Answers to Grant Wiggins' Test for conceptual understanding of mathematics

1) "You can't divide by zero." Explain why not, (even though, of course, you can multiply by zero.)

This statement should be qualified, since there are mathematical structures which admit division by zero. Examples include the extended complex numbers, wheels, and the zero ring. But it is true that division by zero is not defined in any nonzero ring, such as the real or complex numbers. The reason is that the quotient a/b is defined as the unique solution to the equation $bx = a$. The equation $0x = a$ has no solutions when $a \neq 0$and it has multiple solution when $a = 0$. So in no case does the equation have a *unique* solution.

2) "Solving problems typically requires finding equivalent statements that simplify the problem" Explain – and in so doing, define the meaning of the = sign.

A typical pattern when solving equations is to write a chain of equivalent equations, beginning with the equation to be solved, and ending with the solution. Two equations are equivalent if they are both true or both false when the same values are substituted for their variables, i.e. they have the same *solution set*. The equations are connected by rules of inference which guarantee that they are equivalent. An example of a rule of inference is that we may add the same quantity to both sides of an equation. Note that some operations, such as squaring both sides of an equation, may violate equivalence. This can lead to *extraneous solutions* which are eliminated by checking.

The equals sign asserts that two expressions have the same value. The assertion may be true or false; $2 + 2 = 5$ is an example of a false equation. If an equation has variables, then it may be true for some values but not others. An equation that is true for all values of its variables is called an *identity*.

3) You are told to “invert and multiply” to solve division problems with fractions. But why does it work? Prove it.

By definition, dividing by a quantity is equivalent to multiplying by its multiplicative inverse. Since $\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1$ it follows that division by a/b is equivalent to multiplication by b/a.

4) Place these numbers in order of largest to smallest: .00156, 1/60, .0015, .001, .002

1/60, .002, .00156, .0015, .001

5) “Multiplication is just repeated addition.” Explain why this statement is false, giving examples.

One could argue that multiplication of natural numbers is just repeated addition. In Peano arithmetic, a well-known axiomatic system for the natural numbers, multiplication is defined recursively by $a \cdot 0 = 0$ and $a \cdot b' = a + (a \cdot b)$, where b' denotes the successor of b, i.e. $b + 1$. This recursion expresses the idea of repeated addition. But it is difficult to interpret multiplication by 2.5 as repeated addition (how do we add 2.5 times?) and it seems impossible to interpret matrix multiplication in this way. The word “multiplication” is used to describe a multitude of mathematical operations, and while there are connections between them, it seems impossible to give a universal definition.

6) A catering company rents out tables for big parties. 8 people can sit around a table. A school is giving a party for parents, siblings, students and teachers. The guest list totals 243. How many tables should the school rent?

We would need 31 tables to seat everyone on the guest list, since 30 tables would only seat 240. In real life, some people on the guest list will not arrive, and some guests will not be on the list, so the school should rely on its experience with previous events to decide how many tables to rent.

7) Most teachers assign final grades by using the mathematical mean (the "average") to determine them. Give at least 2 reasons why the mean may not be the best measure of achievement by explaining what the mean hides.

This is not true. Most teachers use point totals or some kind of weighted average instead of a simple arithmetic mean; they may also drop low scores or give extra credit. It is true that the mean hides a great deal of information, but the same is true of any statistic; no single number can capture all of the information in a data set.

One drawback of the arithmetic mean is that it is sensitive to outliers, such as zeros for missing assignments. Another drawback is that it treats all assessments equally, regardless of importance, and it ignores trends in performance.

8) Construct a mathematical equation that describes the mathematical relationship between feet and yards. HINT: all you need as parts of the equation are F, Y, =, and 3.

We also need to declare the variables F and Y to avoid ambiguity. If F denotes a foot and Y denotes a yard, then it is correct to say that $Y = 3F$. However, it is more useful to declare that F is the length of an object in feet, and Y is the length of the same object in yards. In this case, F and Y are dimensionless quantities, and we would express the relation between them as $F = 3Y$.

9) As you know, PEMDAS is shorthand for the order of operations for evaluating complex expressions (Parentheses, then Exponents, etc.). The order of operations is a convention. $X(A + B) = XA + XB$ is the

distributive property. It is a law. What is the difference between a convention and a law, then? Give another example of each.

> The distributive property is a rule concerning addition and multiplication in a ring. Order of operations is used to disambiguate expressions. The rules have different domains; the first refers to the mathematical operations themselves, the second is a rule about notation.
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> The fact that we write integers in base ten is a convention; the properties of the integers would not be altered if we used a different base of numeration. But the fact that every integer has a unique decimal expansion **is** a law (more properly, a theorem) because it expresses an intrinsic property of the integers.

10) Why were imaginary numbers invented? [EXTRA CREDIT for 12^{th} graders: Why was the calculus invented?]

> Complex numbers arose from attempts to solve cubic equations. If a cubic polynomial has integer coefficients and three distinct real roots, then in order to express the roots with radicals, it is necessary to extract the square root of a negative number. This inconvenient fact convinced mathematicians to take complex numbers seriously.
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> Calculus arose from a multitude of sources, beginning with the work of Archimedes and Eudoxus, but achieved full development through the work of Newton and Leibniz, spurred by the need to describe the fundamental laws of physics.

11) What's the difference between an "accurate" answer and "an appropriately precise" answer? (HINT: when is the answer on your calculator inappropriate?)

> An estimate is accurate if it differs from the true value by a sufficiently small amount; it is precise if it has a small resolution, e.g. to the nearest millimeter. An estimate is appropriately precise if

its precision does not exceed its accuracy. The statement “I am 1.7277 meters tall” is an example of inappropriate precision, because I cannot measure my height accurately to the nearest 0.1 mm.

12) “In geometry, we begin with undefined terms.” Here’s what’s odd, though: every Geometry textbook always draw points, lines, and planes in exactly the same familiar and obvious way – as if we CAN define them, at least visually. So: define “undefined term” and explain why it doesn’t mean that points and lines have to be drawn the way we draw them; nor does it mean, on the other hand, that math chaos will ensue if there are no definitions or familiar images for the basic elements.

The words *point*, *line*, and *plane* are undefined because they refer to objects of our intuition; they are not defined in terms of other mathematical objects. Euclid famously defined a point as “that which has no part,” but this is not considered a proper definition by modern mathematicians.

By allowing these terms to remain undefined, we are free to interpret the axioms in ways that could not have been anticipated by Euclid. For example, finite geometries have found many practical applications even though they violate our geometric intuitions. Chaos does not ensue because we demand that our axioms must be consistent (without contradictions) and we abandon axioms when they do not appear to be fruitful.

13) “In geometry we assume many axioms.” What’s the difference between valid and goofy axioms – in other words, what gives us the right to assume the axioms we do in Euclidean geometry?

Axioms are valid if they are free from contradictions. Axioms are goofy if they are inconsistent, uninteresting, or useless. Of course a set of axioms could be valid and goofy at the same time. As mathematicians in the 21st century, we have perfect freedom to investigate any consistent axiomatic system. But if we cannot identify an interesting model for the axioms, then others may not be interested in our work.

The axioms of Euclidean geometry are useful because they make predictions about the world that are borne out by observation. If I draw a triangle on a sheet of paper, and I measure the angles, the sum is invariably 180° to within measurement error.

-- David Radcliffe