Episode 8-Mathematics as an Area of Knowledge

The Scope of Mathematics

Our goal today is to gain a better understanding of mathematics as an area of knowledge. We'll begin with the issue of scope. What does mathematics study? Most areas of knowledge have several features in common. They aim to give a true description of the world and they identify generalized relationships and causal connections. They do vary in their methods and the scope of their conclusions because they study different aspects of the shared world. That is, they investigate different subject matter. Now math has both similarities and differences to other areas of knowledge. In terms of similarities, mathematics asked similar questions. How do we know what methods can we use to investigate and what justifications can we offer for our knowledge claims. However, math also has some important differences. Mainly it is completely divorced from the physical world and deals solely with abstraction.

Mathematics as the Discovery of Patterns

It is an area of clarity and certainty, although to what degree it is certain is something we must examine. Above all, mathematics is the study of patterns, abstract patterns that place concepts in a systemized relationship to one another, expressed in a symbolic system that we can manipulate using reason alone. Well, there are many types of patterns. There are real or imagined, visual or mental, static or dynamic, qualitative or quantitative, utilitarian or abstract. And different kinds of patterns are studied in different maths--number theory looks at patterns of numbers and counting. Geometry looks at patterns of shape. Calculus looks at patterns of motion, logic, patterns of reasoning, probability patterns of chance, topology, patterns of closeness and position. So let's look at the degree to which mathematics is involved in the real world.

Is Math Invented or Discovered?

One of the big issues in mathematics is do we discover mathematics? Or do we invent it? Galileo and GH Hardy believe that we discover it. Einstein, on the other hand, questioned the connection between human thought and the world. What we know for certain is that there's an uncanny relationship between mathematical knowledge and the world and its structures. A couple of examples would be pi and Euler's constant, pi, a ratio of a circle's circumference to diameter. They all seem to give us universal knowledge; Euler's constant, for example, can be used to describe radioactive decay, the spread of pandemics, compound interest, behavior and population growth. All of these imply a connection between the world and mathematical calculations. One mathematician has suggested that when we begin a new area of mathematical study, we believe we are creating it. But as we become more immersed in the study, we see parallels with the universe, and then we believe we are discovering it.

Pure and Applied Math

Pure Math

What the preceding examples show is that simultaneously with its movement towards abstractions of the mind, math also establishes connections with the world. One way of dealing with the real or invented question is divide math into the categories of "Pure" and "Applied" math. Pure math deals with algebra, geometry, number theory and topology, all of which do not deal with the physical world or have any direct application. Applied

Mathematics on the other hand, deals with scientific computing, mathematical physics, information theory, control theory, actuarial science, all of which involve practical applications of math to the world. John Kemeny, who you might recall from our discussion in the philosophy of science, explains the difference thusly, he said pure mathematicians deal strictly with the skeletal form of logical arguments. Take for example, the proposition “X, bears a relation R to Y.” Pure math is only interested in the relation R between the two variables, and it does not define what x and y are. It is simply positing a universal relationship are between two variables x and y. This guarantees that if x and y are true, then we can also be certain that the relation R is true. Thus, pure math deals solely with the *logical form* of arguments.

Applied Math

Applied Math is when we decide to specify what the values of x and y mean. Here's an example. Take these three axioms:

1. there is an x such that no other experts the relation are to it.
2. no X has more than 1x bearing the relation R to it.
3. each x bears the relation are to just one other x and never bears this relation to itself.

If we substitute x with “dog” and the relation R is “bitten,” we will get the following

1. there is a dog who is not bitten.
2. No dog gets bitten by more than one dog-each dog bites one other dog but never bites itself.

Now, under this scenario, it would only work if there is an infinite number of dogs, which clearly there aren't. However, if we substitute x with “year,” and the relation R is “proceeds,” then we get a true argument x

1. there's a year AD which is not preceded by any other year AD.
2. No year AD is preceded by more than one year.
3. each year AD proceeds one other such year but never precedes itself.

Once we define our terms as in the two examples above, we pass from the realm of pure math into applied math. And this is normally what goes on in science when a mathematical process is used to explain the behavior of data.

