r'(t) \times k) \, dt$

$\left(v(r(t)) = -ti + 2j, \; r'(t) \times k = j \times k = i \right)$

$\Rightarrow$ Flux $= \int_2^6 (-t) \, dt = -16$

Answer is negative! See from graph that v is "flowing" from right to left across C, but direction of positive flux was established as left to right across C.

271

Outward flux across a closed curve in the plane

Net outward flux = $\oint_C v \cdot n dS$

See 38.2.1, page 273

Divergence of a vector field

Divergence of a vector field is the "outward flux density"

Let

$$v(x, y, z) = v_1(x, y, z)i + v_2(x, y, z)j + v_3(x, y, z)k$$

Be a differentiable function. Then the function:

$$div\ v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \nabla \cdot v$$

Is called the divergence of v. Note that div v is a scalar quantity.

Outward flux across a closed curve in the plane using divergence (flux form of greens theorem):

$$\oint_C v(x, y) \cdot n dS = \iint_D div(v(x, y)) dA$$

Parameterisation in R^3

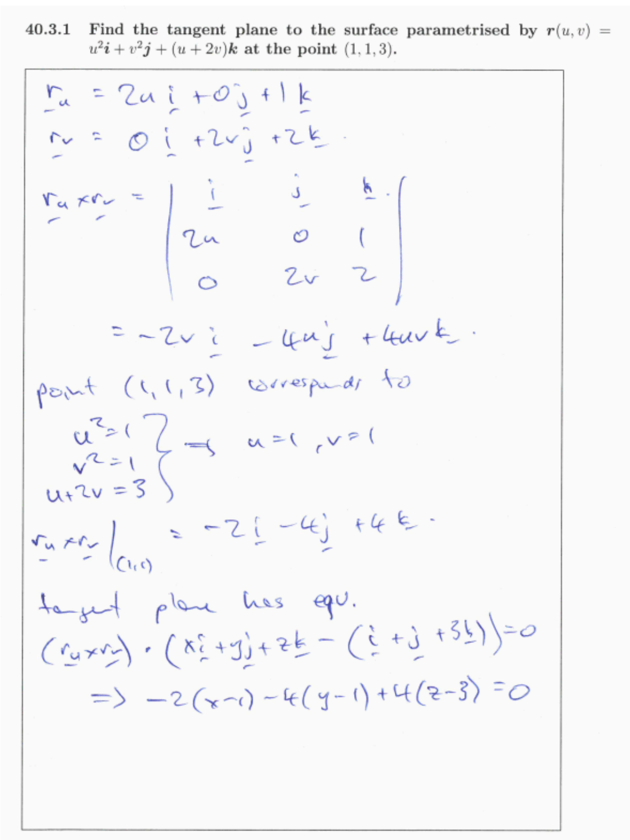
See chapter 40 (page 285) for parameterisation.

Tangent planes

Let S be a surface parameterised by $r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$

The equation of the tangent plane is given by:

$$\left(r_u(a, b) \times r_v(a, b)\right) \bullet ((xi + yj + zk) - r(a, b)) = 0$$



40.3.1 Find the tangent plane to the surface parametrised by $\boldsymbol{r}(u, v) = u^2\boldsymbol{i} + v^2\boldsymbol{j} + (u + 2v)\boldsymbol{k}$ at the point $(1, 1, 3)$.

$r_u = 2u\,i + 0\,j + 1\,k$

$r_v = 0\,i + 2v\,j + 2\,k$

$r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix}$

$= -2v\,i - 4u\,j + 4uv\,k$

point $(1,1,3)$ corresponds to

$u^2 = 1,\ v^2 = 1,\ u + 2v = 3 \Rightarrow u = 1, v = 1$

$r_u \times r_v \big|_{(1,1)} = -2i - 4j + 4k$

tangent plane has equ.

$(r_u \times r_v) \cdot (xi + yj + zk - (i + j + 3k)) = 0$

$\Rightarrow -2(x-1) - 4(y-1) + 4(z-3) = 0$

Surface Integrals

Let S be a smooth parametric surface given by $r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$

$$Surface\ Area = \iint_S dS = \iint_D \left\| r_u \times r_v \right\| du\, dv$$

See page 298 for applications. MAKE SURE DIRECTION IS CORRECT

Variable transformations in double integrals

$$\iint_R f(x,\ y)dx\,dy\ =\ \iint_S f(x(u,\ v),\ y(u,\ v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

Where

$$\frac{\partial(x,y)}{\partial(u,v)}\ =\ \det det\left(\frac{\partial x}{\partial u}\ \frac{\partial y}{\partial u}\ \frac{\partial x}{\partial v}\ \frac{\partial y}{\partial v}\ \right)\ =\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Is called the Jacobian of the transformation T.

A good example is 2020 S2 Q9

Flux integrals and Gauss' divergence theorem

Flux across S = $\iint_S v\cdot n\ ds\ =\ \iint_D v\cdot(r_u\times r_v)dudv$

Gauss' Divergence Theorem:

Let S be a piecewise smooth, orientable, closed surface enclosing a region in V in R^3. Let F(x, y, z) be a vector field whose component functions are continuous partial derivatives in V. Then:

$$\oiint_S F\cdot ndS\ =\ \iiint_V div(F)dV$$

Where n is the outwardly directed unit normal to S. MAKE SURE DIRECTION IS CORRECT

Curl of a vector field

$$curl(v) =\ \nabla\times v\ = \left|i\ j\ k\ \frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\ v_1\ v_2\ v_3\right| = \left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)i\ +\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)j\ +\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)k$$

Note that curl(v) is a vector field.

Curl of a conservative vector field

If F is conservative, curf(F) = 0

Stokes' Theorem

Let S be a piecewise smooth, orientable surface in R^3 and let the boundary of S be a piecewise smooth, simple, closed curve C. Let F(x, y, z) be a continuous vector function with continuous first partial derivatives in some domain containing S. Then:

$$\oint_C F \cdot dr = \iint_S (curlF) \cdot n \, dS$$

Where n is a unit normal vector of S, and the integration around C is taken in the direction using the right hand rule with n.

If normal vector is pointing up, param C anti-clockwise. If normal vector is pointing down, param C clockwise.

To calculate n if param is $S(r, \theta)$:

$$n = S_r \times S_\theta$$

If normal is in the wrong direction:

$n = S_\theta \times S_r$

Example:

12. (*10 marks*) Let C be a simple, closed, smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

can be expressed as a scalar multiple of the area of the region in the plane enclosed by C.

Let S be part of the plane $x+y+z=1$ enclosed by C.

$$\underline{F} = z\underline{i} - 2x\underline{j} + 3y\underline{k} \Rightarrow \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -2x & 3y \end{vmatrix} = 3\underline{i} + \underline{j} - 2\underline{k}$$

which is a constant vector field. A unit normal vector to S (regardless of orientation) is also a constant vector, so that $(\underline{\nabla} \times \underline{F}) \cdot \underline{n} = c$, a constant.

By Stokes' Theorem (with consistant orientation of $\underline{n}$ and C)

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dS$$

$$= c \iint_S dS$$

$$= c \times (\text{surface area of } S),$$

hence the result.