Mathematics as Shared Knowledge

Whether we're dealing with pure or applied math, mathematics is shared knowledge, meaning the individual work of mathematicians is contributing to a communal pool of knowledge. Mathematics uses peer review, and scrutinize es individual efforts for errors in order to determine what constitutes this shared knowledge. However, it differs from other forms of shared knowledge in several important ways. One, mathematicians work individually, unlike many scientists, for example. Also, once the knowledge is established, it us certain and unchangeable, and it is the basis for all subsequent math. Still, while math is done individually, it does not imply this is static research--ideas are borrowed from many other people in many other cultures, and the areas of mathematics are ever-expanding. In conclusion that works less like a science and more like a creative enterprise, more akin to poetry or novel writing.

Mathematics and the Ways of Knowing

Sense Perception

Of course, we can't talk about math without talking about the ways of knowing. Now it is somewhat artificial to separate the ways of knowing, and what we're looking at is the *balance* between the ways of knowing and mathematics, beginning with sense perception. And it's not clear that sense perception plays a very vital role in mathematics. Danzig argues that there's no tangible connection between math and sense perception. He uses the example of the garment wearer, who has no conception of who will wear the clothes, but this does not stop him from his task. Likewise, the mathematician is only interested in the structures or the arguments of the claim, not their ultimate usage. So, looking at sense perception, Alfred North Whitehead, who was one of the greatest mathematicians of the 20th century, said math is certain precisely because mathematics distances itself from the sensory world, which allows it to be free of all of the uncertainty of observation. It allows math to produce results true in all cases, without exception. Of course, uncertainty is a precondition of sensory observations. While we may use our senses, our justifications do not rest on that observation, they rest on reason.

Language

Looking at language, the justification for mathematical conclusions does not rest on ordinary language. We do use language to learn and teach concepts. However, mathematics uses mathematical symbolism, which is a subset of the ideas developed in language. As it has grown it has taken on many of the features of a language itself, as a symbolic system that allows ideas to be manipulated in the mind and communicated to others. What are the characteristics of this mathematical language? Well, mathematical symbols have a very precise meaning. Likewise, the relationship between the symbols are rule-defined this symbolic language is far more precise than natural language because it is a precise statement of an abstract rational argument. That is, it's explicit, and compact. As the example of the Prothagorean theorem can attest, is transferable without the loss of reason. Examples would include the logical process of commutation, reducing fractions, or factoring polynomials. All of them transfer meaning from one form to another without any loss of meaning. Compare that to translating from one language to another. The language of math is completely abstract and conceptual, manipulating its statements using only its own rules. It can lead to conclusions that may not be new, but present knowledge that's new to us. Well, how does the language of math work? Well, it's a purely self-referential and abstract system and this is what makes it certain. It focuses on relationship variables between terms, relationships, such as symmetry, proportion, or sequence. It also provides the vocabulary and grammar enables us to talk about such relationships when we apply maths. We specifically look at what terms and variables mean, and we can then investigate how these patterns operate in the world. Miles Davenport highlights a crucial function of math as the explainer of patterns. Most sciences split or divide knowledge into separate categories in order to better understand them. Math use patterns to lump or connect what often appear to be disparate elements into a single pattern.

Imagination and Intuition

In examining intuition and imagination as ways of knowledge on the surface it appears that math like other areas of knowledge, use both of them to gain new insights. The question is, does it operate in the same manner as other areas of knowledge? Intuition, defined in most ToK texts as the rough and fast grasp of patterns is fraught with faults and is not particularly useful in mathematical knowledge. The solutions of math are often counterintuitive.

Imagination, on the other hand, is a more fruitful avenue. Imagination defined as the capacity to reassemble familiar concepts into new ones, or to project beyond them into fresh conceptualizations, is often at work in math. Mathematics, uses imagination in a different manner, however, than other areas of knowledge. Although mathematicians speak as though their mathematical objects and concepts are real, the subject matter of mathematics is already in the world of the imagination, even before they manipulate it creatively. Mathematicians use propositional imagining; they imagine, that a statement is true, and then trace the implications of considering it to be so much as a scientist does with a hypothesis. Beyond this, not only is imagination used in the generating of the proposed statement, but the testing of it demands that the imagination be engaged because the implications are played out in the realm of the imagination. Cantor is an example. He started with an abstract concept of infinity and then constructed a proof for the existence of multiple infinities. Imagination was at work as he used proof by contradiction or *reductio ad absurdum* to complete the proof. This is both a creative and abstract approach and hear imagination may be used in a different manner than other areas of knowledge.

Reason

Of course, the final and most important way of knowing in mathematics is reason. Reason is the primary way of knowing, as mathematics operates independently of sense perception. So, it is the primary investigative tool. How exactly does reason function in mathematics? Mathematics uses deductive reasoning, which builds new logically derived conclusions from its initial premises. Deduction uses strict rules to perform operations that transform initial premises into new forms that preserve the truth value of the original premises. The premises then provide the content or subject matter to which reasoning is applied. A key issue in math and deductive logic is the difference between validity and truth. Validity is the exclusive domain of deductive logic. Valid means that conclusion follows directly from the premises by applying the rules of logic. Math, especially pure math, is based solely on logical consistency. Truth, on the other hand, is not a mathematical concept. Math done correctly shows an internally consistent pattern. It guarantees that if the premises are true, then we're certain that the conclusion is true. Validity then guarantees the certainty of true premises. This distinction has significant implications for the issue of certainty in mathematics. If mathematics is primarily about structure, about patterns about validity, then the issue becomes how do we establish certainty? This in turn requires an understanding of how the concept of justification has changed over the course of the history of mathematics.

Mathematics and Historical Search for Certainty

Since we know that math builds in a cumulative and building block fashion, the issue becomes how do we justify the premises of any mathematical system? Well, it all started 2300 years ago, when Euclid identified 10 axioms or postulates from which he claimed to be able to deduce the propositions of geometry, he further claimed that they did not require justification because they were self evident. And for 2000 years, this was considered sufficient justification to establish the validity of the system. It was certainly valid meaning self-consistent, but was it true? It would seem that if they were true, they must be the only possible correct axioms. Furthermore, they were useful, readily applicable to the everyday world. In investigating these axioms, the first four appear to be self-evident because they can be illustrated. The fifth axiom, the parallel postulate was impossible to prove because the line could continue forever. It was also impossible to prove it as if it were a theorem. Fortunately, this axiom was only used in one proof and Gauss in the 1800 showed that geometry could be built without it. However, Gauss work got mathematicians thinking about whether there could be alternate sets of axioms and paved the way for non Euclidean geometries, which expose the flaw in the Euclidean system and called the truth (but not the validity) of Euclidean geometry in to question at the beginning of the 18th century. An ingenious approach was suggested by Gauss, that of indirect proof, assuming that the parallel axiom does not follow from the others, then we would not get a contradiction if we deny it. They said let’s assume that we can draw more than one parallel to a given line through a given point and show this leads to an absurdity. The problem was, they detected no absurdity. Out of this came the work of Bolyai and Lobachevsky, who created new geometries. This was further developed by Gauss and Ryman, who also showed there was a third type of geometry, as well as an infinite number of mixed geometries. After Gauss, Euclid’s system was considered valid, but not true in the sense that there could be other consistent systems. This shifted the mathematical understanding of truth away from the self-evident nature of axioms, the ***correspondence theory of truth***, to the idea that a true mathematical system was true, because it was internally consistent, the ***coherence theory of truth***. In this view, Euclidean ,Lobachevskian and Riemannian geometry all produced true mathematical systems, the criterion for truth shifted from self-evidence to self consistency. However, this in turn demanded a new means of establishing truth, and made the concept of proof of paramount importance. Of course, proofs are the rigorous application of logical rules that extend the knowledge contained in the axioms. The new knowledge or theorems are simply the logical implications contained in the premises, but the information is new to us. While mathematical work is individual, it becomes shared knowledge when it's checked by other mathematicians through peer review. The breadth proofs are not only rigorous, they're also economical, which lends them to a degree of elegance. This other criterion that was used as mathematics searched for a third approach to establishing certainty. Thus far, we've examined how proofs work within mathematics that builds upon the shared knowledge confirmed through peer review. Thus far, we've rejected the correspondence theory of truth in favor of coherence theory of truth. However, this came under attack in the 20th century. Thus, we return to the idea of how we justify the axioms or underlying assumptions of mathematical systems? Bertrand Russell showed an internal contradiction within all mathematical systems. This meant that a purely internally coherent system, a requirement of the coherence theory, is impossible. His proof focuses on the efforts to build a set of all sets that are not members in themselves. Russell argued, if I make a catalogue of all the British Library catalogues, and that catalogue is not included, then the set is incomplete. However, if I include that catalogue, then it's no longer a catalogue of all catalogues as an includes itself. So, if the catalogue doesn't include itself, it shouldn't. And if it doesn't include itself, it should. To use an analogy from language. Consider the “Liars Paradox.” All Cretans are liars. If he's telling the truth, does that mean he's lying? If he's lying, does that mean he's telling the truth? The attempt to explain the internal contradictions led to ***Goodell's Incompleteness Theorem***. There is no guarantee he argues that there isn't a contradiction with any axiomatic system. He found this out by trying to refute Russell by grounding mathematics more firmly in logic, but he once again encountered Russell's paradox. Now, this did not affect the view of most mathematicians, because they were perfectly willing to accept that there was absolutely no certainty about the axioms. However, from ToK point of view, it means that we must seek to abandon the coherence theory and we seek another method of justifying the starting point of mathematical inquiry. Goodell leaves us with a clue about how we might proceed. He says that every mathematical system rests upon an assumption or set of assumptions that are themselves outside of that system. So, moving towards considering a new path to the truth in mathematics, math started out with the somewhat naive view that its axioms were self-evident. When it turned out that they were not, it opted for formalism, that math was just a game and as long as you were consistent, then your claims were justified. That is, it adopted a correspondence theory of truth. Russell, Goodell, and Hilpert showed that this was not a sufficient justification because every system runs into internal contradictions when trying to be internally consistent, "Russell Paradox," Thus, every mathematical system is incomplete. It relies on something outside itself to generate its axioms, Hilbert tried to refute Russell using plain geometry, but he relied upon undefined terms such as "point" and line." However, when you specify what a variable is, you're no longer doing pure math, you are relying on definitions that are themselves outside the mathematical system. As math only deals with the abstract relation of "all blank are blank." Once you define blank, you're appealing to a reality outside the mathematical system. Hilbert ends up showing maths cannot be purely self referential. So. what does this imply about mathematical certainty? Well, mathematics cannot be certain solely on the basis of their axioms being self-evident. So we cannot rely on a correspondence theory of truth. Mathematics cannot be proven to be internally self-consistent, so we must abandon a coherence theory of truth, we are left with a ***pragmatic theory of truth***. In his book, *A Philosopher Looks at Science*, John Kemeny argues that the way we gain greater (although not absolute certainty) in mathematics is, ironically, through applied math, by which he means the testing of mathematical hypothesis in science, and seeing if they produce useful information. The more math system is successful in explaining the world without contradicting itself, the more useful and the more certain it is. In the end, there seems to always be a mutual dependence of Pure and Applied Math upon one another.

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As we wind down this review, let's consider mathematics in a larger context in a social context. Mathematics is created by individuals and confirmed through peer review. While it's very abstract, it nevertheless has an intimate relationship with the natural world. However, what is its relationship with the social context within which it's generated? Well, math would appear to be universal, as it is a rational depersonalized approach that has nothing to do with any of us as individuals, but has to do with all of us in its findings. There are six forms of recurring mathematical ideas that span all cultures--counting, locating, measuring, designing, playing and explaining. As for the cultural aspect, it is true that there are innumerable different counting system, each containing a subtly different concept of space and having a different social context, or what the math is used for. The cultural dominance of Western mathematics implies that the so-called Western mathematics draws from diverse traditions, it nevertheless has imposed certain viewpoints on mathematics as rational and objective, a view of the world as a series of discrete objects. it is based on the concept of universality, and cultural neutrality. Ethno-mathematics, a newer branch of mathematics, shows that there are cultures that use math in a more integrated context. In conclusion, math seems to be universal in its thought process, but cultural in this specific forms; it takes social attitudes towards mathematics in considering cultural contexts. We're left with a series of questions regarding mathematics. What role should math play in education and society? The controversy is between practical mathematical calculation or math as a discipline that teaches thinking processes. Invariably, math is used as an entrance barrier, a test to sort out better students, rather than giving them useful information. Is this justified? The crucial issue here is the same one we began with--pure math for the discovery ideas, or applied math for the things that are immediately practical? Of course, we need both, and so does math, as Kemeny has so aptly demonstrated. Thank you for listening to this podcast.